An introduction to contact geometry and topology

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Outline

1. Introduction
2. Some differential geometry
3. Examples, applications, origins
   - Examples of contact manifolds
   - Classical mechanics
   - Geometric ordinary differential equations
4. Fundamental results
5. Ideas and Directions
   - Contact structures on 3-manifolds
   - Open book decompositions
   - Knots and links
   - Surfaces in contact 3-manifolds
   - Floer homology
Overview

An introduction to contact geometry and topology:
- What it is
- Background, fundamental results
- Some applications / “practical" examples
- Some areas of interest / research
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Standing assumptions/warnings:

- All manifolds are smooth, oriented, compact unless otherwise specified.
- All functions smooth unless otherwise specified
- Smooth = $C^\infty$
- Beware sign differences
- Biased!
What is contact geometry?

Contact geometry is...

- Some seemingly obscure differential geometry...

Contact geometry has a sibling: symplectic geometry.
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- ... but actually deeply connected to lots of physical and practical situations!
- A major area of research in contemporary low-dimensional geometry and topology
- Connected to many fields of mathematics:
  - symplectic geometry, Gromov-Witten theory, moduli spaces, quantum algebra, foliations, differential equations, mapping class groups, 3- and 4-manifolds, homotopy theory, homological algebra, category theory, knot theory...

Connected to many fields of physics:
- classical mechanics, thermodynamics, optics, string theory, ice skating...
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Some differential geometry

Let $M$ be a manifold.

**Definition**

A **symplectic form** on $M$ is a closed 2-form $\omega$ such that $\omega^n$ is a volume form.

A **contact form** on $M$ is a 1-form $\alpha$ such that $\alpha \wedge (d\alpha)^n$ is a volume form.
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Immediate consequences:

- Symplectic forms only exist in even dimensions $2n$.
- Contact forms only exist in odd dimensions $2n + 1$. 
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- Insertion of vectors into $\omega$ yields an isomorphism between vectors and 1-forms, $v \leftrightarrow \iota_v \omega = \omega(v, \cdot)$.
- The kernel of a contact form $\xi = \ker \alpha$ is a codimension-1 plane field on $M$. 
Integrability

Frobenius’ theorem:

\[ \alpha \wedge (d\alpha)^n \neq 0 \text{ implies } \xi \text{ is maximally non-integrable} \]
— any submanifold \( S \subset M \) tangent to \( \xi \) must have dimension \( \leq n \).
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A contact structure \( \xi \) on \( M^{2n+1} \) is a maximally non-integrable codimension-1 plane field.
\((M, \xi)\) is a contact manifold.
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A contact structure $\xi$ on $M^{2n+1}$ is a maximally non-integrable codimension-1 plane field. $(M, \xi)$ is a contact manifold.

- A submanifold tangent to $\xi$ is called Legendrian.
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- The kernel of a contact form is a contact structure.
- A contact manifold has many contact forms: as many as smooth nonvanishing functions on $M$. 

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A contact form co-orient $\xi$. 
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Examples

Standard / “only” examples of symplectic & contact structures:

- $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$ and $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$. 

- In $\mathbb{R}^3$: $\alpha = dz - y \, dx$, $\xi = \text{span}\{\partial_y, \partial_x + y \partial_z\}$. No tangent surfaces! Only tangent/Legendrian curves! Legendrian knots and links.
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Another example:

- Cylindrically symmetric standard contact structure on $\mathbb{R}^3$

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\alpha = dz + r^2 \ d\theta, \quad \xi = \text{span} \ \left\{ \partial_r, \ r^2 \partial_z - \partial_\theta \right\}.
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This is equivalent to the standard contact structure.

**Definition**

A **contactomorphism** is a diffeomorphism $f$ between contact manifolds $(M_1, \xi_1) \rightarrow (M_2, \xi_2)$ such that $f_*\xi_1 = \xi_2$. 
Ice skating

The unit tangent bundle $UTS$ of a smooth surface $S$ has a canonical contact structure $\xi$:

- Take $(x, v) \in UTS$, with $x \in S$, $v \in T_x S$
- The contact plane there is spanned by $v$ ("tautological direction") and the fibre direction.

A smooth curve on $S$ lifts uniquely to a Legendrian curve in $UTS$.

Ice skating on an ice rink $S$:
- Status of skater = (position of skater, direction of skates) $\in UTS$
- Ice skater's path is Legendrian iff she does not skid.
- Similar: parking a car, rolling a suitcase.
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Any orientable surface $\omega = \text{any area form}$

Any 3-manifold (Harder)

Not every 4-manifold has a symplectic structure

Not every 3-manifold has a tight contact structure
Symplectic and contact examples

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Hamiltonian mechanics

State of a classical system given by coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Set of states forms the phase space $\mathbb{R}^{2n}$. Energy of states given by Hamiltonian function $H: \mathbb{R}^{2n} \rightarrow \mathbb{R}$.

Hamilton's equations:
$$\dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}.$$
Hamiltonian mechanics

Physics textbook version:

- State of a classical system given by coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$
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Hamiltonian flows

More generally:
- States of system form a **symplectic manifold** \((M, \omega)\)
- Energy of states given by Hamiltonian \(H : M \to \mathbb{R}\)

Hamilton's equations arise from symplectic geometry:

Nondegeneracy of \(\omega\) yields isomorphism between vectors and 1-forms on \(M\):

The vector field corresponding to \(dH\) is the Hamiltonian vector field:

\[
\mathbf{X}_H = \omega(\cdot, X_H)
\]

The flow of \(\mathbf{X}_H\) preserves \(\omega\) and \(H\):

\[
\mathbf{L}_{\mathbf{X}_H} \omega = \mathbf{\iota}_{\mathbf{X}_H} d\omega + d\mathbf{\iota}_{\mathbf{X}_H} \omega = 0 + d(dH) = 0
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Hamilton’s equations arise from symplectic geometry:

- Nondegeneracy of $\omega$ yields isomorphism between vectors and 1-forms on $M$
- The vector field corresponding to $dH$ is the Hamiltonian vector field

$$dH = \omega(X_H, \cdot)$$

---

When $(M, \omega) = (\mathbb{R}^2N, \sum dx_i \wedge dy_i)$ we obtain Hamilton’s equations:

$$X_H = (\frac{\partial H}{\partial y_j}, -\frac{\partial H}{\partial x_j}).$$

The flow of $X_H$ preserves $\omega$ and $H$:

$$L_{X_H} \omega = \iota_{X_H} d\omega + d\iota_{X_H} \omega = 0 + d(dH) = 0.$$
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Hamilton’s equations arise from symplectic geometry:
- Nondegeneracy of \(\omega\) yields isomorphism between vectors and 1-forms on \(M\)
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dH = \omega(X_H, \cdot) \quad X_H \text{ gives dynamics of system}
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X_H = \left( \frac{\partial H}{\partial y_j}, -\frac{\partial H}{\partial x_j} \right).
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Hamiltonian flows

More generally:
- States of system form a symplectic manifold \((M, \omega)\)
- Energy of states given by Hamiltonian \(H : M \to \mathbb{R}\)

Hamilton’s equations arise from symplectic geometry:
- Nondegeneracy of \(\omega\) yields isomorphism between vectors and 1-forms on \(M\)
- The vector field corresponding to \(dH\) is the Hamiltonian vector field

\[
dH = \omega(X_H, \cdot) \quad \text{— } X_H \text{ gives dynamics of system}
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- The flow of \(X_H\) preserves \(\omega\) and \(H\):

\[
L_{X_H} \omega = \iota_{X_H} d\omega + d\iota_{X_H} \omega = 0 + d(dH) = 0.
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Dynamics on a contact manifold: Reeb vector field

Contact geometry has its own version of Hamiltonian mechanics!
- A contact form $\alpha$ on $M$ yields a vector field / dynamics.
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- A contact form $\alpha$ on $M$ yields a vector field / dynamics.
- Since $\alpha \wedge (d\alpha)^n \neq 0$, $d\alpha$ must have 1-dimensional kernel, transverse to $\xi$. 

**Definition**

The Reeb vector field on $(M, \alpha)$ is the unique vector field $R_\alpha$ such that $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$.

**Lemma**

The flow of $R_\alpha$ preserves $\alpha$.

**Proof.**

$L_{R_\alpha} \alpha = \iota_{R_\alpha} d\alpha + d\iota_{R_\alpha} \alpha = 0 + d1 = 0$. 

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A contact vector field is a vector field whose flow preserves \(\xi\).

**Theorem**
Let \((M, \xi)\) be a contact 3-manifold with a contact form \(\alpha\). There is a bijective correspondence \{Contact vector fields on \((M, \xi)\)\} \(\leftrightarrow\) \{Smooth functions \(H: M \to \mathbb{R}\)\} \(X \mapsto \alpha(X)\).
Infinitesimal symmetries of a contact structure

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Contact geometry as solving ODEs by geometry

The standard contact structure $\xi$ on $\mathbb{R}^3$

$$dz - y \, dx = 0 \quad \text{means} \quad "y = \frac{dz}{dx}".$$
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A first-order differential equation can be expressed as

$$F \left( x, z, \frac{dz}{dx} \right) = 0$$

for some smooth function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ and hence (generically) determines a surface $S$ in $\mathbb{R}^3$, with coordinates $x, z, \frac{dz}{dx} = y$. 
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- The intersections of the plane field \( \xi \) with the surface \( S = \{ F = 0 \} \) trace out curves on \( S \) which are solutions to the ODE.
E.g. \( \frac{dz}{dx} = z \), solutions \( z = Ae^x \).

**Definition**

The intersection of a contact structure \( \xi \) with a surface \( S \) gives a **singular 1-dimensional foliation** called the **characteristic foliation** of \( S \).
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2. Some differential geometry
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4. Fundamental results
5. Ideas and Directions
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   - Open book decompositions
   - Knots and links
   - Surfaces in contact 3-manifolds
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Theorem (Darboux theorem (Darboux 1882))

Let $\alpha$ be a contact form on $M^{2n+1}$. Near any $p \in M$ there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ such that $p = (0, \ldots, 0)$ and

$$\alpha = dz - \sum_{j=1}^{n} y_j \, dx_j.$$

"All contact manifolds are locally the same." – no "contact curvature".

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Let $\xi_t$ be a smooth family (an isotopy) of contact structures on $M$. Then there is an isotopy $\psi_t : M \to M$ such that $\psi_t^* \xi_0 = \xi_t$ for all $t$.

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**Weinstein conjecture ’79**

Let $M$ be any odd-dimensional manifold with a contact form $\alpha$. Then the Reeb vector field $R_\alpha$ has a closed orbit.

**Theorem (Taubes ’07)**

*The Weinstein conjecture holds for contact 3-manifolds.*

Proof uses Seiberg–Witten Floer homology.
Questions in 3D contact topology

Given a 3-manifold $M$:
- Classify all contact structures on $M$ up to contactomorphism / isotopy.
- Given a contact 3-manifold $(M, \xi)$:
  - Understand the dynamics of its Reeb vector fields.
  - Understand its Legendrian knots and links.
  - Understand its group of contactomorphisms.

Generally:
- What is a contact structure like near a surface, and how does it change as a surface moves?
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Given a closed 3-manifold $M$, does it have a contact structure?

Answer: YES. (Martinet 1971) — from Heegaard decomposition.

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How many contact structures does $M$ have?

Answer: $\infty$. Non-integrability is an open condition — so any sufficiently small perturbation is still a contact structure.

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How many isotopy classes of contact structures does $M$ have?

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An **overtwisted** contact structure is one that contains a specific contact disc called an **overtwisted disc**.

A non-overtwisted contact structure is called **tight**.
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Classifying overtwisted contact structures reduces to homotopy theory!

**Theorem (Eliashberg 1989)**

\[
\left\{ \text{Overtwisted contact structures on } M \right\} \simeq \left\{ \text{2-plane fields on } M \right\}
\]
Natural next question

How many isotopy classes of tight contact structures does $M$ have?


If $M = M_1 \# M_2$ then contact structures on $M$ correspond to contact structures on $M_1$ and $M_2$.

If $M$ is toroidal then $\infty$.

Interesting question

Given closed irreducible atoroidal $M$, how many isotopy classes of tight contact structures does it have?

Answer: Finitely many (Colin–Giroux–Honda 2003)
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Some examples

- The 3-sphere $S^3$:
  - 1 tight contact structure (Eliashberg 1991).

- Lens spaces $L(p, q)$:
  - The number of tight contact structures is a complicated function of $p$ and $q$ (Honda 2000, Giroux 2001).
  - For example, $L(34, 19)$ has 12 contact structures:
    $$-34 \equiv -2 - 1 - 5 - 1 - 4 \equiv (-2 + 1)(-5 + 1) = 12$$

- For a Poincaré homology sphere $M$, there are no tight contact structures (Etnyre–Honda 1999).
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- $M \# \overline{M}$ where $M$ is Poincaré homology sphere:
  NO tight contact structure (Etnyre–Honda 1999).
Open book decompositions

Theorem (Giroux 2000)

Let $M$ be a closed 3-manifold. There is a bijective correspondence

$\{\text{Contact structures on } M \text{ up to isotopy}\} \leftrightarrow \begin{cases} \text{Open book decompositions of } M \text{ up to isotopy} \\ \text{and positive stabilisation} \end{cases}$

Definition

An open book decomposition of $M$ consists of

an oriented link $B \subset M$ (the binding), and

a map $\pi : M \setminus B \rightarrow S^1$ where the preimage $\pi^{-1}(\theta)$ of every $\theta \in S^1$ is a surface $\Sigma_\theta$ with $\partial \Sigma_\theta = B$.

The surfaces $\Sigma_\theta$ are the pages of the open book decomposition.
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Open book decompositions

An open book can be considered abstractly as a pair \((\Sigma, \phi)\) where \(\Sigma\) is an oriented compact surface with boundary, \(\phi : \Sigma \to \Sigma\) is a diffeomorphism (the monodromy). From this data, we can reconstruct \(M\):

\[ M = \Sigma \times [0,1] \]

\[ (x, 1) \sim (\phi(x), 0) \quad \forall x \in \Sigma, \]

\[ (x, t) \sim (x, t') \quad \forall x \in \partial \Sigma, \]

In fact, from \((\Sigma, \phi)\) can reconstruct \(M\) and \(\xi\). (Roughly, \(\xi\) transverse to the binding, close to parallel to pages.)
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Examples:

- $\Sigma$ a disc, $\phi$ the identity: $S^3$, tight contact structure.
- $\Sigma$ an annulus, $\phi$ a right-handed Dehn twist: $S^3$, tight.
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A stabilisation of an open book $(\Sigma, \phi)$ is an open book $(\Sigma', \phi')$ where $\Sigma'$ is $\Sigma$ with a 1-handle attached along its boundary $\phi' = \phi \circ$ a Dehn twist around a simple closed curve intersecting the co-core of the 1-handle exactly once.
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Let \(\xi, \xi'\) be the corresponding contact structures.

**Proposition**

*If \((\Sigma', \phi')\) is obtained from \((\Sigma, \phi)\)*

- by a positive stabilisation, then \(\xi, \xi'\) are contactomorphic;
- by a negative stabilisation, then \(\xi'\) is overtwisted.

Which \((\Sigma, \phi)\) are tight and which are overtwisted? Wanda (2014): "consistency".
The stabilisation is **positive** (resp. **negative**) if the Dehn twist is right-handed (resp. left-handed).

**Proposition**

*The 3-manifolds constructed from $(\Sigma', \phi')$ and $(\Sigma, \phi)$ are homeomorphic.*

Let $\xi, \xi'$ be the corresponding contact structures.

**Proposition**

*If $(\Sigma', \phi')$ is obtained from $(\Sigma, \phi)$*
  - by a positive stabilisation, then $\xi, \xi'$ are contactomorphic;
  - by a negative stabilisation, then $\xi'$ is overtwisted.

Which $(\Sigma, \phi)$ are tight and which are overtwisted?
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Knots in contact 3-manifolds

Let $K$ be a Legendrian knot in standard $\mathbb{R}^3$, with $\alpha = dz - y \, dx$.

Definition

The front projection of $K$ is its projection onto the $xz$-plane. As $K$ is Legendrian, $y = dz/dx$ along $K$ and the $y$-coordinate can be recovered from the front projection.
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Thurston-Bennequin number

A framing of a knot $K$ in a manifold $M$ is any/all of:

- a trivialisation of the normal bundle of $K$;
- a choice of longitude on torus boundary of a neighbourhood of $K$;
- a choice of “pushoff" of $K$.

Framings “in nature”:

- A Legendrian knot has a contact framing — use contact planes.
- A nullhomologous knot has a surface framing — use a surface bounded by $K$.

Any two framings of a knot differ by an integer.

Definition

Let $K$ be a nullhomologous Legendrian knot in $(M, \xi)$. The Thurston-Bennequin number $tb(K)$ is the difference between the contact framing and the surface framing on $K$. 
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**Proposition (Legendrian surgery)**

Let $K$ be a Legendrian knot in a contact 3-manifold $(M, \xi)$. The manifold $M'$ obtained by $(-1)$-Dehn surgery on $K$ carries a natural contact structure $\xi'$ that coincides with $\xi$ away from $K$.

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**Theorem (Wand 2014)**

If $(M, \xi)$ is tight then $(M', \xi')$ is tight.
Surfaces in contact 3-manifolds

Recall a surface $S$ in $(M, \xi)$ has a singular 1-dimensional foliation $F$, the characteristic foliation, given by $F = \mathcal{T}S \cap \xi$.

Generically, singularities are only of two types: elliptic and hyperbolic. Roughly, $F$ determines the "germ" of a contact structure near $S$. 
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Convex surfaces

**Definition (Giroux 1991)**

An embedded surface $S$ in a contact 3-manifold is **convex** if there is a contact vector field $X$ transverse to $S$.

We can take a neighbourhood $S \times I$ ($I = [0, 1]$), with $X$ pointing along $I$.

- Local coordinates $x, y$ on $S$, and $t$ on $I$, so $X = \frac{\partial}{\partial t}$
Take coordinates $x, y, t$ as above. Then:

The contact form $\alpha(\xi = \ker \alpha)$ is invariant in $I$ direction, so $\alpha = \beta + u \, dt$, where $\beta$ is a 1-form and $u$ a function on $S$.

Characteristic foliation $F = \ker \beta$, with singularities when $\beta = 0$.

$\xi$ is “vertical” $\iff \alpha(\partial / \partial t) = u = 0$.

“Which side of $\xi$ is facing up” corresponds to whether $\alpha(\partial / \partial t) = u$ is positive or negative.

Definition: The dividing set of $S$ is $\Gamma = u - 1(0) = \{ p \in S : X(p) \in \xi \}$.
Take coordinates $x, y, t$ as above. Then:
- The contact form $\alpha (\xi = \ker \alpha)$ is invariant in $l$ direction, so

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*Moral* A contact structure near an embedded surface in a contact 3-manifold can be described by an isotopy class of dividing set. The combinatorial structure of dividing sets and contact structures between them is captured by an algebraic object called the contact category.
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Floer homology

Many powerful invariants have been developed in contact topology using ideas of Floer homology.

- Homology theories constructed from analysis of pseudo-holomorphic curves in a symplectic manifold $(M, \omega)$.
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- Homology theories constructed from analysis of pseudo-holomorphic curves in a symplectic manifold \((M, \omega)\).
- Define an *almost complex structure* on \((M, \omega)\) and consider pseudo-holomorphic maps \(u : \Sigma \rightarrow M\), where \(\Sigma\) is a Riemann surface (Gromov 1985).

\[
\begin{array}{c}
\Sigma \\
\vdots \\
M
\end{array}
\]

\[u \]
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Given appropriate constraints and assumptions, the space of holomorphic curves is a finite-dimensional *moduli space* \(\mathcal{M}\).
- Can do analysis using Fredholm/index theory, compactness theorems, etc.
Boundary of \( \overline{M} \) is stratified: boundary strata are moduli spaces for “degenerate” holomorphic curves (nodal surfaces, etc.)

Analogy: singular homology via Morse complex. Complex generated by critical points of Morse function \( f \). \( \partial \) counts 0-dimensional families of trajectories of \( \nabla f \).
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- **Heegaard Floer homology** is an invariant of 3-manifolds derived from counting holomorphic curves in a symplectic manifold derived from a Heegaard decomposition (Ozsváth–Szábo 2004).

\[ M \rightsquigarrow \widehat{HF}(M) \]

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\begin{align*}
\text{A contact structure } \xi & \text{ on } M \\
\text{determines a contact element } \gamma(\xi) \\
\text{(Ozsváth–Szabó 2005, Honda–Kazez–Matić 2009).} \\
\end{align*}
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Legendrian contact homology associates to a Legendrian link \( L \subset (M, \alpha) \) a differential graded algebra \((A, \partial)\).

Legendrian isotopic \( L, L' \Rightarrow (A, \partial), (A', \partial') \) have isomorphic homology.

(Chekanov, Eliashberg, Ekholm–Etnyre–Sullivan, ...)

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Thanks for listening!

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