The modular data machine

MATRIX 2016-12-15

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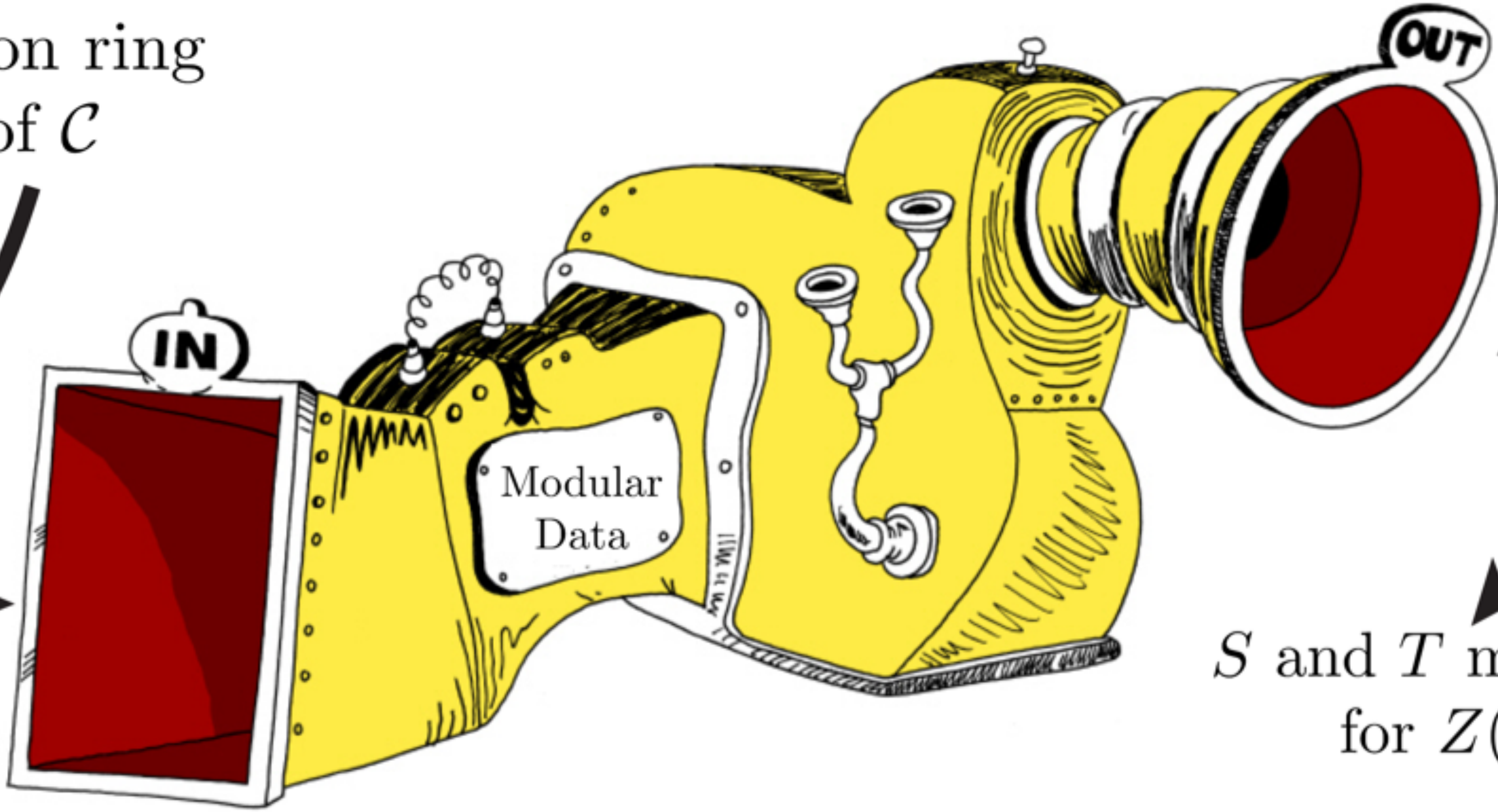
fusion ring of $\mathcal{C}$



S and T matrices for $Z(\mathcal{C})$

fusion categories $\xleftrightarrow[\text{cobordism hypothesis}]{\text{arXiv:1312.7188}}$ 3210-dimensional TQFTs

A 32-dimensional TQFT is a functor

$$\begin{tikzcd}
\{\text{3-manifolds w/boundary}\} \arrow[r, "Z"'] & \{\text{linear maps}\} \\
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\end{tikzcd}$$

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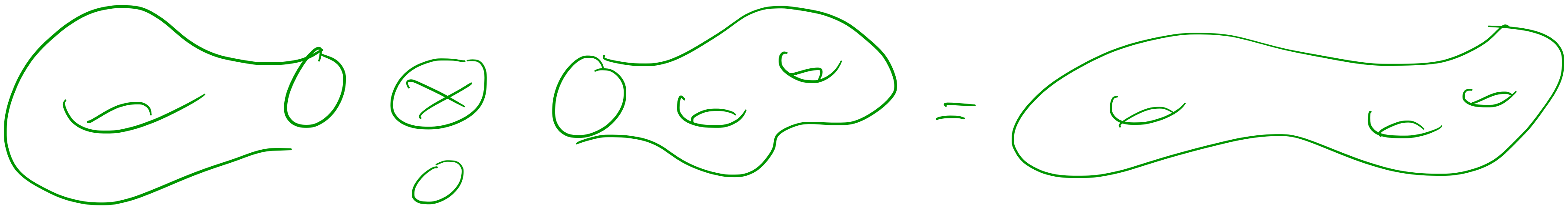
$$\{\text{2-manifolds w/boundary}\} \xrightarrow{Z} \{\text{vector spaces}\}$$

$$\{\text{1-manifolds}\} \xrightarrow{Z} \{\text{categories}\}$$



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\{\text{1-manifolds w/boundary}\} & \xrightarrow[Z]{} & \{\text{categories}\} \\
\{\text{0-manifolds}\} & \xrightarrow[Z]{} & \{\text{2-categories}\}
\end{array}$$

$$\text{fusion categories} \xleftrightarrow[\text{cobordism hypothesis}]{\text{arXiv:1312.7188}} \text{3210-dimensional TQFTs}$$

a fusion category is a

finitely semisimple rigid monoidal category

fusion categories $\longleftrightarrow$ 3210-dimensional TQFTs

cobordism hypothesis

Fusion categories are "noncommutative finite groups".

Thm (Deligne)
A symmetric fusion category (satisfy a growth condition, e.g. that some Schur functor vanishes) is Rep G, for some finite (super) group.

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- finite groups
- quantum groups
- quadratic categories
- 'exotic' subfactors?!

fusion categories $\longleftrightarrow$ 3210-dimensional TQFTs

cobordism hypothesis

a modular tensor category is a
- spherical
- nondegenerately braided
- fusion category

- finite groups
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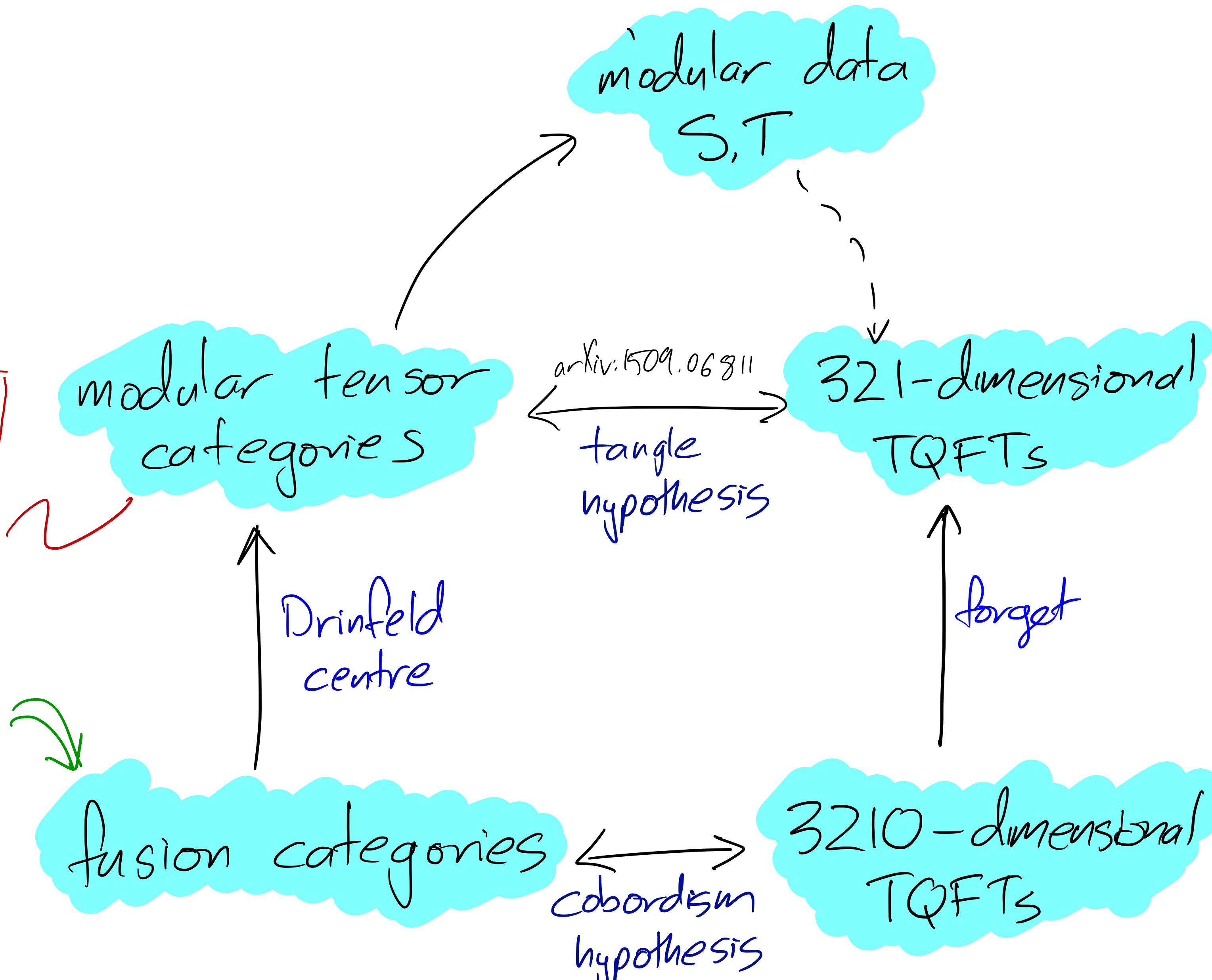

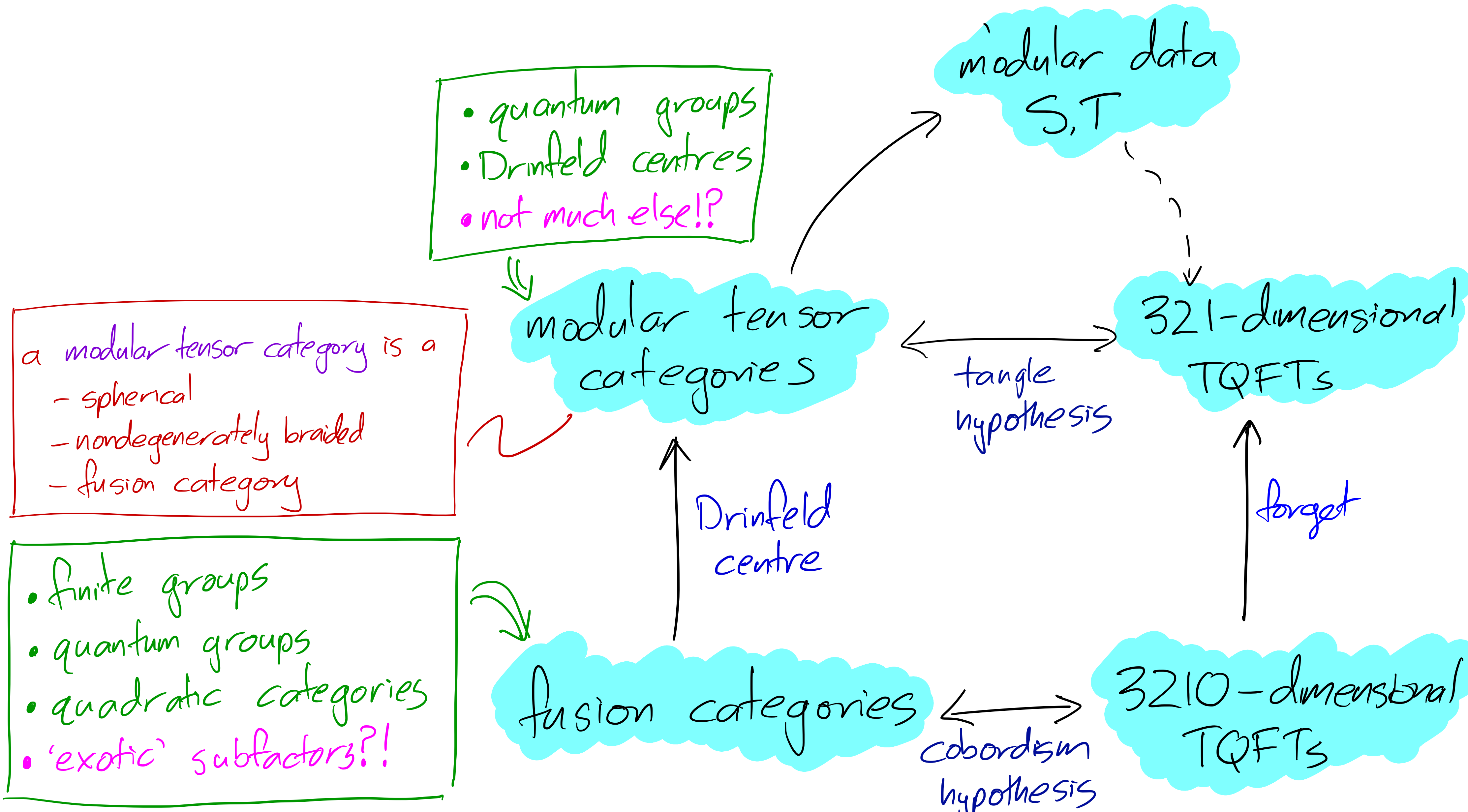
• quantum groups
• Drinfeld centres
• not much else!?
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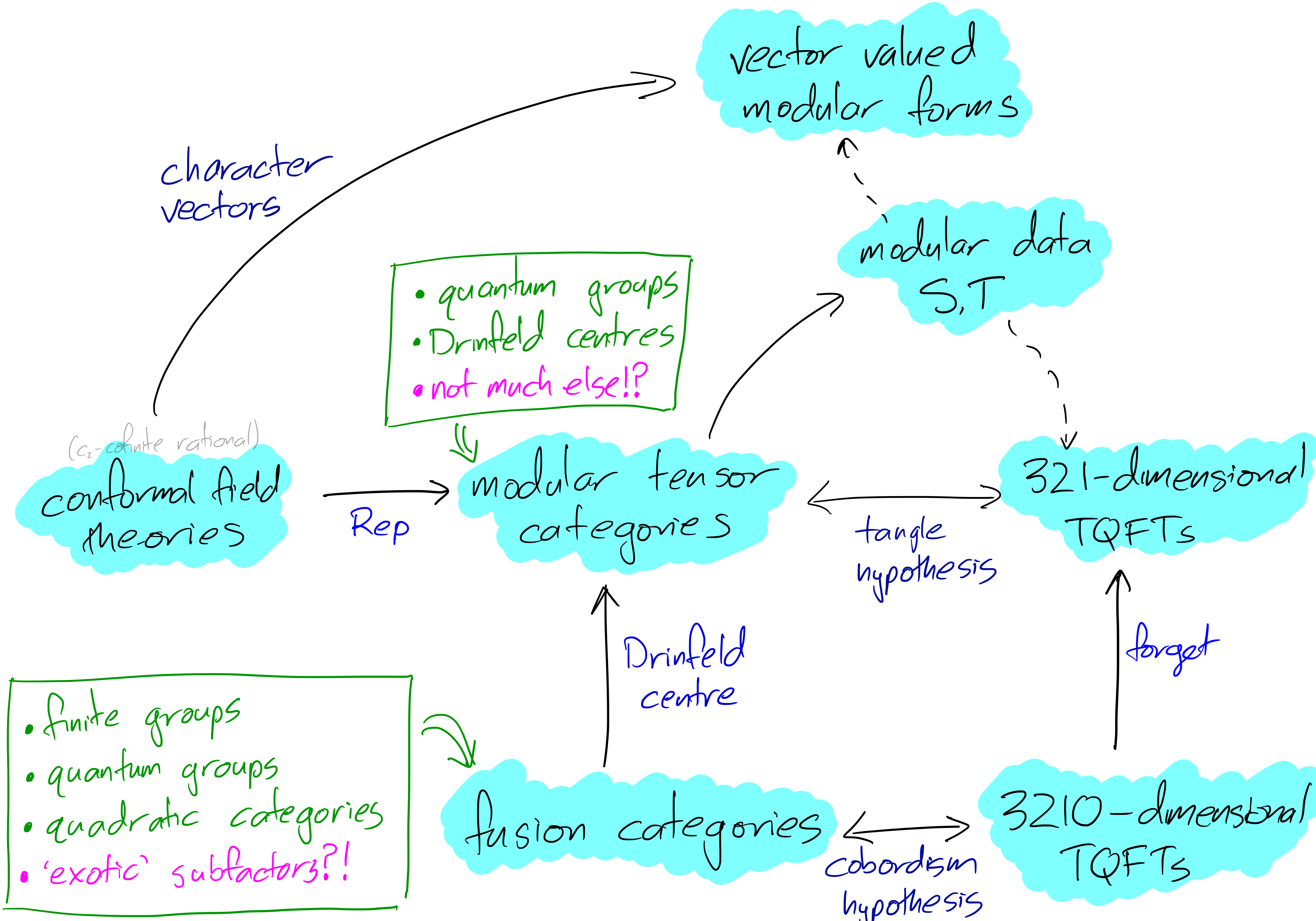
modular data
S, T
modular tensor categories
321-dimensional TQFTs
tangle hypothesis
Drinfeld centre
forget
fusion categories
3210-dimensional TQFTs
cobordism hypothesis


vector valued modular forms
character vectors
modular data S, T
• quantum groups
• Drinfeld centres
• not much else!?
(c_2-cofinite rational)
conformal field theories
Rep
modular tensor categories
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\begin{tikzcd}
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\text{modular tensor categories} \arrow[uur] & \\
\text{fusion categories} \arrow[u, "\text{Drinfeld centre}"'] \arrow[uuur, Rightarrow, bend right] &
\end{tikzcd}
$$

Today's goal is to determine the modular data for a Drinfeld centre $Z(\mathcal{C})$.

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\begin{tikzcd}
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\text{fusion categories} \arrow[u, "\text{Drinfeld centre}"] \arrow[rr, "\text{Grothendieck group}"'] & & \text{“based rings”} \arrow[uul, Rightarrow]
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This is so strongly constrained, in many cases we only need to know the tensor product multiplicities in $\mathcal{C}$.

modular data
S, T

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modular tensor categories

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Drinfeld centre

fusion categories

Grothendieck group

"based rings"

In particular, we can compute the modular data, and the possible character vectors, for the exotic 'extended Haagerup' subfactor.

Modular data consists of

- a diagonal $n \times n$ matrix T with order N
- a unitary, symmetric matrix $S \in M_n(\underbrace{\mathbb{Q}[\zeta_N]}_{})$

the Nth cyclotomic field

n is called the <u>rank</u>

N is called the <u>conductor</u>

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such that:

- $(ST)^3 = S^2 = C$ with C a permutation matrix, $C^2 = I$

$\Longrightarrow$ and so $\rho : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longmapsto S$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longmapsto T$ gives a representation of $SL(2,\mathbb{Z})$

- $S_{1x} > 0$ for all x
- $\rho\begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}$ for ℓ coprime to N is a signed permutation matrix
  $$(G_\ell)_{xy} = \varepsilon_\ell(x)\, \delta_{y, x^\ell}$$

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- $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$
- $\sigma_\ell(S_{xy}) = \varepsilon_\ell(x)\, S_{x^\ell,y}$

  the Galois action

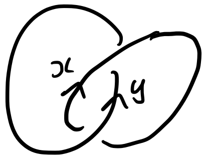
- Verlinde's formula:

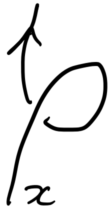
$$N_{xy}^{z} = \sum_w \frac{S_{xw} S_{yw} \overline{S_{zw}}}{S_{1w}} \in \mathbb{Z}_{\geq 0}$$

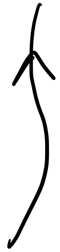
$\Longrightarrow$ and these are the structure constants of a based ring

Theorem

A unitary modular tensor category gives modular data via:

$S_{xy} = \frac{1}{D}$ 

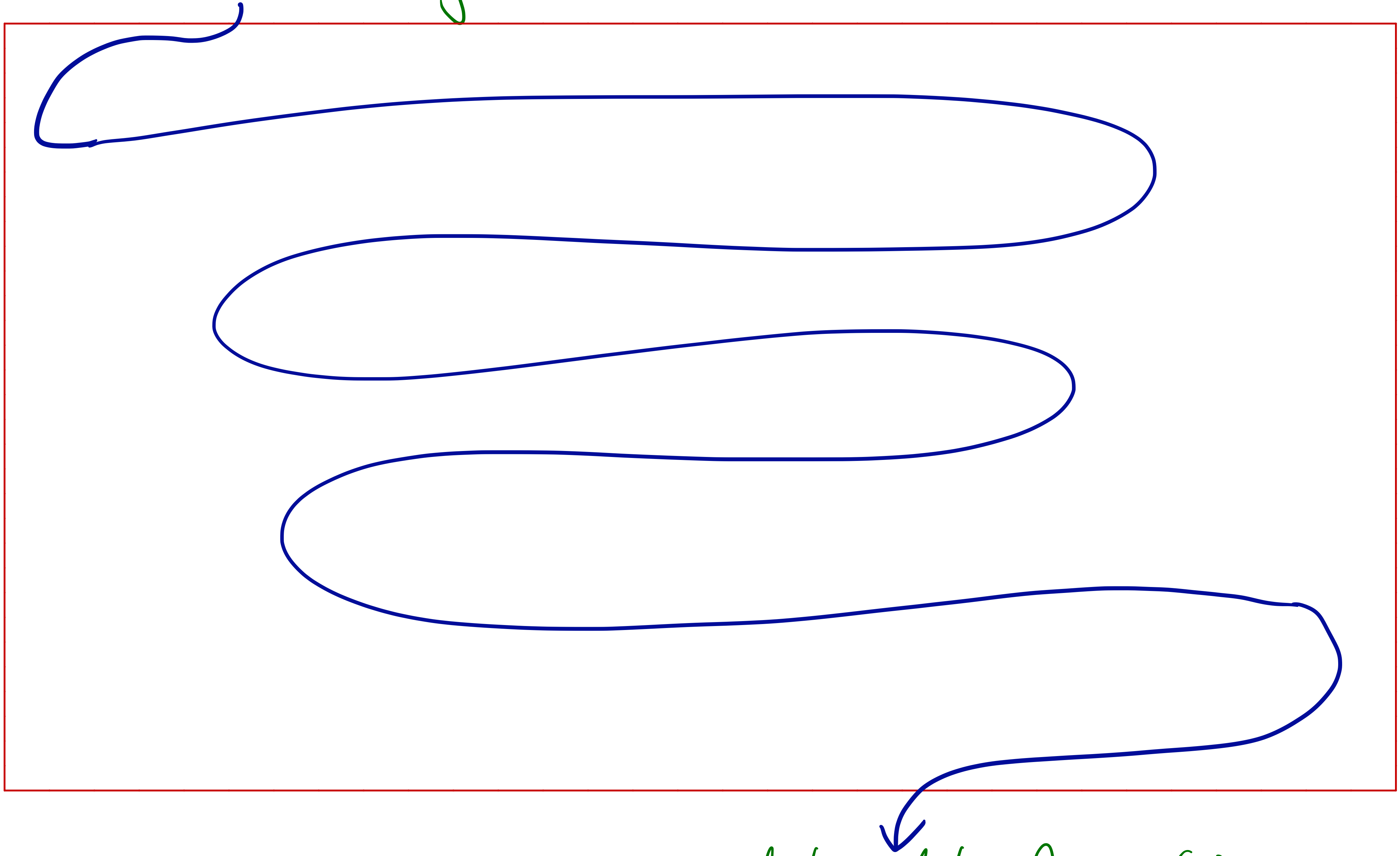


$= T_{xx}$ 

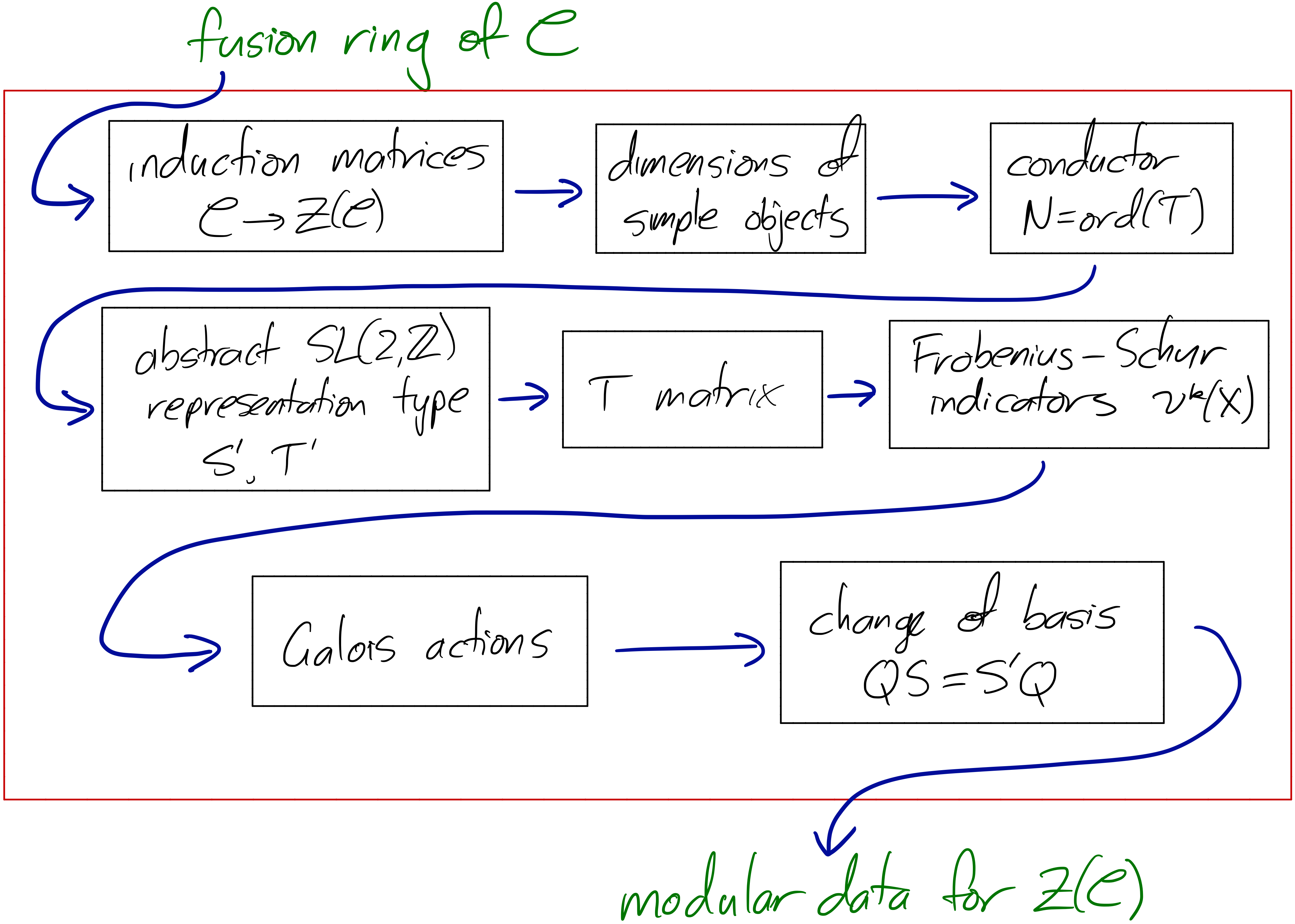
D: the global dimension $\sum_{x \text{ simple}} \dim(x)^2$

Questions
- What are the Drinfeld centres of the exotic subfactors? (In particular 'extended Haagerup')
- Perhaps easier, what is the corresponding modular data?
- Can we describe possible character vectors for a CFT realising these MTCs?

fusion ring of $\mathcal{C}$



modular data for $Z(\mathcal{C})$


fusion ring of $\mathcal{C}$
induction matrices $\mathcal{C} \to Z(\mathcal{C})$
dimensions of simple objects
conductor $N = \mathrm{ord}(T)$
abstract $SL(2,\mathbb{Z})$ representation type S', T'
T matrix
Frobenius-Schur indicators $\nu^k(X)$
Galois actions
change of basis $QS = S'Q$
modular data for $Z(\mathcal{C})$

Induction and restriction

There are adjoint functors $I : \mathcal{C} \underset{\otimes}{\rightleftarrows} Z(\mathcal{C}) : R$

which at the level of Grothendieck groups give matrices

$$A^T : K_0(\mathcal{C}) \rightleftarrows K_0(Z(\mathcal{C})) : A.$$

$R(I(X)) = \bigoplus_{V \in \mathrm{Irr}\,\mathcal{C}} V X V^*$, so the fusion ring of $\mathcal{C}$ determines AA^T

(a symmetric non-negative integer matrix).

<u>Lemma</u> There are finitely many such A.

Now $\dim(Y \in Z(\mathcal{C})) = \sum_{X \in \mathrm{Irr}\,\mathcal{C}} A_{XY} \dim(X \in \mathcal{C})$,

$\dim(Y) \mid \dim(Z(\mathcal{C}))$ as algebraic integers, and $\dim(Y)$ is an Ostrik d-number.

With these restrictions, and a trick when AA^T is not full rank,

it is plausible to enumerate all possible A.

From these, we calculate the dimensions of objects in $Z(\mathcal{C})$.

Conductors

If $N = \prod p_i^{n_i}$, we must have an object x_i with T eigenvalue λ_i where
$$p_i^{n_i} \mid \operatorname{ord} \lambda_i.$$

Then $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$ tells us the Galois orbit of x_i has at least
$$p_i^{n_i - 1}(p_i - 1)/2 \text{ elements} \quad (\text{or } 2^{n_2 - 3} \text{ if } p_i = 2).$$

For a given rank, there are finitely many possible conductors.

Moreover, $S_{1x} = \dfrac{}{D}$  $= \dfrac{\dim x}{D}$, and $\sigma_\ell(S_{1x}) = \pm S_{1x^\ell}$,

gives restrictions on the sizes of Galois orbits.

$SL(2,\mathbb{Z}/N\mathbb{Z})$ representations.

We now enumerate possible (abstract) representation types.

If $N=\prod p_i^{n_i}$, $SL(2,\mathbb{Z}/N\mathbb{Z})=\prod_i SL(2,\mathbb{Z}/p_i^{n_i}\mathbb{Z})$,

and GAP compute character tables in the relevant ranges.

For a representation ρ, write $T(\rho)$ for the set of T-eigenvalues (which we can read off the character table).

We can throw out most representations:

- $\mathrm{lcm}(\mathrm{order}(T(\rho)))=N$
- if λ appears in some irrep, $\{0,\lambda\}$ appear together in some irrep.
- $\#(\lambda \text{ in } T(\rho_{even})) \geq \#(\lambda \text{ in } T(\rho_{odd}))$
- traces of Galois group elements may be constrained by counting fixed points.
- $\#(1 \text{ in } T(\rho)) \geq$ # of simples in the induction of $1_{\mathcal{C}}$.

Write T' for the T-matrix in the abstract representation type.

T-matrices

At this point we have T', so we know the multi-set of T-eigenvalues, but not how they correspond to columns of the induction matrix.

Using $\sigma_\ell(S_{1x}) = \pm S_{1x^\ell}$, $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$

and the top-left entry of $STS = T^{-1}ST^{-1}C$:

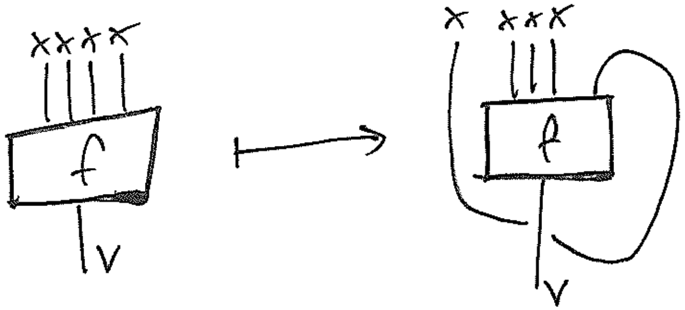
$$\sum_i d_i^2 t_i = \sqrt{\sum_i d_i^2}$$

using the Δ-inequality along the way.

we can enumerate all possible bijections.

Frobenius-Schur indicators

Define $\rho_{\underset{\in\,\mathrm{Irr}\,\mathcal{C}}{X},\,k,\,\underset{\in\,\mathrm{Irr}\,Z(\mathcal{C})}{V}} : \mathrm{Hom}(V \to X^{\otimes k}) \circlearrowleft$



Knowing the T-matrix and the dimensions in the centre, we can compute

$$\mathrm{tr}(\rho_{X,k,V}) = {}_V(ST^kA^T)_X$$

for $V = 1$ (or any V in the Galois orbit of 1).

Since $\rho^k_{X,k,V}$: 

So $\rho_{X,k,V}$ has order dividing $k \cdot \mathrm{ord}(t_V) \overset{\text{def}}{=} k \cdot n_V$,

and hence eigenvalues $\{\zeta^i_{kn_V}\}$

If ℓ is coprime to kn_V, then $\mathrm{tr}(\rho^\ell_{X,k,V}) = \sigma_\ell(\mathrm{tr}\,\rho_{X,k,V})$,

Now assume $V=1$

and if $g = \gcd(\ell, k)$,

$\rho^\ell_{X,k,1} = \rho^{\ell/g}_{X^{\otimes g}, k/g, 1}$; then as $\mathrm{tr}(\rho_{X\oplus Y,k,1}) = \mathrm{tr}(\rho_{X,k,1}) + \mathrm{tr}(\rho_{Y,k,1})$

we know the traces of all powers of $\rho_{X,k,1}$.

Newton's identities tell us the eigenvalues of $\rho_{X,k,1}$, which must all lie in $\sqrt[k]{1}$.

$\Longrightarrow$ There is no "$c=2$" category in Larson's classification of non-self-dual pseudo-unitary rank 4 fusion categories.

Galois actions

We (finally!) determine the details of the Galois group action.

Recall

$$\sigma_\ell(S_{1x}) = \varepsilon_\ell(x)\, S_{1x^\ell} = \varepsilon_\ell(x)\, \frac{\dim x^\ell}{\mathcal{D}},$$

$$\sigma_\ell(S_{1x}) = \sigma_\ell\left(\frac{\dim x}{\mathcal{D}}\right)$$

and $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$.

These constrain the possible Galois actions. Enumerating them all is still a mess, but doable — the problem is that we need to avoid listing Galois actions which only differ by a symmetry of A and T.

The change of basis

Now we obtain explicit S' and T' matrices for the chosen $SL(2,\mathbb{Z}/N\mathbb{Z})$ representation type. (From the RepSn package in GAP.)

There's some change of basis Q so $T'Q = QT$, $S'Q = QS$, with Q and S mostly unknown, Q invertible.

We first solve all the <u>linear</u> equations available:

$$\begin{cases} T'Q = QT \\ STA^t = T^{-1}A^t \quad (\Longleftrightarrow S'QTA^t = QT^{-1}A) \\ S_{1x} = \frac{\dim x}{D}, \quad S_{1\ell_x} = \varepsilon_\ell(1)\,\sigma_\ell(S_{1x}) \\ S^2 = C \quad (\Longleftarrow S'^2 Q = QC) \\ \rho'\begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} Q = QG_\ell \end{cases}$$

(generalised FP-indicators)

$C = \rho\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, so determined by the Galois action.

Now $S'Q = QS$ is a system of quadratic equations (in Q_{ij}, S_{xy} jointly), which (if we're lucky) we can solve (away from $\det Q = 0$).

— Let us calculate! —

What about character vectors?

- If there is a conformal field theory whose representation category realises $Z(\mathcal{C})$, then the graded dimensions of its modules gives a character vector:
  - a vector valued modular form $\mathbb{X}(\tau) \in \mathbb{Q}[q^{-1}, q]]^n$

$$\mathbb{X}_i(\tau) = q^{h_{M_i} - c/24} \sum_{n=0}^{\infty} q^n \dim M^{(i)}_{h_{M^{(i)}}+n} \qquad \left(q = e^{2\pi i \tau}\right)$$

transforming according to the modular data:

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{X}(\tau).$$

We can classify the vector valued modular forms associated with our modular data (for each possible central charge c) which satisfy appropriate integrality and positivity conditions.

Theorem (Gannon-Morrison arXiv:1606.07165)

Any $c=8$ conformal field theory realising $Z(\mathcal{E}\mathcal{H}6)$ has one of four candidate character vectors, with vacuum components:

$$q^{1/3}\mathbb{X}_1(\tau)_{\omega_0} = 1 + 12q + 73q^2 + 346q^3 + 1390q^4 + 4956q^5 + 16715q^6 + 52982q^7 + \cdots$$

$$q^{1/3}\mathbb{X}_2(\tau)_{\omega_0} = 1 + 3q + 22q^2 + 86q^3 + 461q^4 + 1992q^5 + 8343q^6 + 30997q^7 + \cdots$$

$$q^{1/3}\mathbb{X}_3(\tau)_{\omega_0} = 1 + 13q + 83q^2 + 372q^3 + 1460q^4 + 5112q^5 + 17053q^6 + 53651q^7 + \cdots$$

$$q^{1/3}\mathbb{X}_4(\tau)_{\omega_0} = 1 + 4q + 32q^2 + 112q^3 + 531q^4 + 2148q^5 + 8681q^6 + 31666q^7 + \cdots$$

Challenge:

Construct such a CFT!