Exotic quantum symmetries

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2018-10-05
What are quantum symmetries?

- A generalisation of finite groups
- Axiomatised as certain types of tensor categories
- The algebraic data classifying a topological field theory
- Mathematical models of topological phases of matter
What are quantum symmetries?
When $G$ is a finite group,

$\text{Rep}G$ is a category, with

* objects: finite dim $G$-representations $(V, p: G \to \text{End}(V))$
* morphisms: $G$-linear maps
* $\otimes$-products: $\rho_{V \otimes W}(g)(v \otimes w) := \rho_V(g)(v) \otimes \rho_W(g)(w)$
* duals: $\rho_{V^*}(g)(\phi)(v) := \phi(\rho_V(g^{-1})(v))$

a finite set of representations $V_i$ so $\text{Hom}(V_i \to V_j) = \begin{cases} C & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

and every representation can be decomposed as a direct sum of these.
What are quantum symmetries?
A fusion category is a finitely semisimple, rigid monoidal category.

Some features of $\text{Rep}G$ are missing here:

- We don't require $V \otimes W \cong W \otimes V$.
- We don't require $V^{**} \cong V$.
- We don't require a forgetful functor to $\text{Vec}$.

Alternative notions:
- modular tensor categories
- planar algebras
- $\Lambda$-lattices
- vertex operator algebras
Theorem (Deligne)
A symmetric fusion category is $\text{Rep}G$
for some finite (super) group.

Symmetric here means we have $V \otimes W \cong W \otimes V$.
and these isos give an action of the symmetric group.

"Fusion categories are noncommutative finite groups"
What are quantum symmetries?

- the algebraic data classifying a topological field theory
An \((n+1)\)-dimensional topological field theory is a functor:

\[
\begin{array}{ccc}
\{n\text{-manifolds}\} & \overset{}{\xrightarrow{\text{}} } & \{\text{vector spaces}\} \\
\{\text{(n+1)-d cobordisms}\} & \overset{}{\xrightarrow{\text{}} } & \{\text{linear maps}\}
\end{array}
\]

A \underline{local} TFT can be computed "by gluing", and associates higher algebraic data to lower dimensional manifolds.
Lurie’s proof of the Cobordism hypothesis:

A local \((n+1)\)d TFT is determined by its value on a point, and this value may be any fully dualizable \(n\)-category.

Douglas–Schommer-Pries–Snyder:

The fully dualizable 2-categories are the (multi-)fusion categories and hence:

- The algebraic data classifying a topological field theory
- Local 2+1-d TFTs are classified by fusion categories
What are quantum symmetries?

Mathematical models of topological phases of matter
What’s out there?

finite groups:

- $\text{Rep} \, G$
- $\text{Vec} \, G$
- $\text{Vec}^w \, G, \, w \in H^3(G; \mathbb{T})$
- “$G$-extensions”:
  
  $C = \bigoplus_{g} C_g$, $C_g$ a fusion category
  
  $C_g$ a $C_e$-$C_e$-bimodule category

$G$-extension of $C$ are classified \cite{ENO}

by homotopical yoga: $BG \to BBr\text{-}\text{Pic}(C)$
What's out there?

finite groups

quantum groups at roots of unity

For $g$ a complex semisimple Lie algebra,

$U_q g$ is the quantised universal enveloping algebra,

and $\text{Rep} U_q g$ is a braided tensor category.

When $q$ is a root of unity, $\text{Rep} U_q g$ has a natural semisimple quotient, which is a fusion category.
What's out there?

- finite groups
- quantum groups at roots of unity
- Izumi's quadratic categories
  - $G$-categories with a group $G$ of invertible objects, and one other 'orbit'
  - for each group and transitive action, there is a discrete variety parametrising examples
  - overall classification still hard; it look like there's at least one infinite family
What's out there?

- finite groups
- quantum groups at roots of unity
- Izumi's quadratic categories
- the 'extended Haagerup' fusion category

(more on this in a moment!)
What's out there?

- finite groups
- quantum groups at roots of unity
- Izumi's quadratic categories
- the 'extended Haagerup' fusion category

...and not much else!?
Extended Haagerup is a strange object!
- First discovered as a pair of Morita-equivalent fusion categories
- $E_4$ has 6 simple objects

It is generated as a $\Theta$-category by two morphisms

\[ q \in \text{Hom}(e^{\otimes 2}, 1) \]
- satisfying “jellyfish” relations

① \[ \begin{array}{c}
\text{diagrams built out of} \\
\lambda
\end{array} \]

② \[ a \left( \begin{array}{c}
S \quad S \\
\end{array} \right) = \left( \begin{array}{c}
\end{array} \right) \]

③ \[ b \left( \begin{array}{c}
S \quad S \\
\end{array} \right) = \left( \begin{array}{c}
\end{array} \right) \]

To prove \( E\mathcal{H}_1 \) exists we:

- show these relations suffice to evaluate all closed diagrams
- find a “faithful representation” of the quotient.

\( \text{arXiv:0909.4099} \)
We'd really like to understand this example!

- Is it truly exotic?
  - 'Obstructions'?

- Can we find easier constructions?
  - Then find more like it? (at present we don't even have candidates)

Recent work on EH focusses on its

- Drinfeld centre (if \( C \) is a spherical fusion category, \( Z(C) \) is a modular tensor category)

- Brauer-Picard groupoid (the collection of all Morita equivalent fusion categories, and the equivalences)
Drinfeld centre

\[ Z(c) = \mathcal{E}(x, \beta) \mid x \in \mathcal{E}, \quad \beta : x \otimes y \rightarrow y \otimes x^2 \]

• Most of the TFT invariants are determined by \( Z(c) \).

• With Terry Gannon and Kevin Walker, we've pinned down the modular data for \( Z(E) \).  
  
  --- lots of combinatorics, Galois theory, and \( SL(2, \mathbb{Z}/n\mathbb{Z}) \) rep theory!
  
  --- constrains possible CFT realisations of \( Z(E) \)

  --- tantalising hints of a connection to \((F_4)_4\)!
**Brauer-Picard groupoid**  \((C \cong \mathcal{O} \text{ if } Z(C) \cong Z(\mathcal{O}))\)

- At most four fusion rings for categories Morita equivalent to \(\mathcal{E} \mathcal{K}_2\) (combinatorics)

- At most one category for each ring, and no Morita auto-equivalences (uniqueness of the \(\mathcal{E} \mathcal{K}_1 - \mathcal{E} \mathcal{K}_2\) bimodule)

- Then construct everything remaining:

\[
\begin{array}{ccc}
\mathcal{E} \mathcal{K}_1 & \leftrightarrow & \mathcal{E} \mathcal{K}_2 \\
\uparrow & & \uparrow \\
\mathcal{E} \mathcal{K}_3 & \leftrightarrow & \mathcal{E} \mathcal{K}_4
\end{array}
\]

(a generalisation of the original construction, using a beautiful skein theory for the 2-category \(\mathcal{E} \mathcal{K}_{12}\))

Our method generalises the graph planar algebra embedding theorem (and unifies distinct perspectives of it from the subfactor and $\mathfrak{g}$-category literatures).

Given a graph $\Gamma$ (e.g. $\xymatrix@C=1pc{\bullet & \bullet \ar[l]_1} )$ and a ‘weight’ $d: V(\Gamma) \to \mathbb{C}$, there is an associated multifusion category $G_\mathbb{C}(\Gamma)$.

- The original construction of $\mathcal{E}^{\mathcal{X}_s} \mathcal{X}^{\mathcal{X}_s}$ built a faithful map into such a category.
- It’s long been known that every fusion category $\mathcal{C}$ admits such an embedding for $\Gamma$ one of the ‘principal graphs’ for $\mathcal{C}$. 
- We extend this to multilocus categories, and turn it into a classification:

\[ \exists \text{embeddings } E \rightarrow G(G) \xrightarrow{3} \exists \text{module categories for } C \text{ with principal graph } \Gamma^3 \]

(as well as pivotal/unitary versions)

- We build this from scratch, so it's accessible with soaking first in the subfactor literature.

- Finally, even this is too hard for \( C = \mathcal{E}4 \), or \( \mathcal{E}2 \), but the beautiful skein theory of the multilocus category \( \mathcal{E}12 \) allows us to construct \( \mathcal{E}4 \otimes_3 \) and \( \mathcal{E}4 \otimes_4 \) as modules over it!
• What next?
  - Sadly, $\text{E}Xl_3$ and $\text{E}Xl_4$ did not reveal connections with other fusion categories
    (previously, apparently exotic examples have been ‘explained’ by their BP groupoid)
    → what's going on with $(F_4)_4$?
      - grafting? twisted equivariantisation?
    → continue the search for more examples?