Monoidal categories enriched in braided monoidal categories

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- What are these things?
- What do they look like?
- What are they for?

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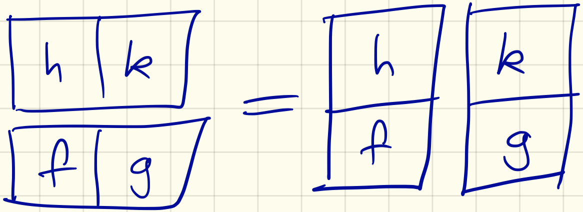
In a monoidal category, we have objects and morphisms, and we can form tensor products of objects and of morphisms.

The important axiom is the exchange relation



or in symbols:

$$(f\otimes g)\circ(h\otimes k) = (f\circ h)\otimes(g\circ k).$$

We could also express this as a commutative diagram:

The important axiom is the exchange relation

$$\boxed{\begin{array}{c|c} h & k \end{array}}\atop\boxed{\begin{array}{c|c} f & g \end{array}} \;=\; \boxed{\begin{array}{c} h \\ \hline f \end{array}}\,\boxed{\begin{array}{c} k \\ \hline g \end{array}}$$

or in symbols:

$$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k).$$

$$\begin{tikzcd}
\mathcal{C}(a \to b) \times \mathcal{C}(c \to d) \times \mathcal{C}(b \to e) \times \mathcal{C}(d \to f) \arrow[rr, "(-\otimes-)\times(-\otimes-)"] \arrow[d, "\text{switch the 2nd and 3rd factors}"] & & \mathcal{C}(ac \to bd) \times \mathcal{C}(bd \to ef) \arrow[d, "-\circ-"] \\
\mathcal{C}(a \to b) \times \mathcal{C}(b \to e) \times \mathcal{C}(c \to d) \times \mathcal{C}(d \to f) \arrow[dr, bend right, "(-\circ-)\times(-\circ-)"'] & & \mathcal{C}(ac \to ef) \\
& \mathcal{C}(a \to e) \times \mathcal{C}(c \to f) \arrow[ur, "-\otimes-"'] &
\end{tikzcd}$$

We're now going to enrich our monoidal categories in some other category $\mathcal{V}$.

We no longer have morphisms, or even a set of morphisms. Instead, for each pair of objects a, b,

$\mathcal{C}(a \to b)$ is an object of $\mathcal{V}$,

and we ask that composition

$$\mathcal{C}(a \to b) \times \mathcal{C}(b \to c) \longrightarrow \mathcal{C}(a \to c)$$

and tensor product

$$\mathcal{C}(x \to y) \times \mathcal{C}(w \to z) \longrightarrow \mathcal{C}(xw \to yz)$$

are both morphisms in $\mathcal{V}$.

What sort of categories can we use as $\mathcal{V}$?

Certainly $\mathcal{V}$ will need to be a monoidal category, to make sense of these products

$$\mathcal{C}(a \to b) \times \mathcal{C}(b \to c) \xrightarrow{- \circ -} \mathcal{C}(a \to c)$$

$$\mathcal{C}(x \to y) \times \mathcal{C}(w \to z) \xrightarrow{- \otimes -} \mathcal{C}(xw \to yz)$$

We'll also need a way to implement the 'switch' in the exchange relation:

$$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k)$$

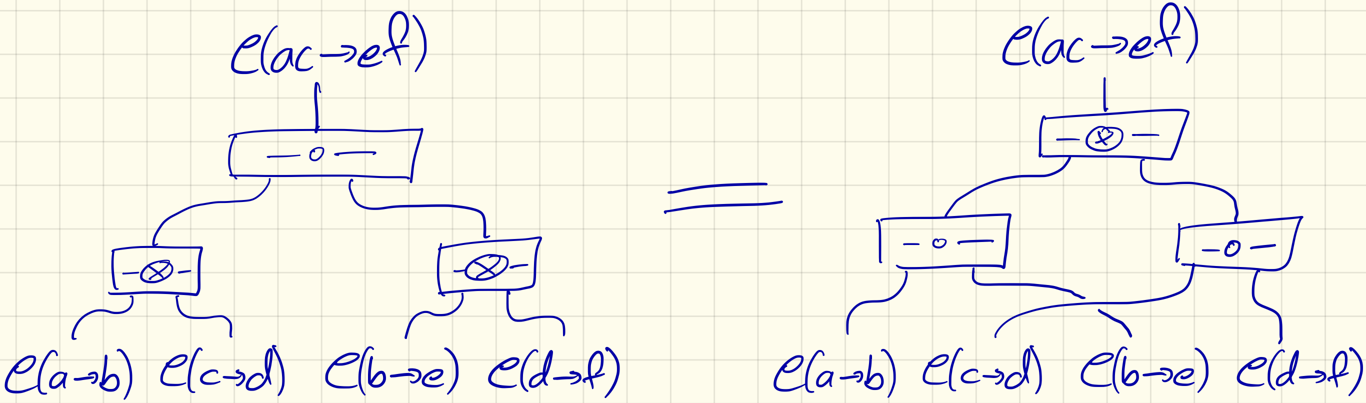
Usually one studies the situation where $\mathcal{V}$ is a symmetric monoidal category.

We've realised that much, but not all, of the theory can be developed for

$\mathcal{V}$ a braided monoidal category.

and even better there are interesting and useful examples.

The exchange relation is



We call such a gadget a $\mathcal{V}$-monoidal category.

One can define monoidal functors between $\mathcal{V}$-monoidal categories, and natural transformations between these.

(In good situations, these form a 2-category enriched in $\mathcal{V}$.)

Curiously, products of $\mathcal{V}$-monoidal categories don't seem to work.

We can classify $\mathcal{V}$-monoidal categories in terms of classical data.

Define $\mathcal{C}^{\mathcal{V}}$, an 'honest' monoidal category, with

$$\mathcal{C}^{\mathcal{V}}(a \to b) = \mathcal{V}(1_{\mathcal{V}} \longrightarrow \mathcal{C}(a \to b)).$$

There is a functor

$$\begin{aligned} \mathrm{tr} : \mathcal{C}^{\mathcal{V}} &\longrightarrow \mathcal{V} \\ x &\longmapsto \mathcal{C}(1 \to x). \end{aligned}$$

<u>Theorem</u> (Morrison–Penneys 2016)

$$\left\{ \begin{array}{l} \mathcal{V}\text{-monoidal categories } \mathcal{C} \\ \text{(such that tr admits} \\ \quad \text{a left adjoint)} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{braided oplax monoidal} \\ \text{functors } \mathcal{V} \to Z(\mathcal{T}) \\ \text{for some monoidal } \mathcal{T}, \\ \text{(such that } \mathcal{V} \to Z(\mathcal{T}) \to \mathcal{T} \\ \quad \text{admits a right adjoint)} \end{array} \right\}$$

$$\left\{ \mathcal{V}\text{-monoidal categories } \mathcal{C} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{braided oplax monoidal} \\ \text{functors } \mathcal{V} \to Z(\mathcal{T}) \\ \text{for some monoidal } \mathcal{T}, \end{array} \right\}$$

This gives a generalisation of de-equivariantisation:

when we have $\mathrm{Rep}\,G \subset Z(\mathcal{T})$, $\mathcal{T}$ monoidal,

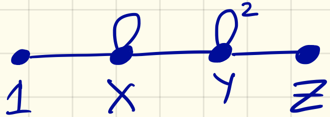
there is another monoidal category $\mathcal{T}/\!\!/G$, 'the quotient by G'.

We can think of this via our theorem —

- first use the bijection to produce a $\mathrm{Rep}\,G$-enriched category
- second use the lax functor $\mathrm{Rep}\,G \to \mathrm{Vec}$ to obtain the honest monoidal category $\mathcal{T}/\!\!/G$.

Examples

AdE_8 is an interesting fusion category with four simple objects



In $Z(AdE_8)$, we find a copy of the Fibonacci category, so we can 'quotient' by it, producing a Fib-enriched monoidal category $AdE_8 /\!/_{Fib}$, with two simple objects $1, \mathcal{X}$, and

$$\mathcal{X}^2 \cong 1 \oplus \mathcal{X} \oplus \mathcal{X}$$

(something that could not happen in the world of fusion categories)

but where one of the maps $\mathcal{X}^2 \to \mathcal{X}$ is 'τ-graded':

$$AdE_8 /\!/_{Fib}(\mathcal{X}^2 \to \mathcal{X}) = 1 \oplus \tau$$

What next?

- more examples ('genuinely oplax'?)
- simpler constructions of exotic tensor categories
- $\mathcal{V}$-complete $\Longleftrightarrow$ strong monoidal functor
- coherence theorems, strictification
- semisimplicity
- idempotent completion
- Drinfeld centres
- bimodule categories, Brauer-Picard groupoids
- fusion ring, principal graphs, connections
- classification