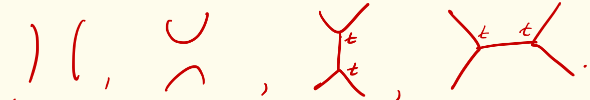
Today (pivotal) categories generated by a trivalent vertex.

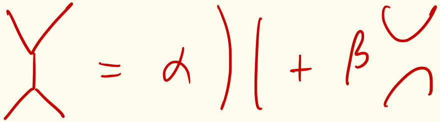
Example Say we had a $_\wedge$ pivotal category $\mathcal{C}$, with a self-dual $_\wedge$ simple object X, generated by a morphism $t : X\otimes X \to X$, and $\dim \mathcal{C}(X\otimes X \to X\otimes X) = 2$.

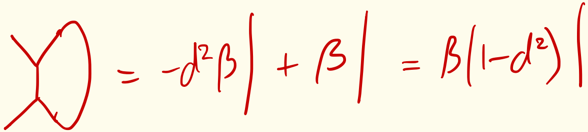
What could we say?

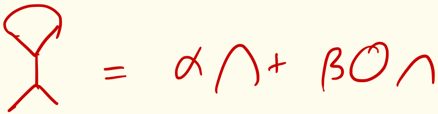
Inside $\mathcal{C}(X\otimes X \to X\otimes X)$, we have



(id$_{X\otimes X}$) (ev $\circ$ coev)







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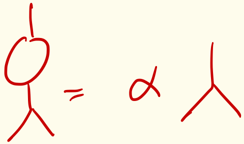
zero

so $\alpha + d\beta = 0$, $\alpha = -d\beta$



||

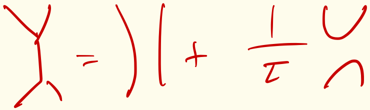
$\beta(1-d^2)$ 

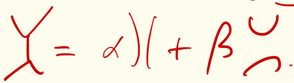
$\Longrightarrow -d\beta = \beta(1-d^2)$

so $d^2 = d+1$

$d = \frac{1\pm\sqrt{5}}{2}$.

We can take $\alpha = 1$, $\beta = d^{-1}$, so



<u>Lemma</u> There is a unique <u>non-degenerate</u> pivotal category generated by  satisfying 

<u>Lemma</u> If you have relations sufficient to evaluate all closed diagrams, you're unique. with simple tensor unit

<u>Proof</u> Pivotal categories, have a unique maximal ideal, which is the ideal of negligible elements.

<u>Proof</u> a non-negligible element in an ideal forces the ideal to be everything.

Does such a category exist? Fib, $(G_2)_1$ (30th-root of unity) $SO(3)_{\zeta(5)}$.

We'd like to develop a uniform approach to questions like this.

Consider the free spherical category $\mathcal{F}$ on some collection of generating morphisms.

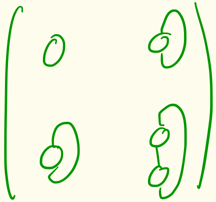
Any quotient $\mathcal{C} = \mathcal{F}/\mathcal{I}$ having a simple tensor unit gives an invariant $\mathbb{k}^{\text{Polyhedra}}$

determined by evaluating each polyhedron as an endomorphism of $\mathbb{1}$.

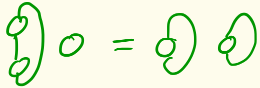
In fact, there's at most one nondegenerate quotient over each point in $\mathbb{k}^P$!

Dimension bounds on hom spaces in $\mathcal{C}$ cut out subvarieties.

__Example__ First $\{1, \phi\}$ in $\mathcal{C}(X \to X)$.
The matrix of inner products is

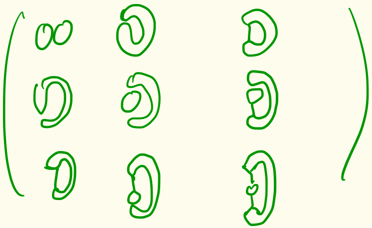


If this is rank one, we must have



In fact, since $|$ is nonzero, $\phi = b|$ for some b, and by normalising we can take $b=1$.

Consider $\{)(, \substack{\cup \\ \cap}, Y\}$. The matrix of inner products is



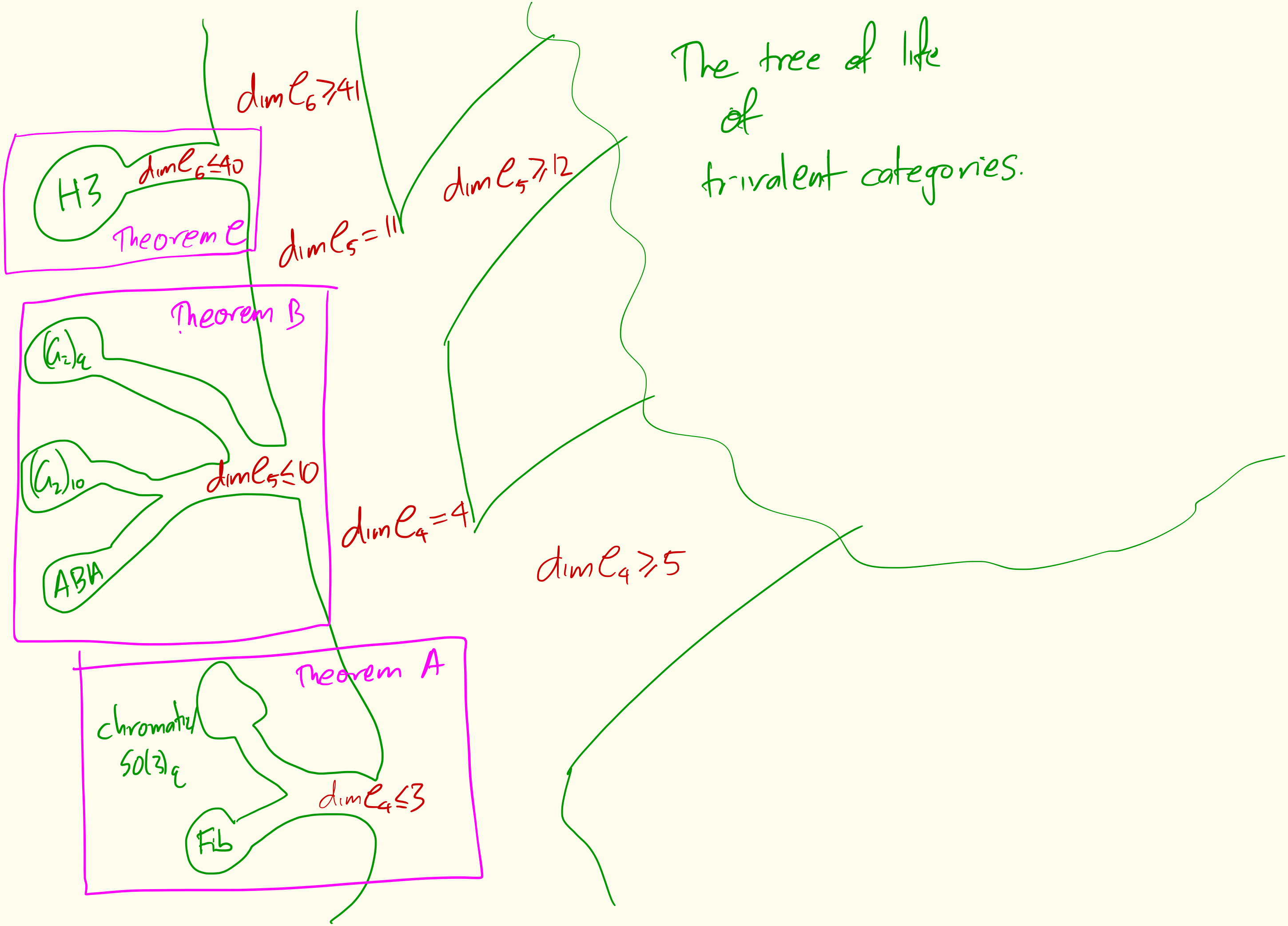
$$= \begin{pmatrix} d^2 & d & db \\ d & d^2 & 0 \\ db & 0 & db^2 \end{pmatrix} = \begin{pmatrix} d^2 & d & d \\ d & d^2 & 0 \\ d & 0 & d \end{pmatrix}$$

with determinant $d^5 - d^3 - d^4 = 0$

so $d^3(d^2 - d - 1) = 0$.

The hypothesis that $\mathcal{C}$ is nondegenerate ensures $d=0$,
so we get $d = \frac{1+\sqrt{5}}{2}$.

The tree of life of trivalent categories.



Theorem A $\dim \mathcal{C}_4 \leq 3 \implies P_{SO(3)} = d + t - dt - 2 = 0$

and $\mathcal{C} = SO(3)_q$ if $(d,t) \neq (-1, \frac{3}{2})$

or $\mathrm{Rep}\,OSp(1|2)$ if $(d,t) = (-1, \frac{3}{2})$.

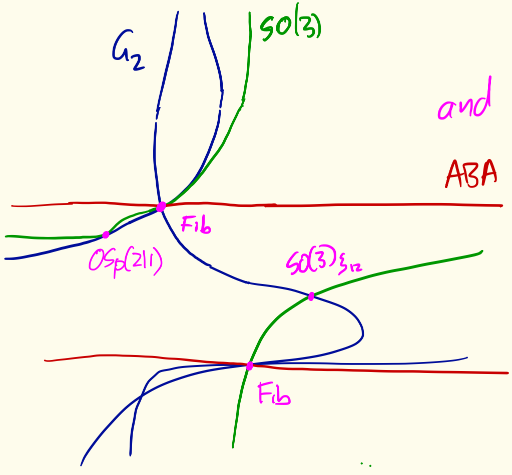
Theorem B $\dim \mathcal{C}_4 = 10$ and $\dim \mathcal{C}_5 \leq 10 \implies$

$P_{ABA} = t^2 - t - 1 = 0$ and $\mathcal{C}$ is an "ABA category"

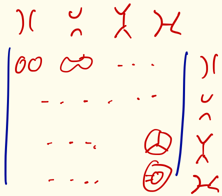
or $P_{G_2} = d^2t^5 - 2dt^5 + 4dt^4 - dt^3 + 6dt^2 + 4dt$
$+ d + t^5 - 4t^4 + t^3 + 7t^2 - 2 = 0$

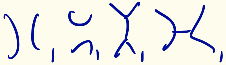
and $\mathcal{C}$ is $(G_2)_q$ with $d = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10}$

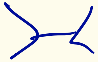
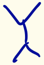
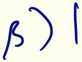
$$t = -\frac{q^2 - 1 - q^{-2}}{q^4 + q^{-4}}$$



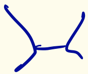
Sketch of A

$\dim \mathcal{C}_4 \leq 3 \longrightarrow \det$  $= 0 \Rightarrow P_{SO(3)} = d+t-dt-2=0.$

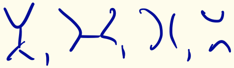
There must be a relation amongst 
and with a little work it must be of the form

 $= \alpha$  $+ \beta$  $+ \gamma$ .

This let's us evaluate any closed diagram, so there is at most one nondegeneate category over each point of $P_{SO(3)}$.

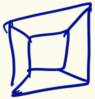
In fact we can derive  $-$  $+ \frac{1}{d-1}\Big($  $-$  $\Big) = 0$

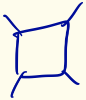
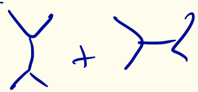
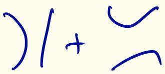
The $SO(3)_q$ categories realise all but the point $(d=-1, t=\frac{3}{2})$, which is realised by $OSp(2|1)$.

Proof of B If  are dependent, we can deduce $\dim \mathcal{C}_4 \le 3$.

Thus they are a basis.

$\Delta(4,0) \neq 0$, so $P_{SO(3)} \neq 0$ and $d+t+dt \neq 0$.

$\Delta(4,1) = 0$, so  $= \dfrac{2d + 2dt - 4dt^2 + 2dt^4 + 2d^2t^4}{d+t+dt}$

and  $= \dfrac{dt^2+t^2-1}{dt+d+t}$ () $+ \dfrac{-t^2+t+1}{dt+d+t}$ ().

Lemma 1 $M^{\square}(n,k)$ has entries rational functions in d, t, if $n+2k<12$.

Proof $n+2k<12 \Rightarrow$ there are fewer than 12 faces in each polyhedron

$\Rightarrow$ there's a square or smaller

Lemma 2 If there is a linear relation amongst $D(n,k)$,

$\Delta(n',k') = 0$ for all $n' \geq n$ and $k' \geq k$.

$$\Delta(5,0) = d^{10} P_{ABA}^{2} P_{SO(3)}^{4} Q_{1,2}$$

$$\Delta^{\square}(5,1) = d^{11} P_{ABA}^{3} P_{SO(3)}^{5} P_{G_2} Q_{1,1}^{-2} \quad (\text{if } \dim \mathcal{C}_4 = 4)$$

If $\dim \mathcal{C}_5 \leq 10$, $\Delta^{\square}(5,1) = 0$, so $P_{ABA} = 0$ or $P_{G_2} = 0$.

Now we turn to uniqueness, and prove: for each (d,t) with $P_{ABA} = 0$ or $P_{G_2} = 0$, there is at most one category with $\dim \mathcal{C}_4 = 4$, $\dim \mathcal{C}_5 \leq 11$.

Say a face is 'small' if it is a pentagon or smaller.

__Lemma 3__ a boundary connected open planar trivalent graph with $n \leq 5$ boundary points and no internal small faces has no internal faces.

__Easier lemma__ A closed planar trivalent graph has a small face.

__Proof__ Given each n-gon \$$6-n$ dollars. By Euler characteristic, there are \$12 in total. Show me the money!

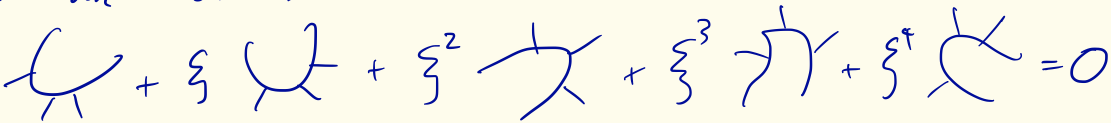
<u>Lemma</u> 4 given a relation reducing each $(n \leq 4)$-gon, and some relation in $D^{\square}(5,1)$, there is a relation reducing the pentagon.

<u>Cor</u> we can evaluate all closed graphs, so there is a unique such nondegenerate category.

<u>Lemma</u> 5 If $\dim \mathcal{C}_4 = 4$, with a relation in $D^{\square}(5,1)$, then $\mathcal{C}_5$ is spanned by $D(5,0)$.

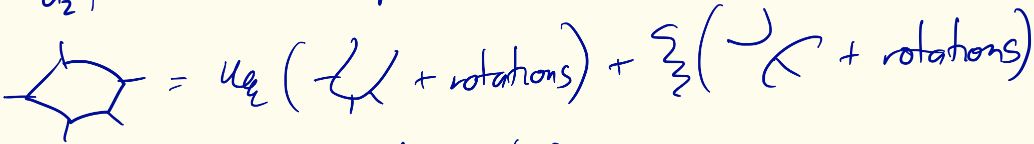
<u>Proof</u> by Lemmas 3,4 anything in $\mathcal{C}_5$ can be reduced to $D(5,0)$.

Relations • On $P_{ABA} = 0$, if $D^{\square}(5,1)$ spans $\mathcal{C}_5$, there are relations



(and the c.c.)

• On P_{G_2}, if $D^{\square}(5,1)$ spans $\mathcal{C}_5$



Proof: just look at the kernel of $M^{\square}(5,1)$.

The proof of uniqueness

If $D^{\square}(5,1)$ is dependent, then lemma 5 says $D(5,0)$ spans, so $D^{\square}(5,1)$ spans.

Otherwise, since $\dim \mathcal{C}_5 \leq 11$, it also spans.

Thus we have the relations above, and the corollary gives uniqueness.

Realisation

ABA as a free product

$(G_2)_q$ via Kuperberg's spider

(or, very recently, Ostrik-Snyder show

$\mathrm{Rep}\, U_q \mathfrak{g}_2 \cong (G_2)_q$ even at (most)

roots of unity.)

Sketch of $\mathcal{C}$

$\dim \mathcal{C}_4 = 4$, $\dim \mathcal{C}_5 = 11$, $\dim \mathcal{C}_6 \leq 40$

If $D(6,0)$ is dependent, $\Delta(6,0) = \Delta^{\square}(6,1) = \Delta^{\square}(6,2) = \Delta(7,0) = 0$

by Gröbner bases, we reduce to finitely many points on the G_2 or $SO(3)$ curves, and by the earlier uniqueness results in fact $\dim \mathcal{C}_5 = 10$.

There are 41 diagrams in $D^{\square}(6,1)$, so

$\Delta^{\square}(6,1) = \Delta^{\square}(6,2) = 0$

Thus we are on $P_{SO(3)}$ or P_{G_2} (contradicting $\dim \mathcal{C}_5 = 11$)

- on an elliptic curve $Q_{2,3} =$
  on which we can calculate $\Delta^{\square}(7,1)$ and $\Delta^{\square}(7,2)$,
  obtaining $d = \frac{3+\sqrt{13}}{2}$, $t = -\frac{2}{3}d + \frac{5}{3}$ or one of 96 horrific points at which rank $M^{\square}(6,2) = 8$
- at finitely many other points, at which $\Delta^{\square}(7,1)$ and $\Delta^{\square}(7,2)$ don't vanish

<u>Uniqueness</u>

<u>Lemma</u> every closed planar trivalent graph has a pentapent or a hexapent.

<u>Lemma</u> every open planar trivalent graph with at most 6 boundary points is in $D^{\square}(n,1)$ or has an internal pentapent or hexapent.

<u>Lemma</u> if $\dim \mathcal{C}_4 = 4$, and we can reduce pentapents and hexapents, $D^{\square}(n,1)$ spans $\mathcal{C}_n$ for $n \leq 6$.

<u>Lemma</u> if $\dim \mathcal{C}_4 = 4$, $\dim \mathcal{C}_5 = 11$, $\dim \mathcal{C}_6 \leq 40$, then $D^{\square}(6,1)$ spans.

if $\dim \operatorname{span} D^{\square}(6,1) \leq 39$, $D(6,0)$ independent means there are relations amongst pentaforks hence reductions.

From this we look at the kernel of $M^{\square}(6,1)$ to obtain relations, and use these to show we can evaluate by reducing pentapents and hexapents.

<u>Realisation</u> some sneaky arguments showing #13 can't be anywhere earlier on the list.