Today *(pivotal)* categories generated by a trivalent vertex.

**Example**: Say we had a *pivotal* category $E$, with a self-dual object $X$, generated by a morphism $t : X \otimes X \to X$, and $\dim E(X \otimes X \to X \otimes X) = 2$.

What could we say?

Inside $E(X \otimes X \to X \otimes X)$, we have $\otimes$, $\cup$, $X$, $X^t$, $X^t$, $(\otimes \otimes X)$, $(ev \circ coev)$.

\[
\begin{align*}
Y &= \alpha \circ \beta \cup \\
\alpha \wedge + \beta \otimes \alpha
\end{align*}
\]

\[
\begin{align*}
\phi &= \alpha \wedge \beta \circ \alpha \wedge \beta \\
\Rightarrow -d \beta &= \beta (1 - d^2) \\
\Rightarrow d^2 &= d + 1 \\
\Rightarrow d &= \frac{1 + \sqrt{5}}{2}.
\end{align*}
\]
We can take $\alpha=1$, $\beta=d^4$, so

$$X = \lambda + \frac{i}{\pi} \nu$$

**Lemma** There is a unique non-degenerate pivotal category generated by $X$ satisfying $X = \alpha Y + \beta Z$.

**Lemma** If you have relations sufficient to evaluate all closed diagrams, you're unique, with simple tensor unit.

**Proof** Pivotal categories have a unique maximal ideal, which is the ideal of negligible elements. Let $a$ be a non-negligible element in an ideal. Then the ideal is everything.

Does such a category exist? $\text{Fib}$, $(G_2)_{1/2}$, $SO(3)\times SU(2)$. 
We'd like to develop a uniform approach to questions like this.

Consider the free spherical category $F$ on some collection of generating morphisms.

Any quotient $E = F/I$ having a simple tensor unit gives an invariant $\text{Polyhedra}$ determined by evaluating each polyhedron as an endomorphism of $1$. In fact, there's at most one nondegenerate quotient over each point in $\text{IRP}^!$.

Dimension bounds on hom spaces in $C$ cut out subvarieties.
Example

First \( e^1, e^3 \) in \( \mathcal{E}(x \to x) \).

The matrix of inner products is
\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\end{pmatrix}
\]
If this is rank one, we must have
\[
\begin{pmatrix}
0 \\
0 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
\]

In fact, since \( I \) is nonzero, \( \phi = b \) for some \( b \), and by normalising we can take \( b = 1 \).

Consider \( \{ e^1, e^2, e^3 \} \). The matrix of inner products is
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
d^2 & d & db \\
d & d^2 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
d^2 & d & d \\
d & d^2 & 0 \\
\end{pmatrix}
\]
with determinant \( d^5 - d^3 - d^4 = 0 \)
so \( d^3 (d^2 - 1) = 0 \).

The hypothesis that \( \mathcal{E} \) is nondegenerate ensures \( d \neq 0 \),
so we get \( d = \frac{\sqrt{3}}{2} \).
The tree of life of trivalent categories.
Theorem A \[ \dim E_4 \leq 3 \implies P_{\text{so}(3)} = d + t - dt - 2 = 0 \]

and \( E = \text{so}(3)_q \) if \((d,t) \neq (-1,\frac{3}{2})\)

or \( \text{RepOSp}(1|2) \) if \((d,t) = (-1,\frac{3}{2})\).

Theorem B \[ \dim E_4 = 0 \text{ and } \dim E_5 \leq 0 \implies P_{\text{ABA}} = t^2 - t - 1 = 0 \]

and \( E \) is an "ABA category"

or \[ P_{\text{G}_2} = d^2 + 5d^4 - 2dt^5 + 4dt^4 - dt^3 + 6dt^2 + 6dt \]

\[ + d + t^5 - 4t^4 + t^3 + 7t^2 - 2 = 0 \]

and \( E \) is \( (G_2)_q \) with \( d = q^{10} + q^8 + q^6 + q^4 + q^2 + q^0 \)

\( t = -\frac{q^2 - 1 - q^{-2}}{q^9 + q^{-9}} \)
**Sketch of \( A \)**

\[ \dim E_4 \leq 3 \implies \det \begin{vmatrix} \chi & \chi & \chi \\ \chi & \chi & \chi \\ \chi & \chi & \chi \end{vmatrix} = 0 \implies P_{50(3)} = dt^2 - dt - 2 = 0. \]

There must be a relation amongst \( \chi, \gamma, \chi, \chi, \gamma \), and with a little work it must be of the form

\[ Y^2 = \alpha \chi^2 + \beta \gamma \chi. \]

This lets us evaluate any closed diagram, so there is at most one nondegenerate category over each point of \( P_{50(3)} \).

In fact we can derive

\[ Y^2 - \chi + \frac{1}{d-1} \left( \frac{1}{2} \chi \right) = 0 \]

The \( \text{SO}(3) \) categories realise all but the point \( (d = -1; \ t = \frac{3}{2}) \), which is realised by \( \text{OSp}(2|1) \).
Proof of B If \( \lambda_1, \lambda_2, \lambda_3 \) are dependent, we can deduce \( \dim E \leq 3 \).

Thus they are a basis.

\[ \Delta(4,0) \neq 0, \text{ so } P_{5x3} \neq 0 \text{ and } d + t + dt = 0. \]

\[ \Delta(4,1) = 0, \text{ so } \begin{bmatrix} 2d + 2dt - 4dt^2 + 2dt^3 + 2d^2t^2 \\ d + t + dt \end{bmatrix} \]

and

\[ \begin{bmatrix} d^t + t^2 + 1 \\ d + d + t \end{bmatrix} \left( \lambda + \lambda^2 \right) + \frac{-t^2 + t + 1}{d + d + t} \left( \lambda^3 + \lambda \right). \]

Lemma 1 \( M^2(n,k) \) has entries rational functions in \( d, t \),

if \( n + 2k < 12 \).

Proof \( n + 2k < 12 \Rightarrow \) there are fewer than 12 faces in each polyhedron

\( \Rightarrow \text{there's a square or smaller} \).

Lemma 2 If there is a linear relation amongst \( D(n,k) \),

\[ \Delta(n', k') = 0 \text{ for all } n' > n \text{ and } k' > k. \]
\[ \Delta(5,0) = d^{10} P_{\text{ABA}}^2 P_{\text{so(3)}} Q_{1,2} \]

\[ \Delta^0(5,1) = d^{11} P_{\text{ABA}}^3 P_{\text{so(3)}}^5 P_{\text{a}_2} Q_{1,1}^{-2} \quad (\text{if } \dim E_4 = 4) \]

If \( \dim E_5 \leq 10 \), \( \Delta^0(5,1) = 0 \), so \( P_{\text{ABA}} = 0 \) or \( P_{\text{a}_2} = 0 \).

Now we turn to uniqueness, and prove:

- for each \((d,t)\) with \( P_{\text{ABA}} = 0 \) or \( P_{\text{a}_2} = 0 \), there is at most one category with \( \dim E_4 = 9 \) and \( \dim E_5 \leq 11 \).

Say a face is 'small' if it is a pentagon or smaller.

**Lemma 3** A boundary connected open planar trivalent graph with \( n \leq 5 \) boundary points and no internal small faces has no internal faces.

**Easier Lemma** A closed planar trivalent graph has a small face.

**Proof** Give each \( n \)-gon \$6-n \) dollars. By Euler characteristic, there are \$12 in total. Show me the money!
Lemma 4: Given a relation reducing each \((n \leq 4)\)-gon, and some relation in \(D^n(5,1)\), there is a relation reducing the pentagon.

Cor: We can evaluate all closed graphs, so there is a unique such nondegenerate category.

Lemma 5: If \(\dim C_a = 4\), with a relation in \(D^n(5,1)\), then \(C_5\) is spanned by \(D(5,0)\).

Proof: By Lemmas 3, 4 anything in \(C_5\) can be reduced to \(D(5,0)\).
Relations. On $P_{ABA} = 0$, if $D^{0}/5,i)$ spans $E_{5}$, there are relations

$$
\xi + \xi^{2} + \xi^{3} + \xi^{4} + \xi^{5} \neq 0
$$

(and the c.c.)

On $P_{Pz}$, if $D^{0}/5,i)$ spans $E_{5}$

$$
\bigcirc = iE_{5} (\xi + \text{rotations}) + \xi (\xi + \text{rotations})
$$

Proofs: Just look at the kernel of $M^{0}/5,i)$.

The proof of uniqueness

If $D^{0}/5,i)$ is dependent, then Lemma 5 says $D^{5,0}$ spans, so $D^{0}(5,i)$ spans.

Otherwise, since $\dim E_{5} \leq 11$, it also spans.

Thus we have the relations above, and the corollary gives uniqueness.
Realisation

ABA as a free product

$(G_2)_q$ via Kuperberg's spider

(or, very recently, Ostrik-Snyder show

$\text{Rep}_{U_q G_2} \simeq (G_2)_q$ even at (most)

roots of unity.)
Sketch of $\mathcal{E}$

$\dim E_4 = 4$, $\dim E_5 = 11$, $\dim E_6 \leq 40$

If $D(6,0)$ is dependent, $\Delta(6,0) = \Delta^D(6,1) = \Delta^D(6,2) = \Delta(7,0) = 0$

by Gröbner bases, we reduce to finitely many points

on the $G_2$ or $SO(3)$ curves, and by the earlier

uniqueness results in fact $\dim E_5 = 10$.

There are 41 diagrams in $D^P(6,1)$, so

$\Delta^P(6,1) = \Delta^P(6,2) = 0$

Thus we are on $P_{40(3)}$ or $P_{G_2}$ (contradicting $\dim E_5 = 11$)

- on an elliptic curve $Q_{2,3} =$
  - on which we can calculate $\Delta^P(7,1)$ and $\Delta^P(7,2)$,
    obtaining $d = \frac{3 + \sqrt{13}}{2}$, $z = -\frac{2}{3} d + \frac{5}{3}$ or one of 96 horimic
    points at which $\text{rank } M^P(6,2) = 8$

- at finitely many other points, at which
  $\Delta^P(7,1)$ and $\Delta^P(7,2)$ do not vanish
Uniqueness

Lemma every closed planar trivalent graph has a pentapent or a hexapent.

Lemma every open planar trivalent graph with at most 6 boundary points is in \( D^0(n, i) \) or has an internal pentapent or hexapent.

Lemma if \( \dim E_4 = 4 \), and we can reduce pentapents and hexapents, \( D^0(n, i) \) spans \( E_n \) for \( n \leq 6 \).

Lemma if \( \dim E_4 = 4 \), \( \dim E_5 = 11 \), \( \dim E_6 \leq 40 \), \( D^0(n, i) \) independent means \( D^0(6, 1) \) spans.

From this we look at the kernel of \( N^0(6, 1) \) to obtain relations, and use these to show we can evaluate by reducing pentapents and hexapents.

Realisation some sneaky arguments showing 113 can't be anywhere earlier on the list.