topological matter

Slides online at https://tqft.net/talks

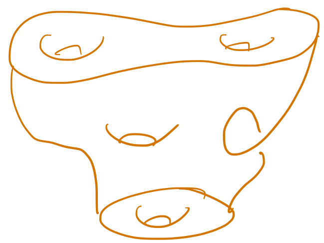
(anyonic)
topological matter

(anyonic)
topological matter

topological quantum field theories

n-manifolds

vector spaces



n+1 dimensional
cobordisms

linear maps

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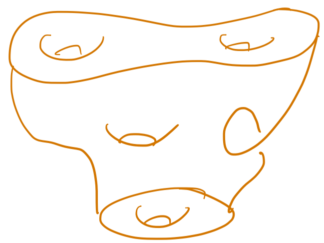
(extended) topological quantum field theories

. . . lower dimensional manifolds . . .

more complicated algebraic data

n-manifolds

vector spaces



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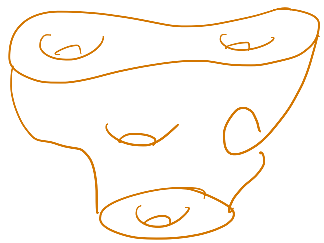

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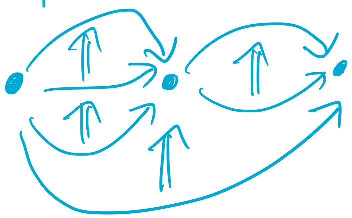

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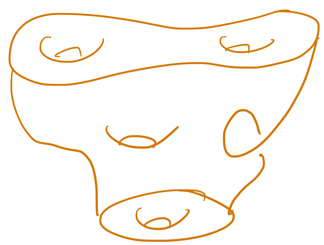

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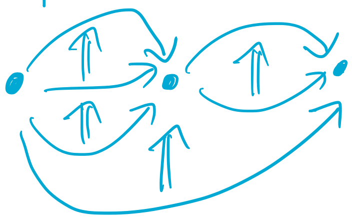

(anyonic) topological matter

approximate mathematical description?

(extended) topological quantum field theories

... lower dimensional manifolds ...

$\rightsquigarrow$

more complicated algebraic data

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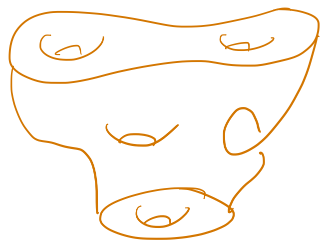

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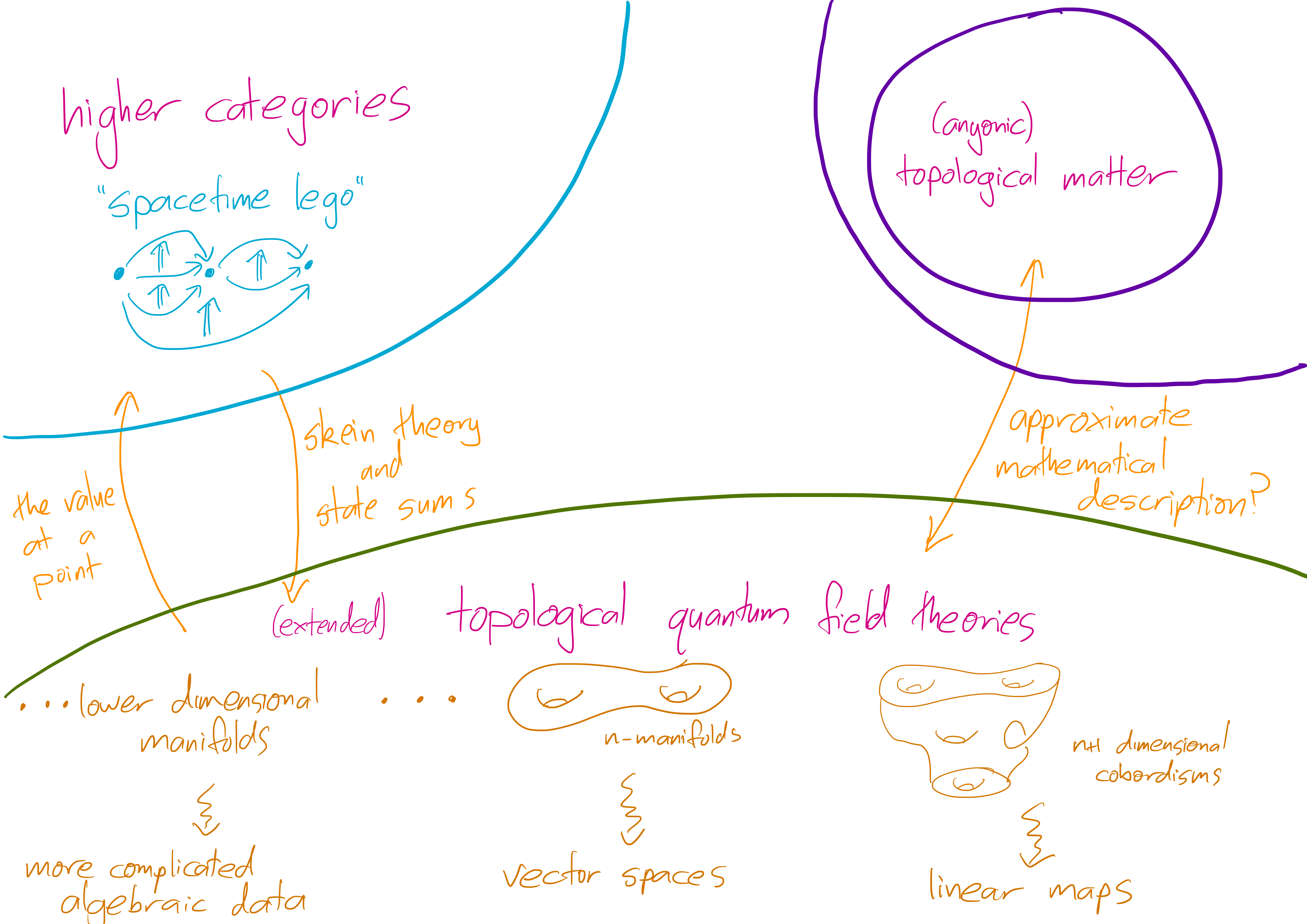
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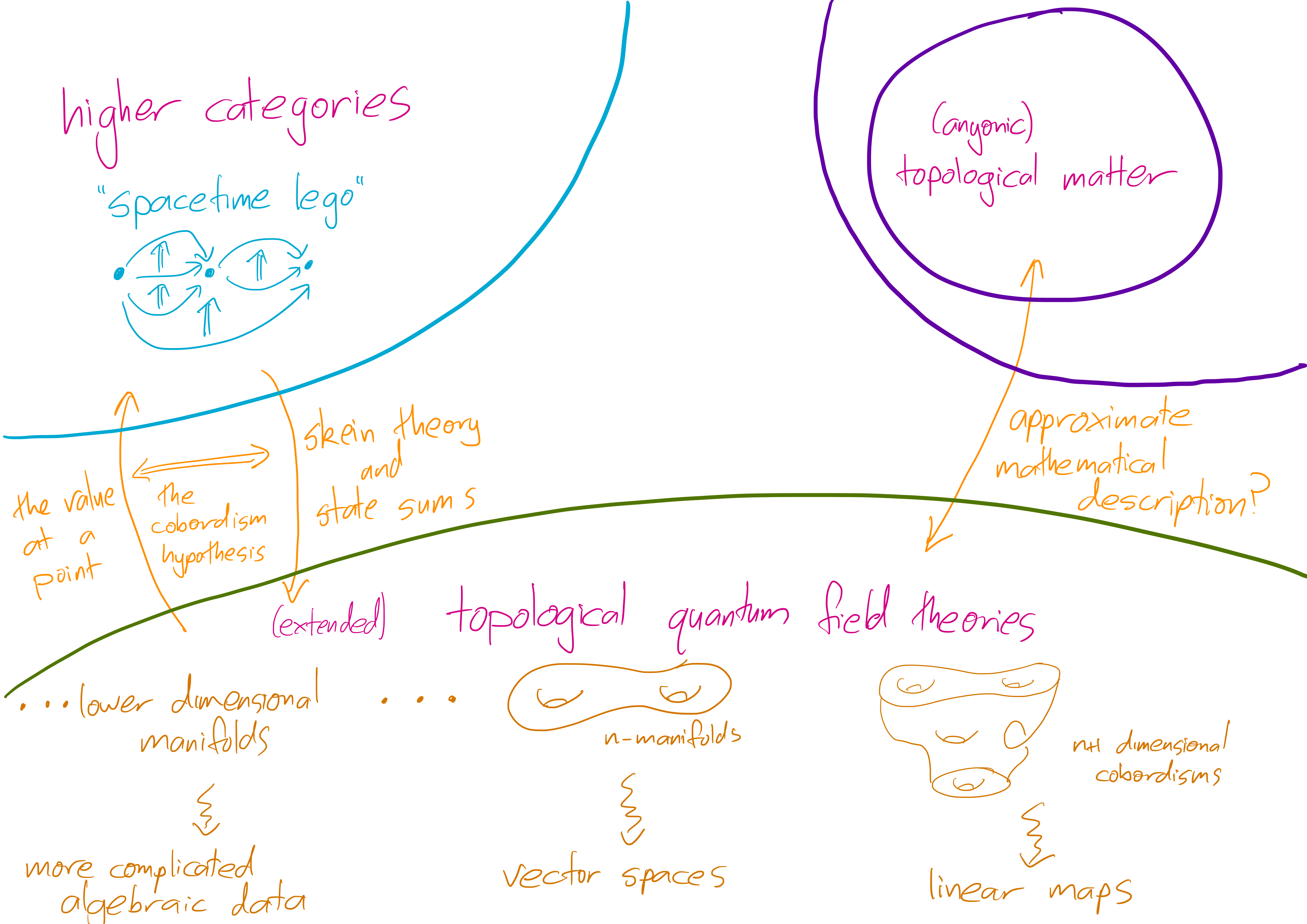
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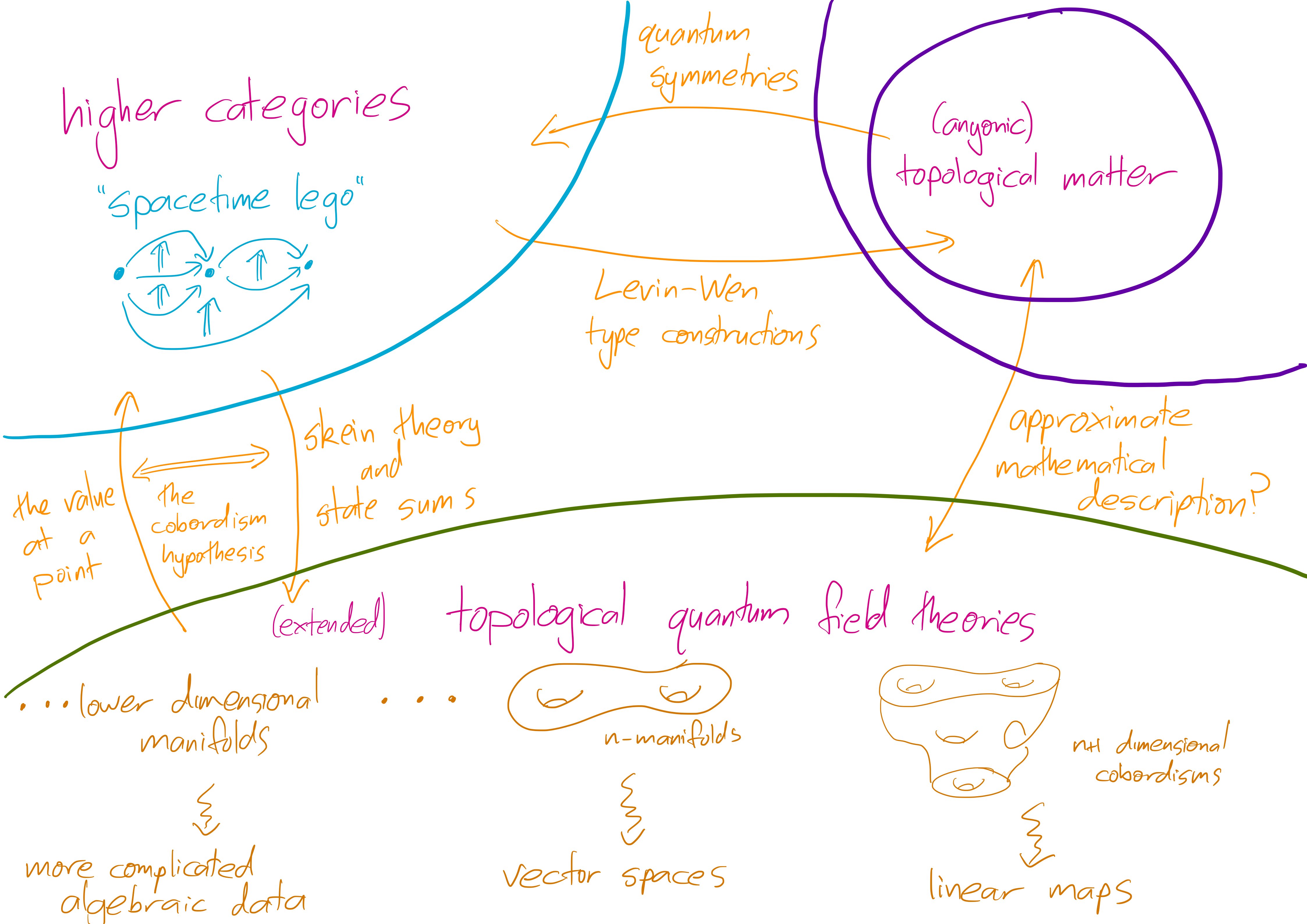

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A Topological Quantum Field Theory
is a functor F

$$\{n\text{-manifolds}\} \longrightarrow \{\text{vector spaces}\}$$

$n=1$:

$n=2$:

A Topological Quantum Field Theory
is a functor $\mathcal{F}$

$$\left\{ \begin{array}{c} n{+}1\text{-dimensional} \\ \text{cobordisms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{linear maps} \\ \mathcal{F}(W) : \mathcal{F}(M^{in}) \to \mathcal{F}(M^{out}) \end{array} \right\}$$

$$\circlearrowright \qquad\qquad\qquad\qquad \circlearrowright$$

$$\{n\text{-manifolds}\} \longrightarrow \{\text{vector spaces}\}$$

A Topological Quantum Field Theory

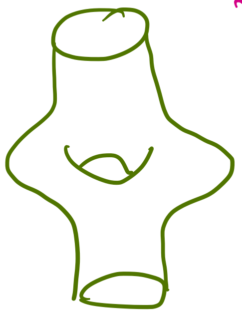
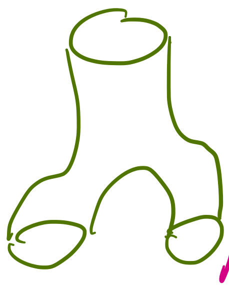
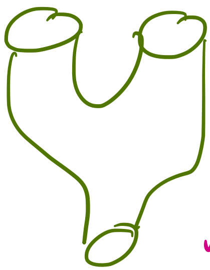
is a functor F

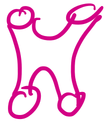
$\{$ n+1-dimensional cobordisms  $\}$ $\longrightarrow$ $\{$ linear maps $F(W): F(M^{in}) \longrightarrow F(M^{out})$ $\}$

$\{$ n-manifolds $\}$ $\longrightarrow$ $\{$ vector spaces $\}$

What does it mean to say that this is a functor?

If we stack two cobordisms on top of each other, we should get the composite of the two linear maps:

$F\Big($  $\Big) = F\Big($  $\Big) \circ F\Big($  $\Big)$

$F\big($  $\big) = F\big($  $\big) \circ F\big($  $\big)$

A Topological Quantum Field Theory

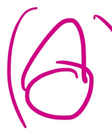
is a "symmetric monoidal" functor F

$\{$n+1-dimensional cobordisms  $\} \longrightarrow \{$linear maps $F(W): F(M^{in}) \longrightarrow F(M^{out})\}$

$\{$n-manifolds$\} \longrightarrow \{$vector spaces$\}$

disjoint unions of n-manifolds should be sent to tensor products of vector spaces!

$$F(\bigcirc\ \bigcirc\ \bigcirc) = F(\bigcirc) \otimes F(\bigcirc) \otimes F(\bigcirc).$$

$F($$)$: $F(\bigcirc) \longrightarrow F(\emptyset) = \mathbb{C}$

A Topological Quantum Field Theory

is a "symmetric monoidal" functor $\mathcal{F}$

{ n+1-dimensional cobordisms 

} $\longrightarrow$ { linear maps $\mathcal{F}(W): \mathcal{F}(M^{in}) \longrightarrow \mathcal{F}(M^{out})$ }

{n-manifolds} $\longrightarrow$ {vector spaces}

This is a mathematician's birds-eye view of physics:

- for each "shape of the universe", a Hilbert space describing the possible state of the universe
- for each "time evolution", a map describing how states transform.

The theory is <u>topological</u> if two cobordisms which are diffeomorphic (rel ∂) give the same linear map.

As a consequence, two diffeomorphic manifolds have isomorphic vector spaces.

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As a consequence, two diffeomorphic manifolds have isomorphic vector spaces.

Have we thrown the baby out with the bathwater?

Our "physics" can't see any geometry or any matter.

① Initially, TQFTs were intended as toy models of QFT (to get practice!) and to explain the Jones polynomial in knot theory.

② Later, with much prodding from the physicists, we realised they were actually real physics.

Example

A 2+1-dimensional TQFT $\mathcal{F}$ assigns

- a vector space $\mathcal{F}(\Sigma)$ to each surface Σ, and
- a linear map $\mathcal{F}(M) : \mathcal{F}(\Sigma^{in}) \to \mathcal{F}(\Sigma^{out})$ to each 3-d cobordism

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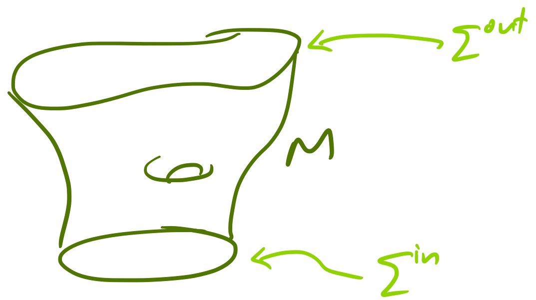
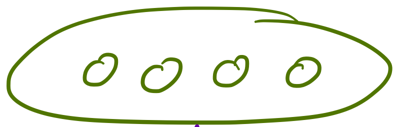
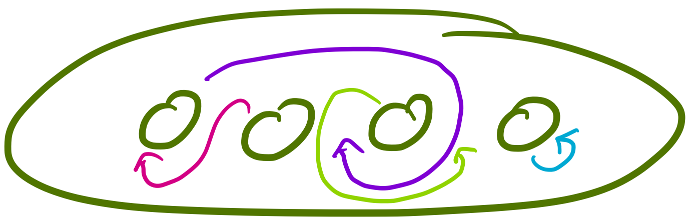


In particular, a vector space for each punctured disc .

Given any way of braiding the punctures around each other



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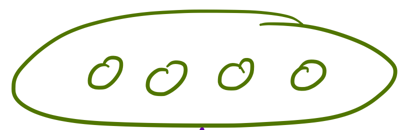
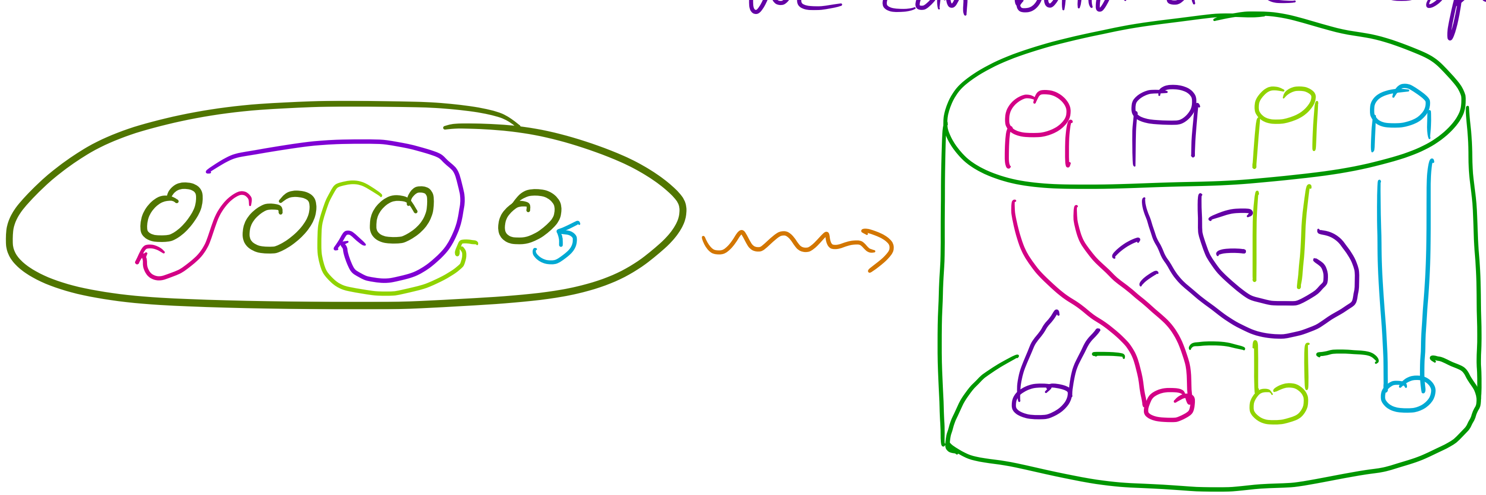
In particular, a vector space for each punctured disc .

Given any way of braiding the punctures around each other we can build a corresponding cobordism



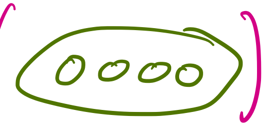
This gives a <u>representation</u> of the braid group on the vector space associated to a punctured disc.

For many interesting 2+1-dimensional TQFTs, the image of the braid group is <u>dense</u> in the unitary group of this vector space.

This gives a <u>representation</u> of the braid group on the vector space associated to a punctured disc.

For many interesting 2+1-dimensional TQFTs, the image of the braid group is <u>dense</u> in the unitary group of this vector space.

The initial plans for anyonic quantum computation used this fact:

- have the computer scientists express their quantum algorithm as a unitary operator
- "compile" their program for this hardware, by approximating that unitary by a braid group element
- for each run of the algorithm, prepare the initial state in $\mathcal{F}($$)$ according to the input.
- physically(?!) braid the punctures around each other
- perform a measurement to read off the answer.

As a prelude to the classification of TQFTs, let's begin by studying (0+1)-dimensional TQFTs.

Every (unoriented) 0-manifold is a disjoint union of points.

We require $\mathcal{F}(\underbrace{\bullet\ \bullet\ \bullet\ \bullet}_{n\text{-points}}) \cong \mathcal{F}(\bullet)^{\otimes n}$,

so we just need to pick a vector space $\mathcal{F}(\bullet)$.

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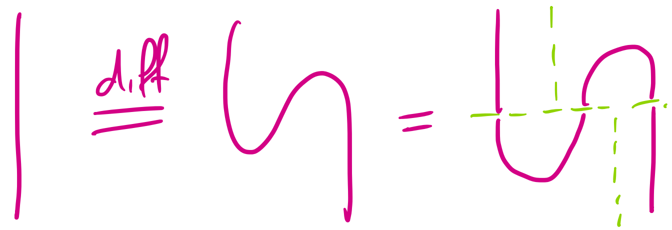
so we just need to pick a vector space $F(\bullet)$.

Will any vector space do? We need to look at 1-manifolds!

We need maps $F(\cap) : F(\bullet) \otimes F(\bullet) \longrightarrow F(\emptyset)$ just $\mathbb{C}$

and $F(\cup) : \mathbb{C} \longrightarrow F(\bullet) \otimes F(\bullet)$.

Moreover since



we must have $id_{F(\bullet)} = \left(id_{F(\bullet)} \otimes F(\cap)\right) \circ \left(F(\cup) \otimes id_{F(\bullet)}\right)$.

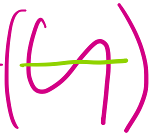
and similarly for

$\mathcal{F}(\cap) : \mathcal{F}(\bullet) \otimes \mathcal{F}(\bullet) \longrightarrow \mathbb{C}$
$\mathcal{F}(\cup) : \mathbb{C} \longrightarrow \mathcal{F}(\bullet) \otimes \mathcal{F}(\bullet)$
$| \overset{\text{diff}}{=}$ 

Say $\mathcal{F}(\cup) : 1 \longmapsto \sum_{ij} f_{ij}\, e_i \otimes e_j$ (with $\{e_i\}$ a basis)

and $\mathcal{F}(\cap) : e_i \otimes e_j \longmapsto g_{ij}$.

Notice since $\cap \overset{\text{diff}}{=}$  and $\cup \overset{\text{diff}}{=}$ , both f and g are symmetric.

Then $\mathcal{F}($$) : e_k \longmapsto \sum_{ij} f_{ij}\, e_i \otimes e_j \otimes e_k \longmapsto \sum_{ij} f_{ij}\, g_{jk}\, e_i \overset{\text{we want}}{=} e_k$

This shows (g_{jk}) is the inverse of (f_{ij}).

$\mathcal{F}(\cap) : \mathcal{F}(\bullet) \otimes \mathcal{F}(\bullet) \longrightarrow \mathbb{C}$

$\mathcal{F}(\cup) : \mathbb{C} \longrightarrow \mathcal{F}(\bullet) \otimes \mathcal{F}(\bullet)$

$| \overset{\text{diff}}{=} \backsim$

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This shows (g_{jk}) is the inverse of (f_{ij}).

Moreover, $\mathcal{F}(\bigcirc) = \mathcal{F}(\cap) \circ \mathcal{F}(\cup)$ is some map $\underset{\mathcal{F}(\emptyset)}{\mathbb{C}} \longrightarrow \underset{\mathcal{F}(\emptyset)}{\mathbb{C}}$,

so must itself be (multiplication by) some complex number.

Now $\mathcal{F}(\cap) \circ \mathcal{F}(\cup) : 1 \longmapsto \sum f_{ij}\, e_i \otimes e_j \longmapsto \sum f_{ij}\, g_{ij} = \sum_{ij} f_{ij}\, g_{ji} = \sum_i \delta_{ii}$

$= \dim \mathcal{F}(\bullet)$.

Thus we see the vector space is finite dimensional.

This is a very general phenomenon:
to construct a TQFT, we'll need to provide a bunch of algebraic ('higher categorical') data,
and in order to be able to construct the top-dimensional piece of the TQFT, this data will have to satisfy some very restrictive conditions, which are

"higher dimensional analogues of finite dimensionality"

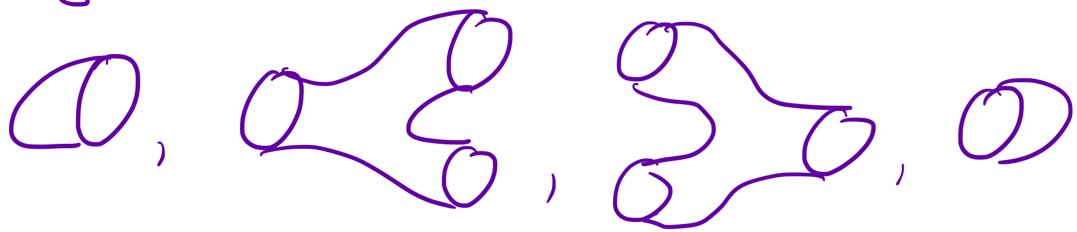
(e.g. semisimplicity, fusion, modularity...)

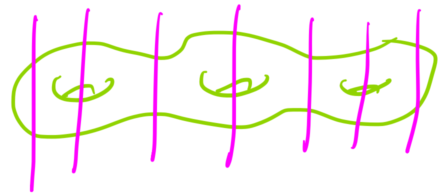
Extended topological field theories.

Sometimes we would like to be able to cut our n-manifolds up into pieces, and use these to calculate the value of the TQFT.

A "local" physical theory should always allow this.

E.g. every 2-manifold can be built out of



(e.g. )

It's more complicated in higher dimensions, but we always have <u>handle decompositions</u>.

To do this, we're going to have to assign more algebraic data to $(n-1)$-manifolds, and then to $(n-2)$-manifolds, all the way down to 0-manifolds.

Example A 2|0-dimensional TQFT $\mathcal{F}$ should associate

- a linear category $\mathcal{F}(\bullet)$ to a point
- a vector space $\mathcal{F}(\bigcirc)$ to each 1-manifold

  (up to disjoint unions, the circle is the only closed 1-manifold)
- linear maps to 2-dimensional cobordisms.

additionally satisfying the property that

$$\mathcal{F}(\bigcirc) = \mathcal{F}(\bullet) / _{ab=ba}.$$

Example A 2|1|0-dimensional TQFT $\mathcal{F}$ should associate

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additionally satisfying the property that

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In fact, one can do this if and only if $\mathcal{F}(\bullet)$ is a semisimple algebra $\mathcal{F}(\bullet) = M_{n_1 \times n_1}(\mathbb{C}) \oplus M_{n_2 \times n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k \times n_k}(\mathbb{C})$

and then $\mathcal{F}(\bigcirc) = \mathbb{C}^k$.

If we hadn't "extended down to a point", $\mathcal{F}(\bigcirc)$ could have been any commutative Frobenius algebra.

In general, for a fully extended $(n+1)$-dimensional TQFT,

for a closed $(n-j)$-manifold we will associate a linear j-category,

(Here a linear 0-category is a vector space, and

a linear (-1)-category is a number.)

and for an $(n-j)$-manifold W with "incoming boundary" Σ^{in}, and

"outgoing boundary" Σ^{out}, (both of which are $(n-j-1)$-manifolds,

with associated $(j+1)$-categories, we will associate

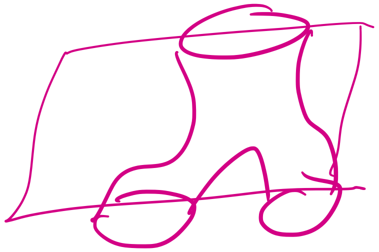
a linear j-category $F(W)$ equipped with "actions" of $F(\Sigma^{in})$ and $F(\Sigma^{out})$.

When we can slice W into two pieces W_1, W_2, along some

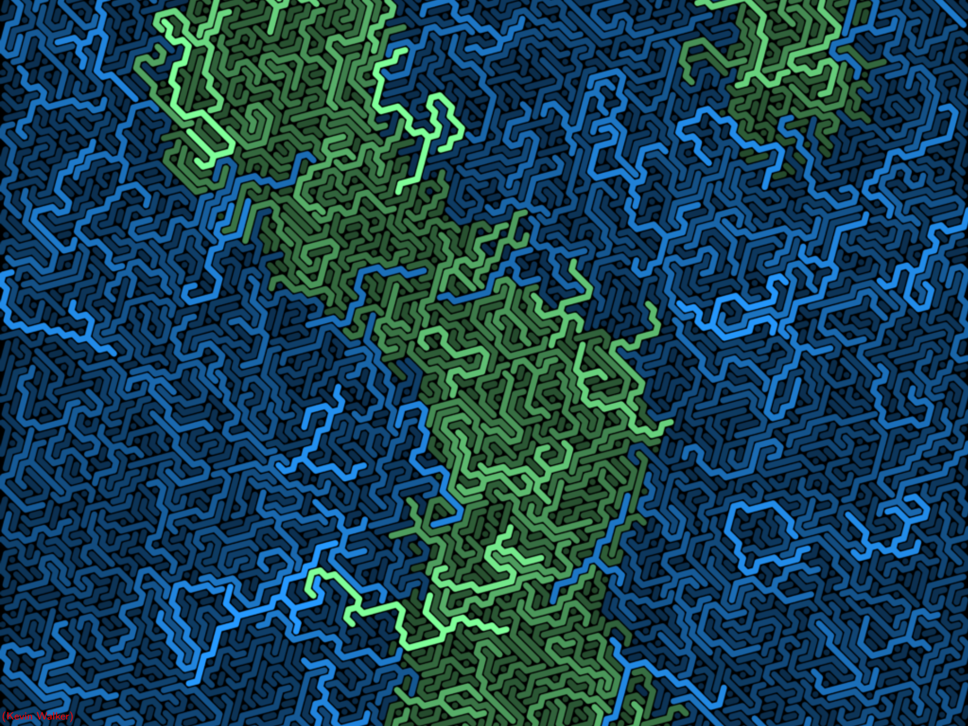
codimension 1 locus Σ, we have a <u>gluing formula</u>

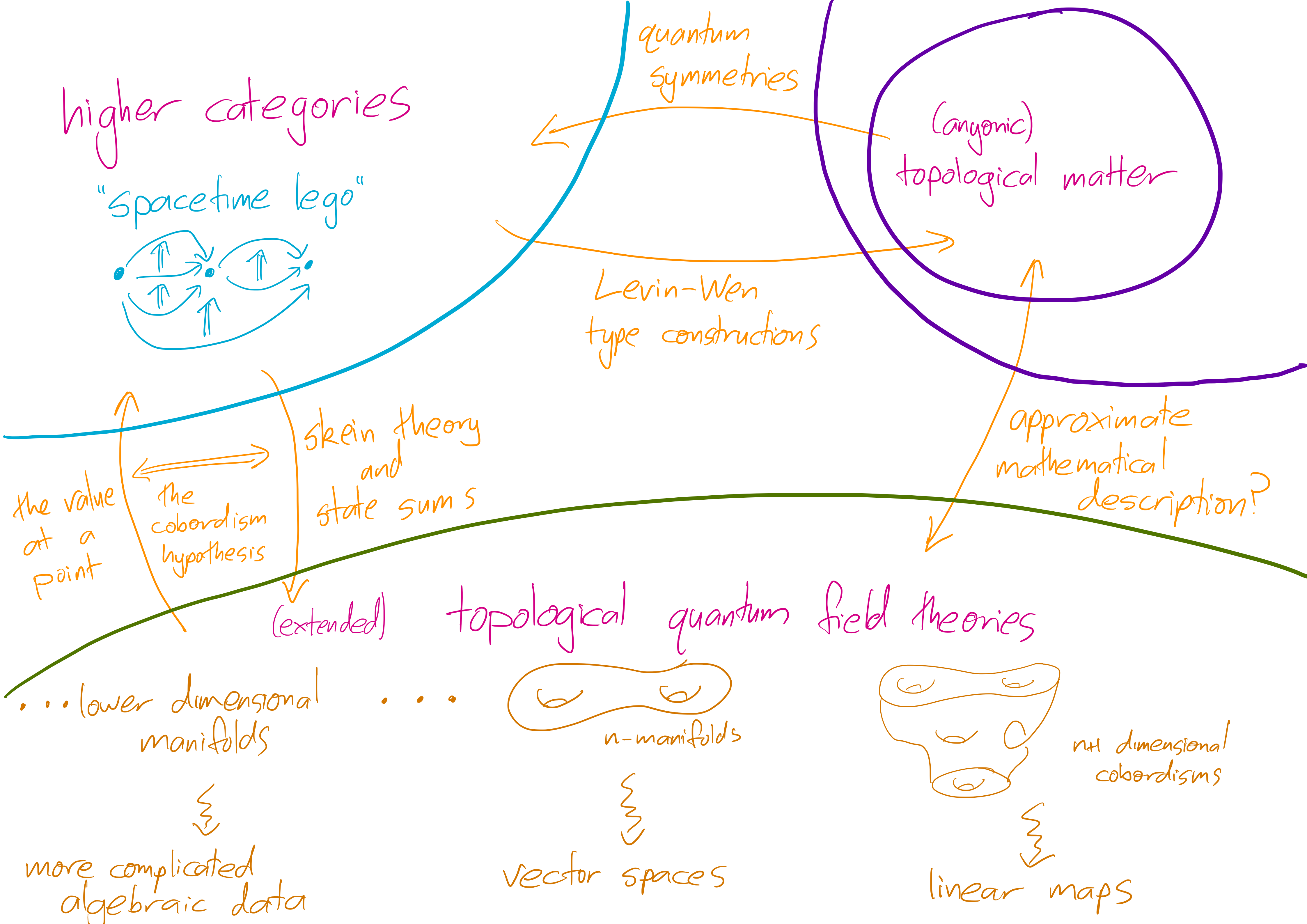
$$F(W) = F(W_1) \underset{F(\Sigma)}{\otimes} F(W_2).$$

In 2+1 dimensions:

 will be a vector space, with an action of $\mathcal{F}(\bigcirc)$, and an action of $\mathcal{F}(\bigcirc\bigcirc)$

 will be a ~~category~~ algebra


(Kevin Walker)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A *category* $\mathcal{C}$ consists of:

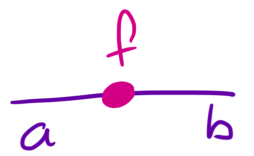
- a set $\mathcal{C}^0$ of "objects", or "states", or "0-morphisms".
- for each pair of objects $a, b \in \mathcal{C}^0$,

  a set $\mathcal{C}^1(a \to b)$ of "morphisms" (or "transitions", or "1-morphisms")

  from a to b.
- a rule for composing morphisms

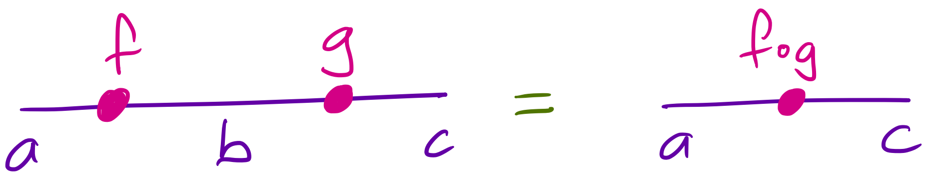
$$\mathcal{C}^1(a \to b) \times \mathcal{C}^1(b \to c) \longrightarrow \mathcal{C}^1(a \to c).$$

String diagrams:

We draw a morphism $f : \mathcal{C}^1(a \to b)$ as 

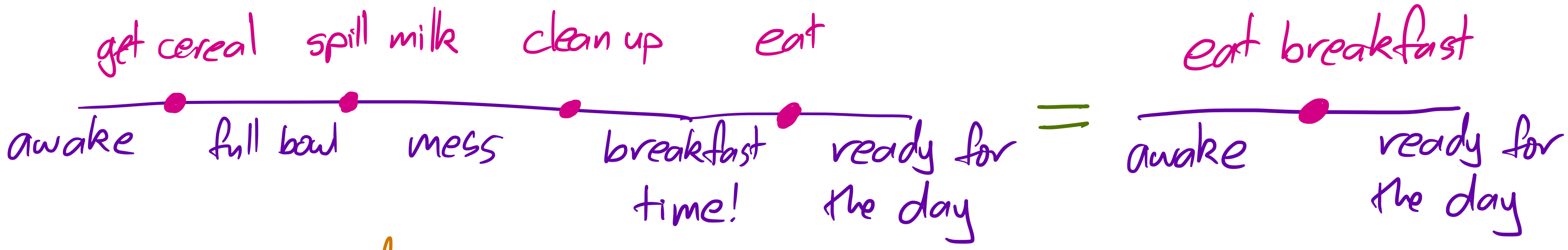
,

and compose morphisms by drawing a sequence of beads on a string:

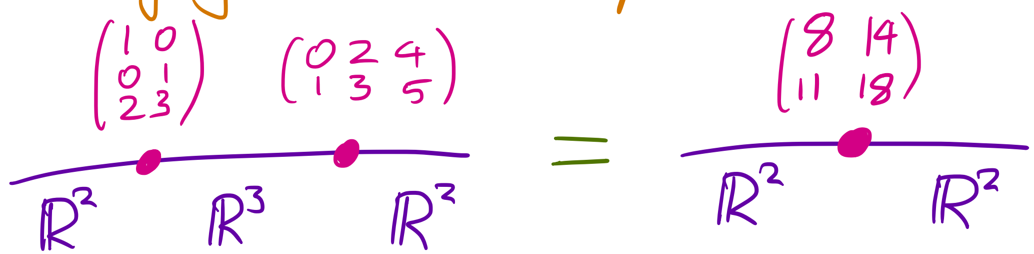


Examples:

① "states and transitions"



② "the category of vector spaces"



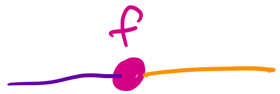
(here $\mathcal{C}^0 = \{$all vector spaces$\}$
$\mathcal{C}^1(V \to W)$ consists of
all linear maps $V \to W$)

That's everything you need to know about 1-category theory!

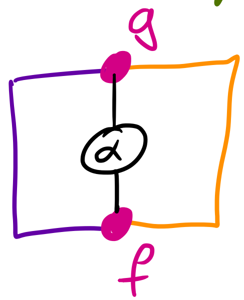
A 2-category consists of:

- a set $\mathcal{C}^0$ of "0-morphisms"

- for each $a, b \in \mathcal{C}^0$, a set $\mathcal{C}^1(a \to b)$ of "1-morphisms"



- for each $a, b \in \mathcal{C}^0$, and $f, g \in \mathcal{C}^1(a \to b)$, a set $\mathcal{C}^2(f \to g)$ of "2-morphisms"



together with rules for stacking these together like Lego bricks.

Examples

① "vector spaces with $\otimes$-product"

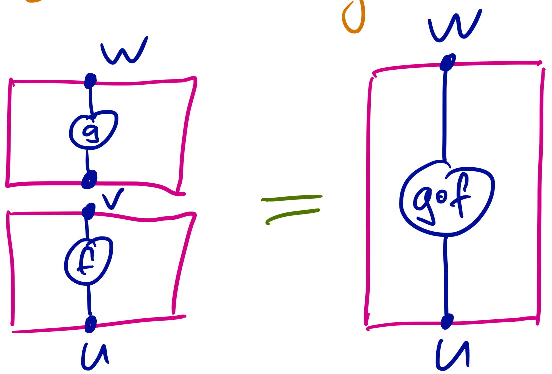
$$e^0 = \{ \bullet \}$$ "turtle"

$$e^1(\bullet \to \bullet) = \{\text{all vector spaces}\}$$

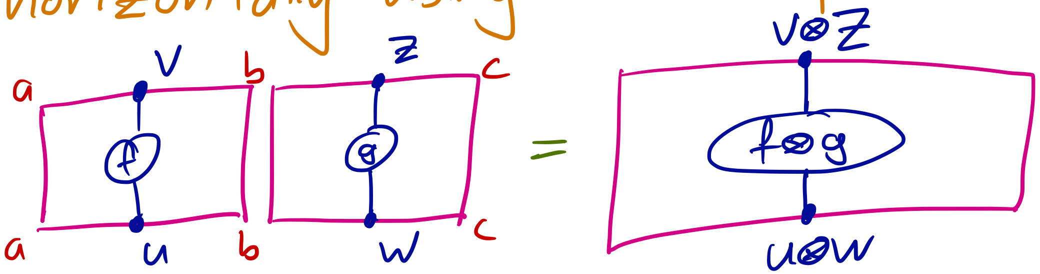
with composition 

$$e^2(V \to W) = \{\text{all linear maps from } V \text{ to } W\}.$$

We stack lego blocks vertically using composition:



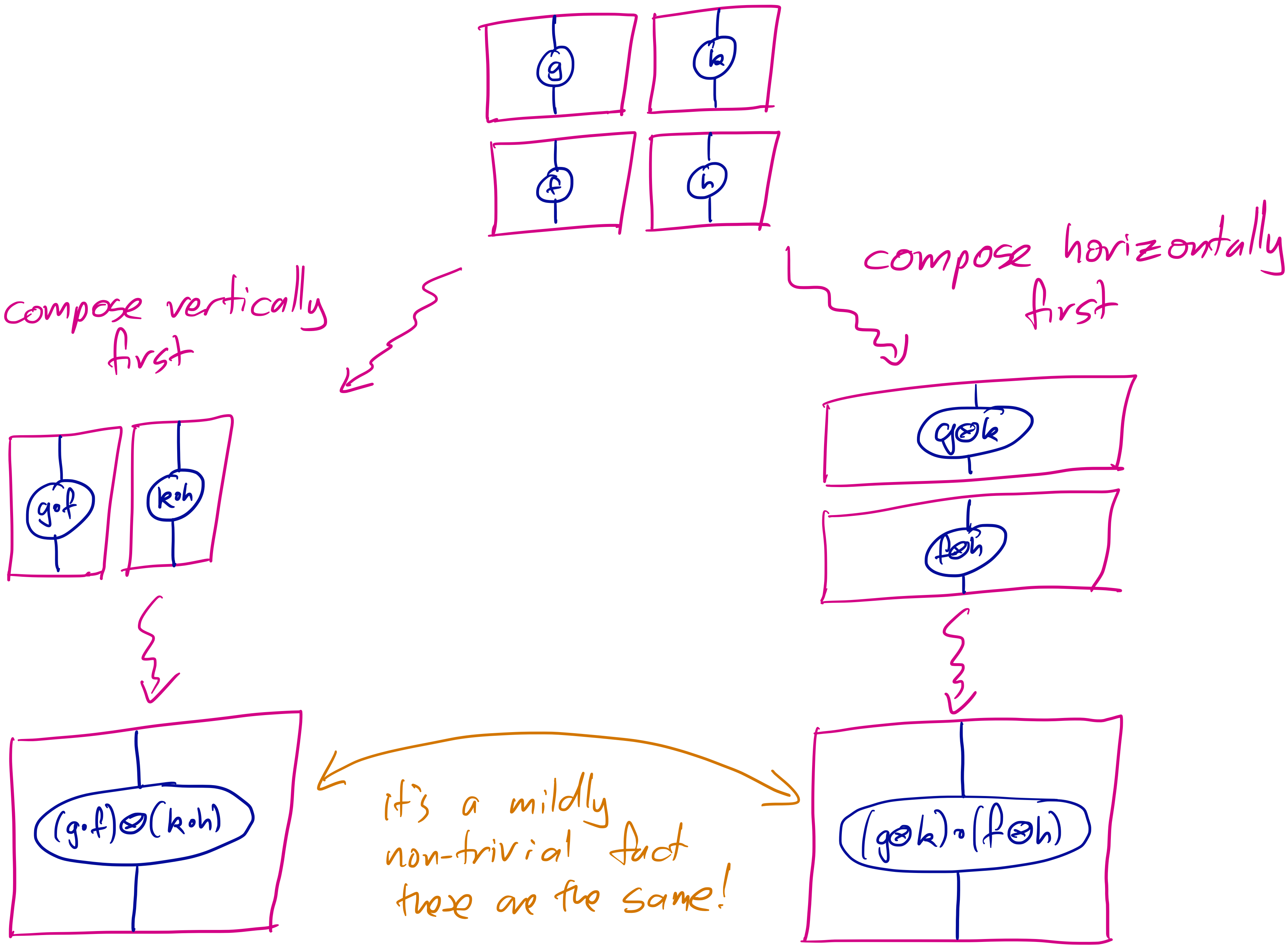
and horizontally using tensor product:



If $\{e_i\}$ is a basis for U
$\{f_j\}$ —"— W

$$(f \otimes g)(\underbrace{e_i \otimes e_j}_{\text{a basis vector for } U \otimes W}) := f(e_i) \otimes g(e_j).$$

Notice that composition and tensor products of linear maps satisfy the "laws" dictated by Lego blocks:



② "The toric code"

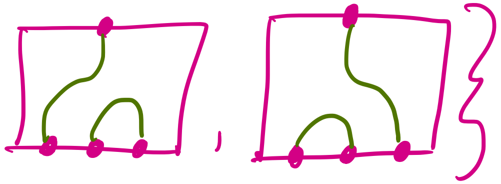
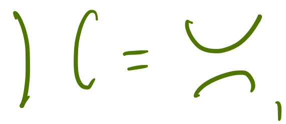
$$e^0 = \{\bullet\}$$

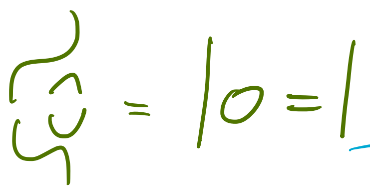
$$e^1 = \{ \text{—},\ \text{—}\bullet\text{—},\ \text{—}\bullet\bullet\text{—},\ \dots \}$$

$$e^2(\underbrace{\text{—}\bullet\bullet\bullet\text{—}}_{n \text{ points}} \to \underbrace{\text{—}\bullet\text{—}}_{m \text{ points}}) = \begin{cases} \mathbb{C} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd} \end{cases}$$

Both horizontal and vertical composition are just multiplication in $\mathbb{C}$.

We can represent these 2-morphisms by matchings of the points:

$e^2(\bullet\bullet\bullet \to \bullet) = \mathbb{C}\{$  $\}$ modulo the relations , $\bigcirc = \emptyset$.



We're interested in linear categories, where the top-level morphisms form vector spaces. (TQFT, rather than TFT.)

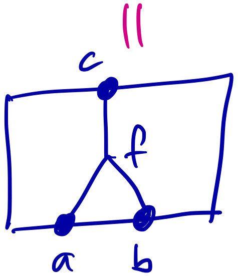
(in fact, Hilbert spaces, with a $*$-operation compatible with reflection)

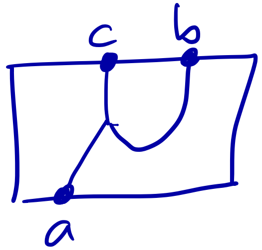
We'll be interested in higher categories which are "isotropic", that is, there's no real distinction between vertical and horizontal directions. (for $n=2$, "pivotal")

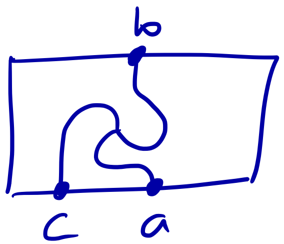
We formalise this by asking that we have isomorphisms

$$e^2(ab \to c) \cong e^2(a \to cb) \cong e^2(ca \to b)$$

$$\qquad \| \qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$

$\{$ 

$\}$ $\{$ 

$\}$ $\{$ 

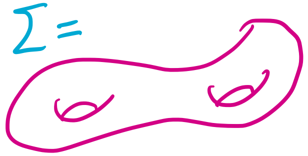
$\}$

which we think of as "rotating" morphisms.

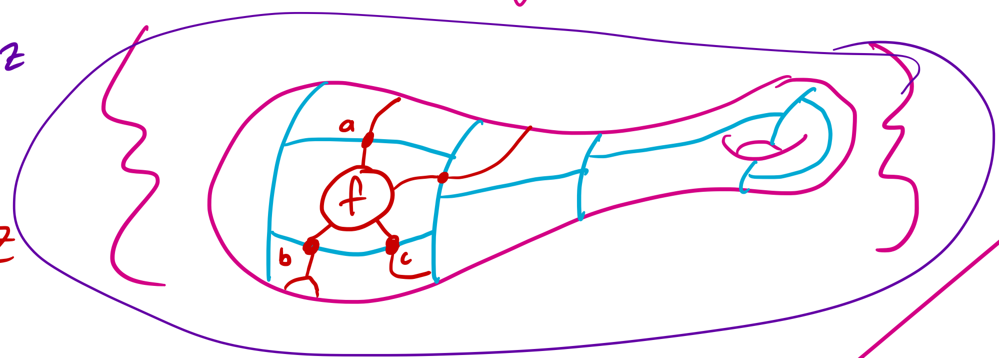
With these, it doesn't really matter what the <u>source</u> and <u>target</u> of a morphism are: just the labelling of the circular boundary.

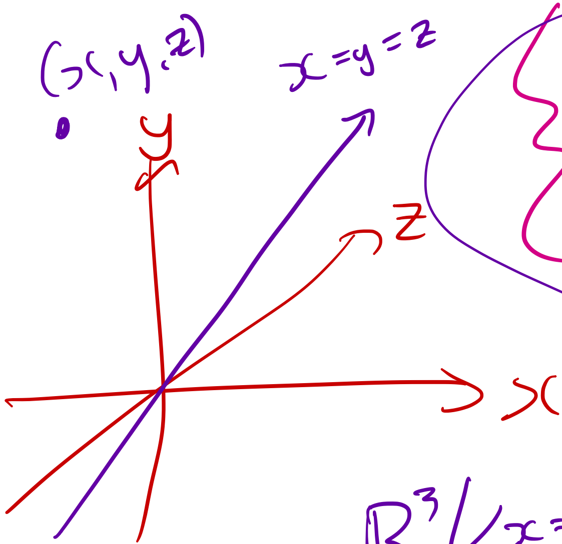
From higher categories to topological field theories.

From a 2-category, we can now associate a vector space to every 2-manifold:

$e\Big(\Sigma =$  $\Big) = \bigoplus \bigoplus \bigotimes e^2(B;c)$

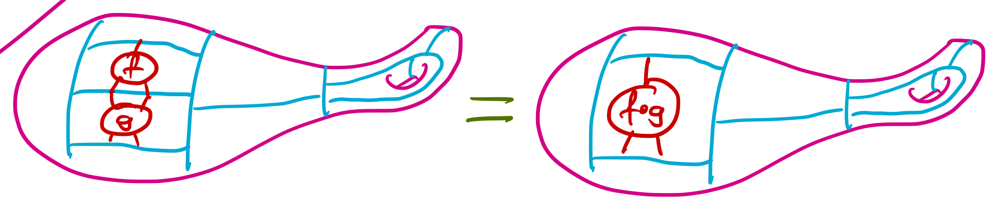
- $\bigoplus$: all ways to chop Σ into balls
- $\bigoplus$: all ways to label the boundaries of those balls with 1-morphisms
- $\bigotimes$: balls B with boundary c





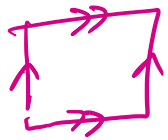
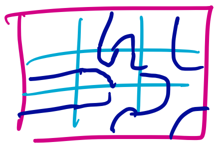
$\mathbb{R}^3/\langle x=y=z\rangle$

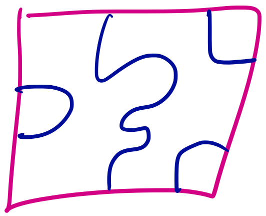
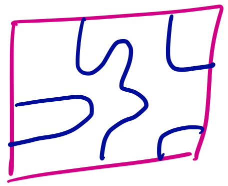
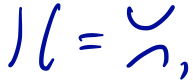
$(1,2,3) \sim (4,5,6)$

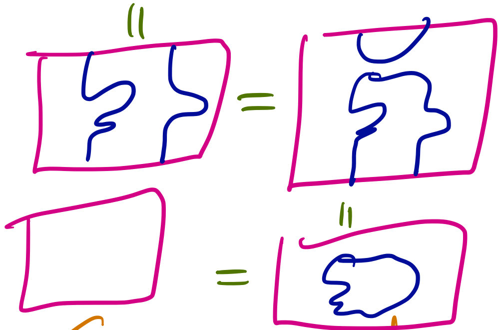


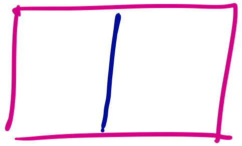
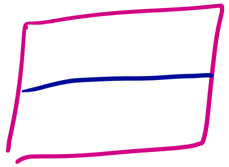
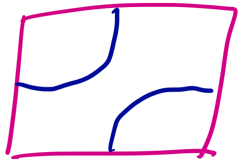
That was a bit scary!

Let's compute the vector space for the toric code on the torus:

$e($  $) = e($  $) = \mathbb{C}\{$  $\}\,/$ gluing together balls

 $= \mathbb{C}\{$  $\}\,/$ isotopy on the torus and , $\bigcirc = \emptyset$.



$= \mathbb{C}\{$ , , ,  $\}$

So we get a 4-dimensional vector space.

In general for the toric code

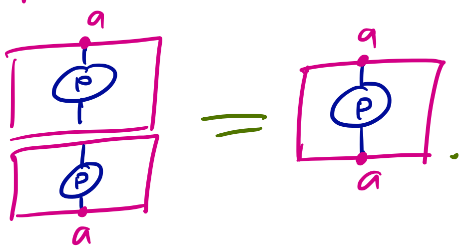
$$e(\Sigma) = \mathbb{C}\{H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})\}.$$

The big question: can we consistently define linear maps for each 3-dimensional cobordism?

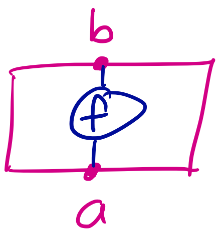
(that is — when do we get a TQFT?)

To answer this, we'll need to study <u>projections</u> in a 2-category.

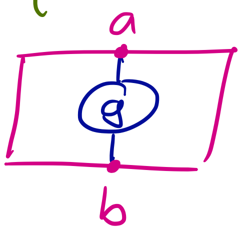
We say $p \in \mathcal{C}^2(a \to a)$ is a projection if



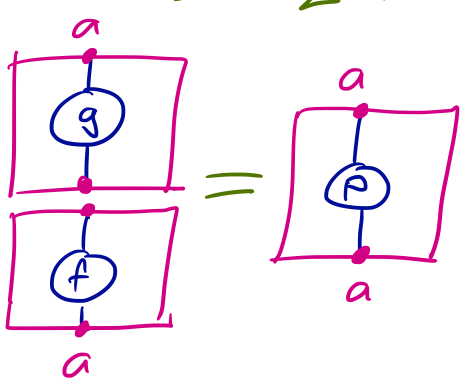
Two such projections $p \in \mathcal{C}^2(a \to a)$ and $q \in \mathcal{C}^2(b \to b)$ are said to be <u>equivalent</u> if there exist 2-morphisms



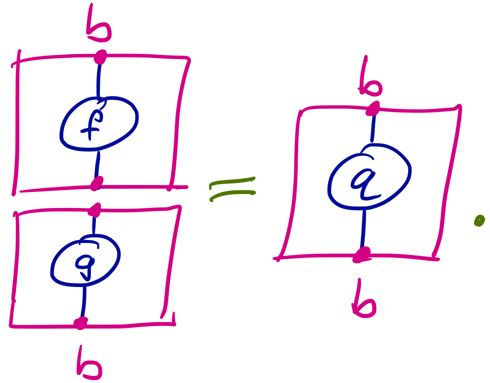
and



such that



and



A projection $p : \mathcal{C}^2(a \to a)$ is <u>simple</u> if

$$\mathrm{End}(p) :\equiv \left\{ \boxed{\overset{a}{\underset{a}{f}}} \;\middle|\; \boxed{\begin{matrix} f \\ p \end{matrix}} = \boxed{f} = \boxed{\begin{matrix} p \\ f \end{matrix}} \right\}$$

is a 1-dimensional vector space.

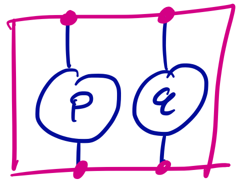
We say that a (unitary) 2-category is fusion if it has finitely many equivalences classes of simple projections.

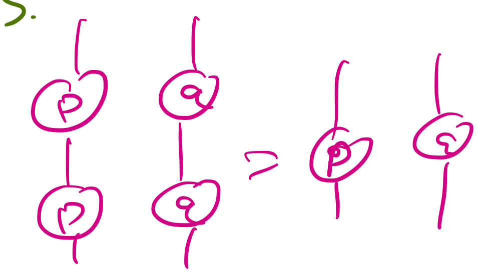
<u>Theorem</u> (The cobordism hypothesis in 2+1 dimensions)

$$\left\{ \begin{matrix} \text{2+1-d fully extended} \\ \text{TQFTs} \end{matrix} \right\} \longleftrightarrow \left\{ \text{fusion 2-categories} \right\}$$

What do fusion 2-categories look like?

We can (partially) describe 2-categories by fusion rules.

If p and q are projections, so is  .



However there's no need for $p \otimes q$ to be simple, even if p and q are. Generally, $\mathrm{End}(p \otimes q)$ is some multi-matrix algebra $M_{n_1 \times n_1}(\mathbb{C}) \oplus M_{n_2 \times n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k \times n_k}(\mathbb{C})$,

and this means for some collection of simple projections r_i

$$p \otimes q \cong \underbrace{r_1 \oplus \cdots \oplus r_1}_{n_1 \text{ times}} \oplus \underbrace{r_2 \oplus \cdots \oplus r_2}_{n_2 \text{ times}} \oplus \cdots\cdots \oplus r_k \oplus \cdots \oplus r_k$$

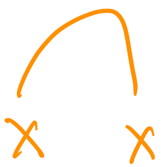
We think of simple projections as 'particle types', and this says "a p and a q particle can fuse together and form an r_1 particle (n_1 different ways), or an r_2 particle, and so on"

__Example__ Is there a fusion 2-category with two particles, $\mathbb{1}$ and X, satisfying

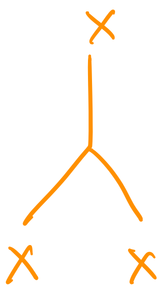
$$\mathbb{1}\otimes\mathbb{1}\cong\mathbb{1},\quad \mathbb{1}\otimes X\cong X\otimes\mathbb{1}\cong X, \text{ and } X\otimes X\cong\mathbb{1}\oplus X\,?$$

(Let's assume $\mathbb{1}$ and X are actually objects in the category, not just projections.)

The two "fusion channels" for $X\otimes X$ imply we have morphisms



and



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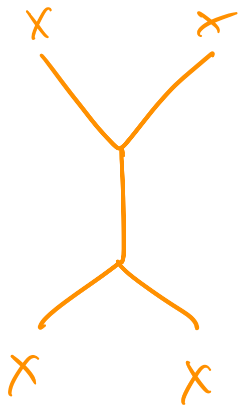
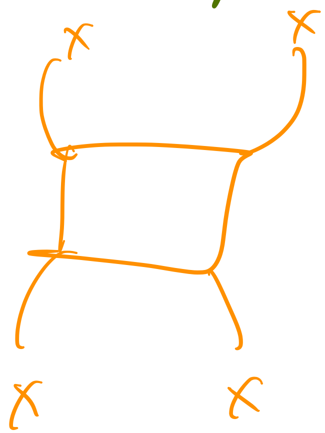
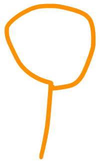


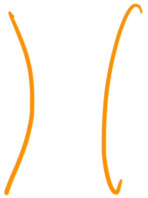
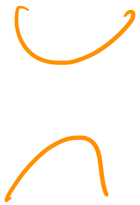
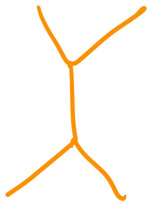


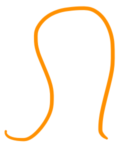
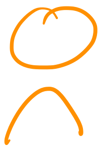
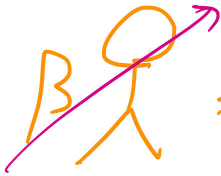
Since 1 and X are inequivalent simple objects, they cannot have any maps between them:

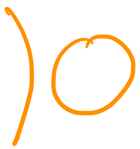
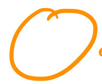
 $=$ zero.

Similarly, since X is simple  $= z \,|$ for some scalar z, which we can normalise to be $z = 1$.

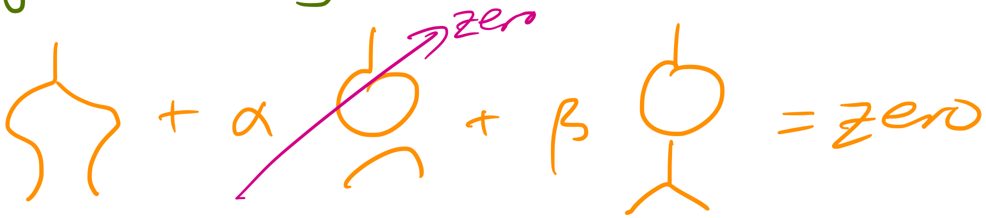
Since $\mathrm{End}(X \otimes X)$ is two-dimensional, we must have a linear relation

 $+ \alpha$  $+ \beta$  $=$ zero

Then  $+ \alpha$  $+ \beta$  $=$ zero, so α  $= -1$.

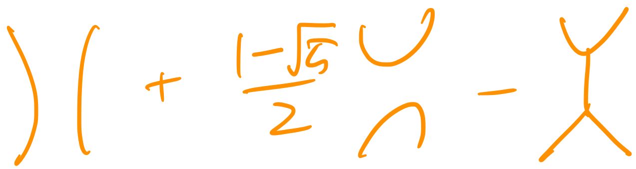
Then  $+ \alpha$  $+ \beta$  $=$ zero, so  $+ \alpha + \beta =$ zero.

Finally, stacking a ⅄ on top we get

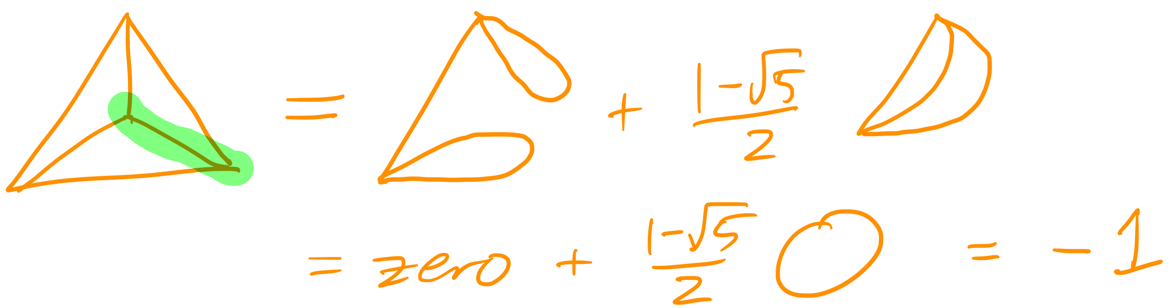


so $\beta = -1$. We now have $-\alpha^{-1} + \alpha - 1 =$ zero, which we can solve to find $\alpha = \frac{1 \pm \sqrt{5}}{2}$, the golden ratio!

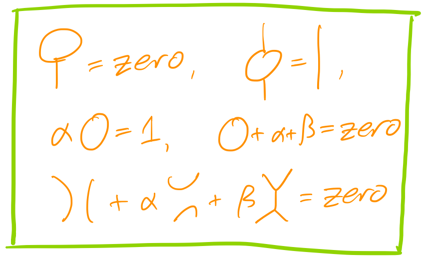
This relation



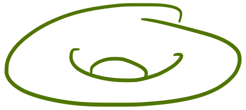
tells us everything about the category, e.g. we can evaluate all closed diagrams:

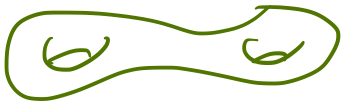


This category is called the golden category or the Fibonacci category, and it is perhaps relevant to the $\nu = \frac{5}{2}$ FQHE.



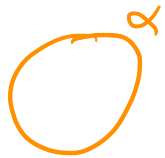
Again, the TQFT vector spaces can be computed as for the toric code:

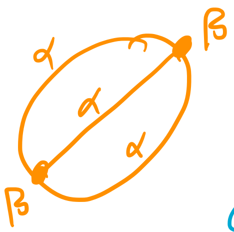
$\mathrm{Fib}\Big($  $\Big) \cong \mathbb{C}^4$ (same as the toric code)

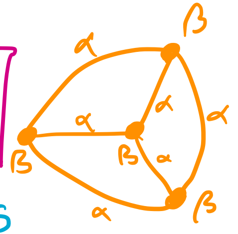
$\mathrm{Fib}\Big($  $\Big) \cong \mathbb{C}^{25}$ (only $\mathbb{C}^{16}$ for the toric code)

The linear maps for cobordisms can be computed via a state sum: (Ocneanu, and Turaev-Viro)

$$Z(M^3) = \underbrace{\sum}_{\substack{\text{label 2-cells}\\ \text{by simple}\\ \text{objects}\\ \alpha}} \underbrace{\sum}_{\substack{\text{label 1-cells}\\ \text{by 2-morphisms}\\ \beta}} \underbrace{\prod}_{\text{3-cells}} \mathcal{D}^{-1} \underbrace{\prod}_{\text{2-cells}} \bigcirc_\alpha \underbrace{\prod}_{\text{1-cells}} \theta \underbrace{\prod}_{\text{0-cells}} \cdots$$



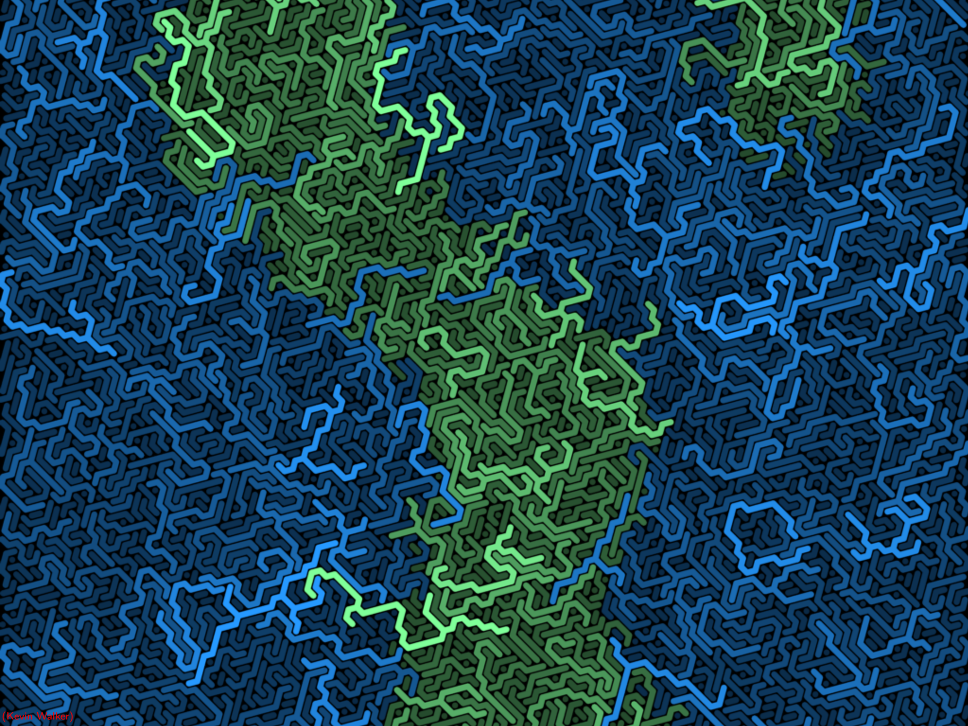




$Z(M^3)$: closed, for simplicity

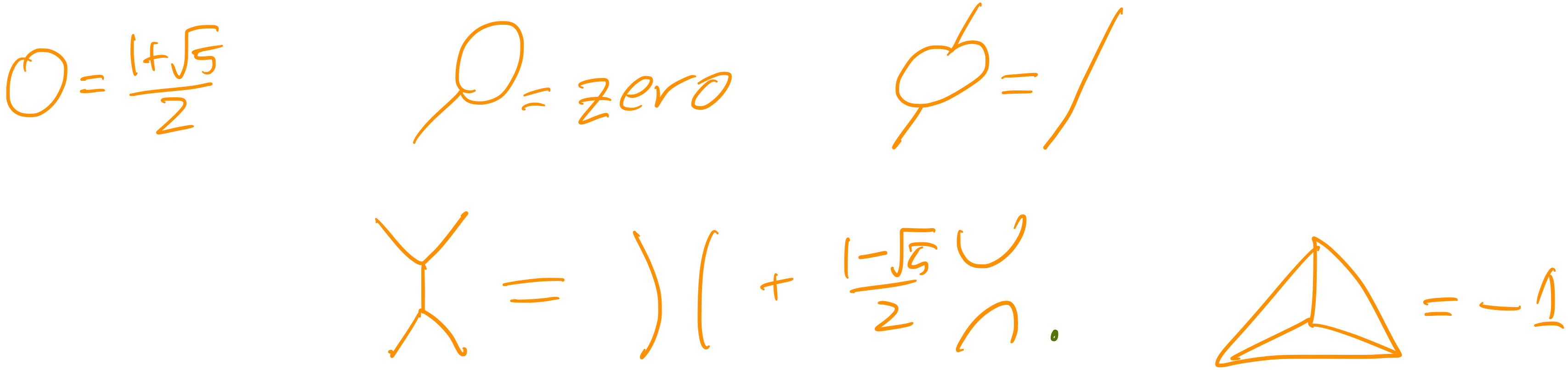
$\mathcal{D}^{-1}$: global dimension

(tomorrow we'll talk about how this relates to Levin-Wen Hamiltonians)

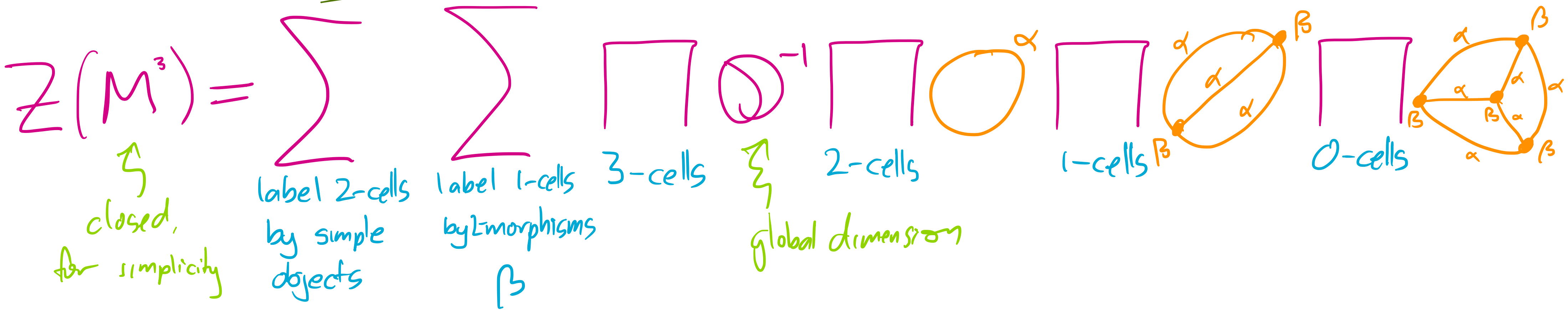


(Kevin Walker)

Recall last time we proved that a fusion 2-category with fusion rules $X^2 = 1 + X$ satisfied the following local identities:



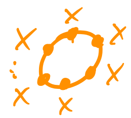
These allowed us to evaluate all closed diagrams and hence compute the linear maps associated to cobordisms in the associated fully extended 2+1-d TQFT, via a state sum formula:



In fact, the only fusion 2-categories with two particle types are the two we've discussed:

the toric code and the golden category.

The classification of fusion categories is hard but we have some results:

- complete classification for ≤ 3 particle types, almost complete for 4.
- complete classification if there is a "self-dual" particle with dimension ≤ 2.29 (dimension is the asymptotic growth rate of )
- partial classifications e.g. if there is a particle

$$X \otimes X \cong 1 \oplus X \oplus A \oplus B$$

So far, all known examples come from

- quantum groups at roots of unity

  (e.g. $\mathrm{Fib} \cong SO(3)_1$)

- finite groups

  (e.g. the toric code is $\mathrm{Rep}\, \mathbb{Z}/2\mathbb{Z}$)

- various constructions starting from these

- Izumi's "quadratic categories"

- a few "sporadic" examples

  are these really exceptional? or just the first signs of a big mess?

What about higher dimensions?

We don't know how to continue the sequence

0+1d	1+1d	2+1d
{finite dimensional vector spaces}	{semisimple categories}	{fusion 2-categories}

classifying fully extended n+1-dimensional TQFTs.

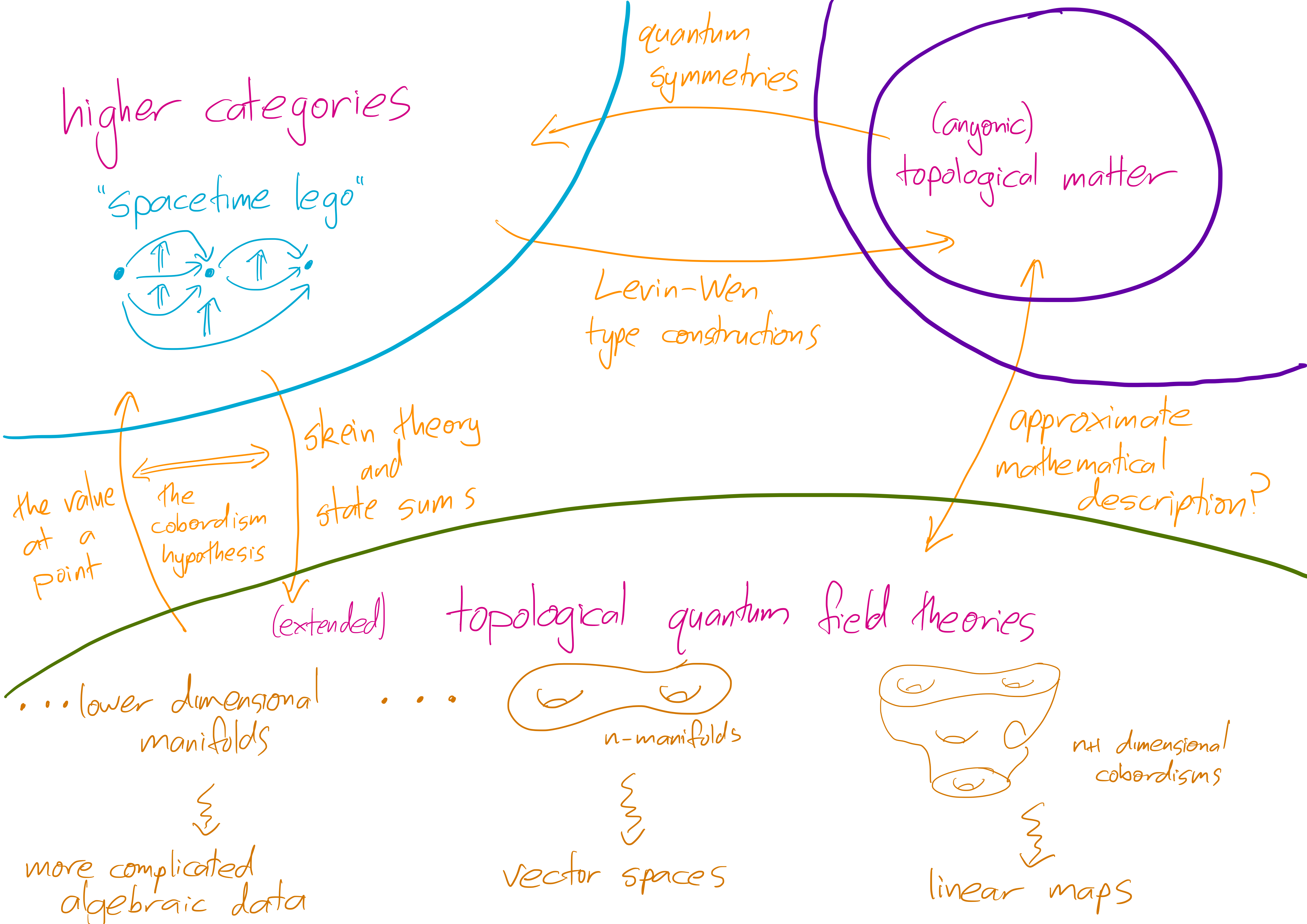
The cobordism hypothesis gives an abstract characterisation

(as "fully dualizable objects in nCAT")

but even in the physically interesting 3+1-d case we would like

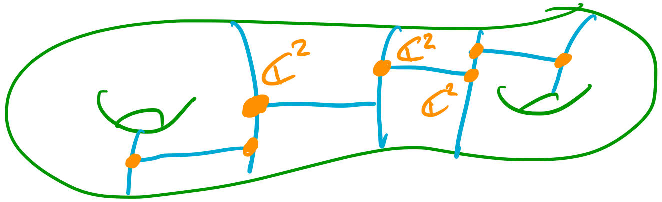
both interesting examples and

an explicit description of fully dualizability.


higher categories
"spacetime lego"
quantum symmetries
(anyonic) topological matter
Levin-Wen type constructions
skein theory and state sums
the value at a point
the cobordism hypothesis
approximate mathematical description?
(extended) topological quantum field theories
...lower dimensional manifolds
...
n-manifolds
n+1 dimensional cobordisms
more complicated algebraic data
vector spaces
linear maps

Today: Given a fusion 2-category $\mathcal{C}$ corresponding to some fully extended 2+1-d TQFT,

we will construct a local lattice Hamiltonian (~ Levin - Wen)



$$\mathcal{H} = \bigotimes_{\text{vertices}} \mathbb{C}^n$$

$$H = \sum_{\text{edges}} H_e + \sum_{\text{faces}} H_f$$

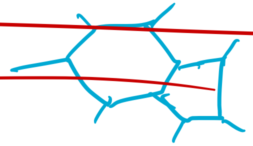
with the property that the ground state is degenerate, and canonically identified with the TQFT vector space for the underlying surface.

Even better, we will be able to identify the excited states in terms of the Drinfeld centre of $\mathcal{C}$.

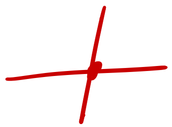
Suppose $\mathcal{C}$ has n (e.g. $n=2$) simple particle types.

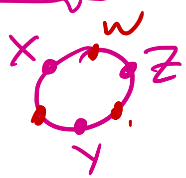
It will be convenient to put local Hilbert spaces on both vertices and edges.

On each edge, put a copy of $\mathbb{C}^n$, with basis labelled by particle types.

~~For simplicity (but this is not important) let's assume our graph on the surface is trivalent~~  .

On each vertex, put a copy of $V = \bigoplus_{\substack{w,X,Y,Z \\ \text{particle types}}} \underbrace{\mathcal{C}^2(1 \xrightarrow{w} XYZ)}$.





(In fact, it's good enough to put $\mathbb{C}^k$, where

$$k = \max_{X,Y,Z} \dim \mathcal{C}^2(1 \to XYZ).)$$

Example For the toric code this means putting a copy of $\mathbb{C}^2$ on each edge.

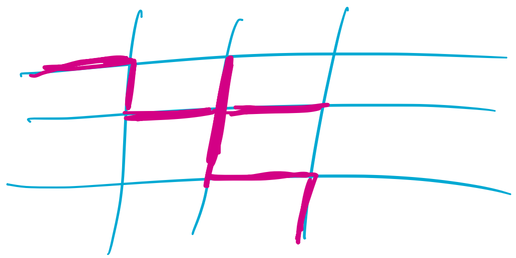
(The inequivalent projections in the toric code category are

$$P = \phi \sim || \sim |||| \sim \cdots \quad \text{and} \quad Q = | \sim ||| \sim |||||)$$

Since $e^2(1 \to XYZ) = \begin{cases} \mathbb{C} & \text{if an even number of } X, Y, Z \text{ are } Q \\ 0 & \text{otherwise} \end{cases}$

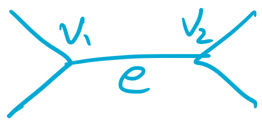
we can get away with just a copy of $\mathbb{C}$ at each vertex (which is the same as putting nothing there).

In fact, this is still the case if we allow higher valence vertices.



Next we'll describe the Hamiltonian in two steps:

$$H = \sum_{\text{edges}} H_e + \sum_{\text{faces}} H_f .$$

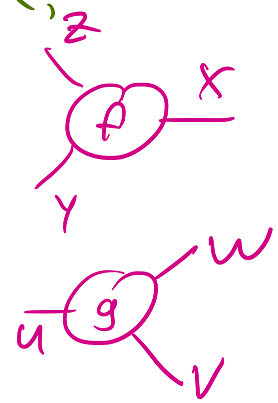
For a given edge , H_e acts on

$$\mathcal{H}_e \otimes \mathcal{H}_{v_1} \otimes \mathcal{H}_{v_1} = \mathbb{C}^n \otimes \left(\bigoplus_{X,Y,Z} e^2(1 \to XYZ) \right) \otimes \left(\bigoplus_{U,V,W} e^2(1 \to UVW) \right)$$

A typical element here is $P \otimes f \otimes g$, where P is a particle type,

$f \in e^2(1 \to XYZ)$ for some particle types X, Y, Z

$g \in e^2(1 \to UVW)$ —— " —— U, V, W.

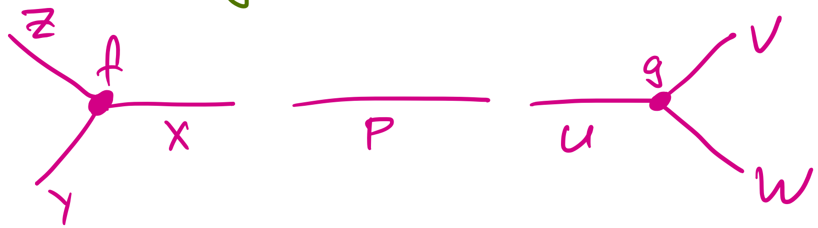


H_e is a projection (i.e. it has eigenvalues 0 and 1)

$$H_e(P \otimes f \otimes g) = \begin{cases} 0 \cdot P \otimes f \otimes g & \text{if } P = X = U \\ 1 \cdot P \otimes f \otimes g & \text{otherwise.} \end{cases}$$

To understand this, we should imagine the local state

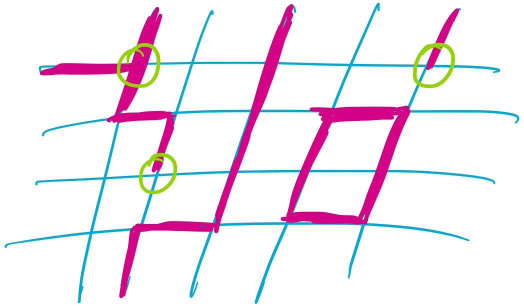
$P \otimes f \otimes g$ as



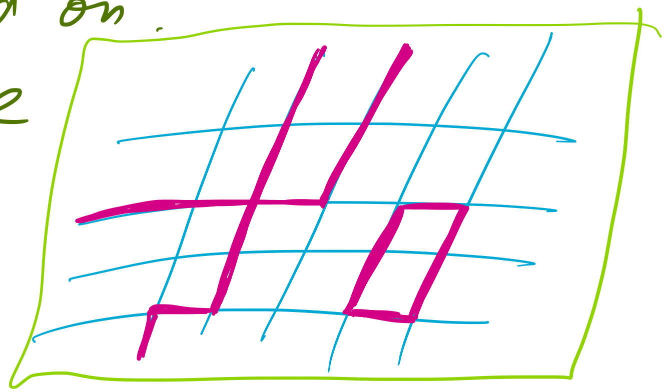
This Hamiltonian says the low-energy states are the ones where $X = P = U$, i.e. everything matches up.

We still need to define the face terms, but let's understand the ground state "so far"

Example For the toric code, with $\mathbb{C}^2$ on each edge, we can represent a basis vector in $\mathcal{H} = \bigotimes_{\text{edges}} \mathbb{C}^2$ as a subset of the edges "turned on".

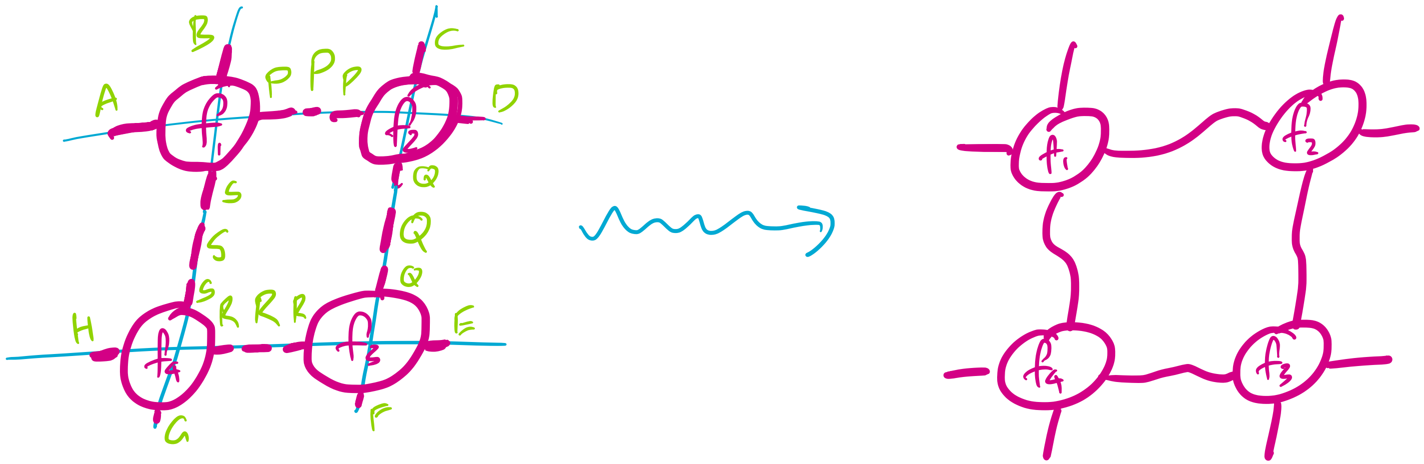


The ground states are those at which every vertex has an even number of edges turned on.

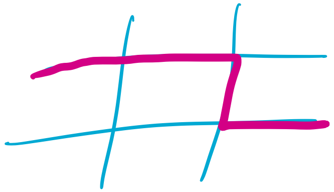
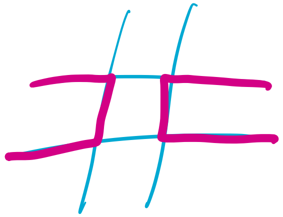
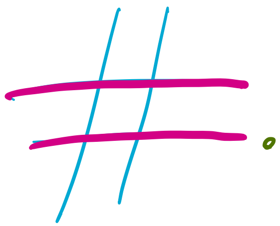


Notice that given any groundstate for the edge Hamiltonian, we can interpret it as an element of the TQFT vector space for the surface we are looking at!

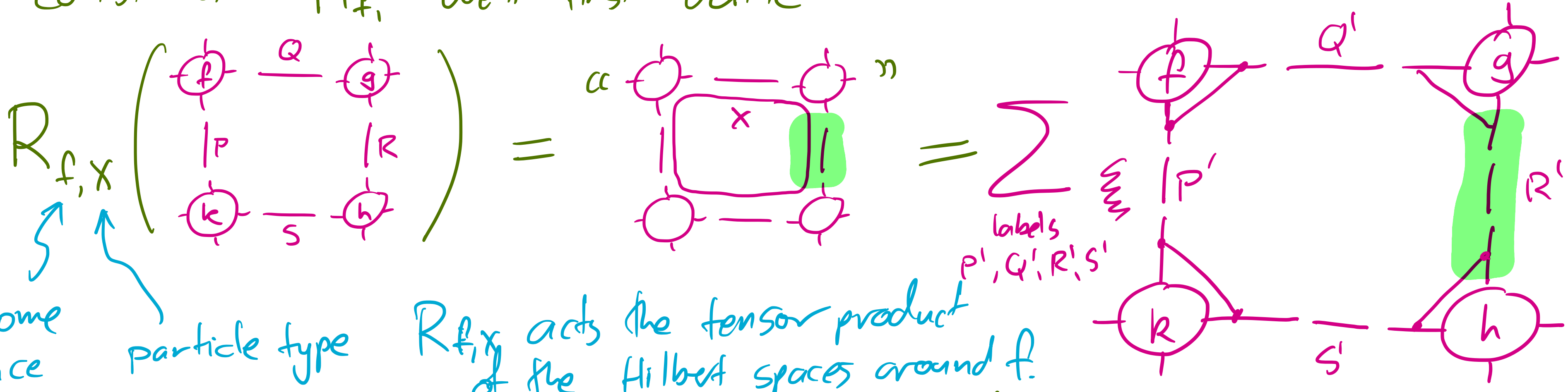
Each state looks like (a linear combo of)



This is certainly not a 1-1 map.

For example, in the toric code  and  represent the same vector, as do  and .

To construct H_f, we'll first define



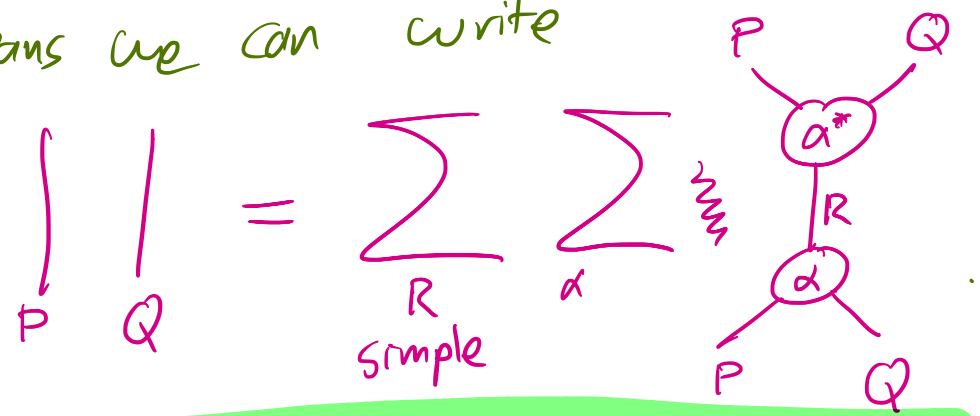
some face

particle type

$R_{f,x}$ acts the tensor product of the Hilbert spaces around f.

What's going on here? Recall the product of two simple objects is generally not simple.

This means we can write



We do this in the green box above, and at each edge of the face.

We then define $H_f = I - D^{-1} \sum_{\substack{x \\ \text{simple}}} d_x R_{f,x}$.

Lemma: all the terms H_e, H_f commute with each other (so the groundstate of H is the intersection of all the ground states).

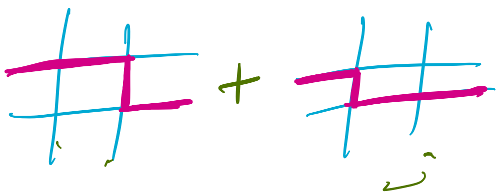
Theorem: on the groundstate of H, the "interpretation" map gives an isomorphism to the TQFT Hilbert space.

$$R_{f,P} = I$$

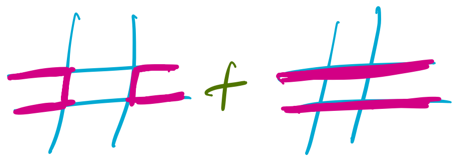
Example: In the toric code $R_{f,Q}$ flips all the edges adjacent to f.

$$H_f = \tfrac{1}{2} I - \tfrac{1}{2}\,\mathrm{Flip}.$$

Thus

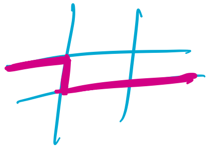


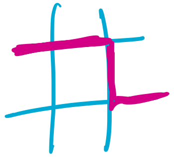
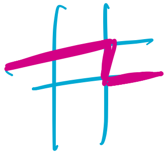
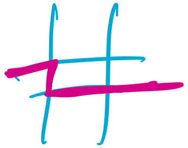
and

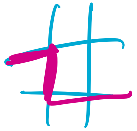
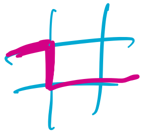
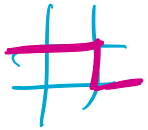


are in the groundstate, but the individual terms are not.

$$H_\ell = \frac{1}{2} I - \frac{1}{2} \text{Flip}.$$

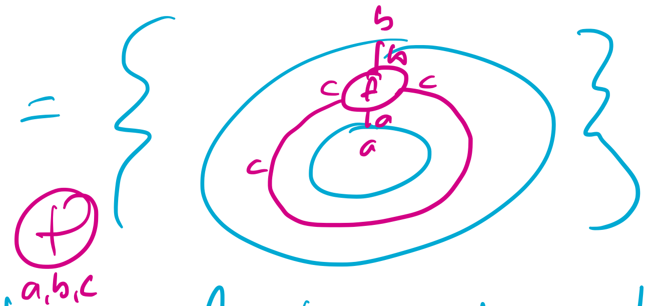
 + 

H_ℓ  $= \frac{1}{2}$  $- \frac{1}{2}$ 

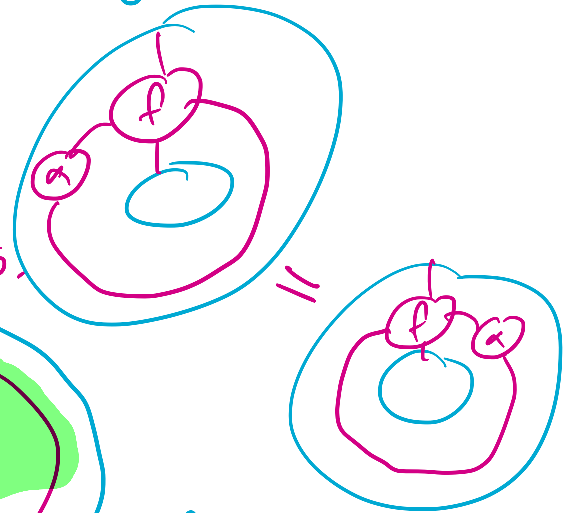
H_ℓ  $= \frac{1}{2}$  $- \frac{1}{2}$ 

We can also describe the local excitations in terms of category theoretic invariants of $\mathcal{C}$.

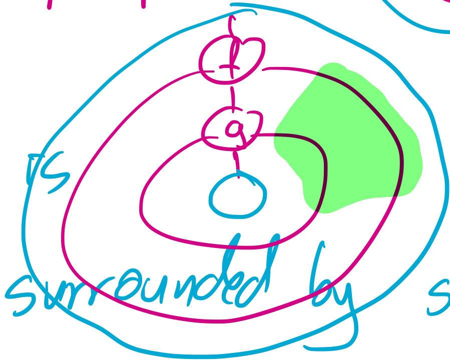
The tube algebra of $\mathcal{C}$ is the finite dimensional algebra of diagrams drawn on an annulus:

$\mathrm{Tube}(\mathcal{C}) = \{$  $\}$.

a,b,c should be simple particles.



A representation of the tube algebra is "a gadget that can sit a point, surrounded by stuff from $\mathcal{C}$".



We can make a 3-category

$$Z(\mathcal{C}) = \begin{array}{c} \{\bullet\} \\ \circlearrowleft \\ \{\bullet\} \\ \circlearrowleft \\ \{\mathrm{RepTube}(\mathcal{C})\} \\ \circlearrowleft \\ \{\text{intertwiners}\} \end{array}$$

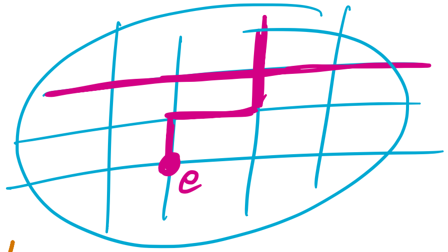
called the Drinfeld centre of $\mathcal{C}$.

Example: there are 4 simple particle types in Z(toric code) and we can use these to give examples of energy 1 states in the toric code Hamiltonian.

The particle types form a $\mathbb{Z}/2 \times \mathbb{Z}/2$ under product; let's name them 1, e, m, em.

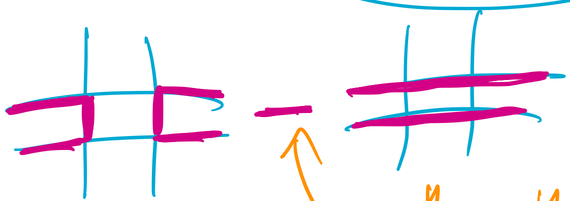
1 does not correspond to an excitation: it's the ground state.

e corresponds to states



where the parity condition is violated.

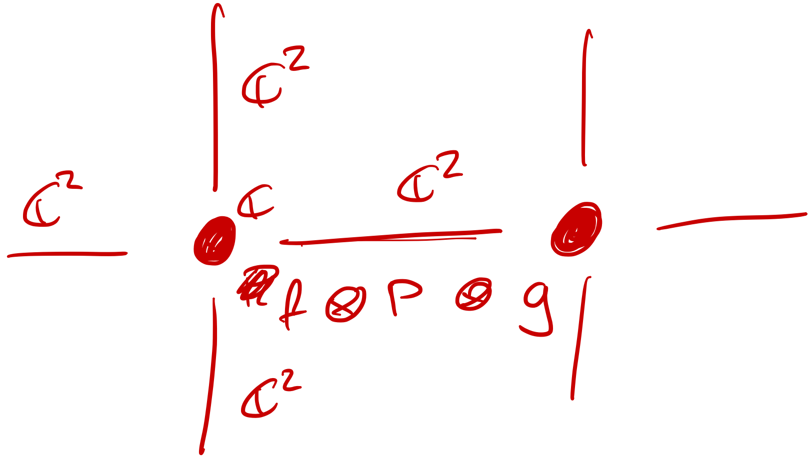
m to states



rather than +

Future directions

- describe domain walls between topological phases via bimodule categories
- describe interactions between domain walls via generalisations of the Drinfeld centre
- explain the connection between the Levin-Wen Hamiltonian and tensor network ground states via higher dimensional algebra.
- reconstruct a higher category from a topological phase without going via the TQFT and the cobordism hypothesis.



$$H_e(f\otimes P\otimes g) = \begin{cases} 0 \; f\otimes P\otimes g \\ 1 \cdot f\otimes P\otimes g \end{cases}$$

if things match, and f sits in the subspace it's meant to.