Exotic quantum symmetries

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(Kevin Walker)

What are quantum symmetries?

axiomatised as certain types of tensor categories

a generalisation of finite groups

the algebraic data classifying a topological field theory

mathematical models of topological phases of matter

a generalisation of finite groups

What are quantum symmetries?

a generalisation of finite groups

When G is a finite group,

$\mathrm{Rep}\,G$ is a category, with

* objects: finite dim'l representations $(V, \rho : G \to \mathrm{End}(V))$
* morphisms: G-linear maps
* $\otimes$-products: $\rho_{V\otimes W}(g)(v\otimes w) := \rho_V(g)(v) \otimes \rho_W(g)(w)$
* duals: $\rho_{V^*}(g)(\phi)(v) := \phi(\rho_V(g^{-1})(v))$
* a finite set of representations V_i so $\mathrm{Hom}(V_i \to V_j) = \begin{cases} \mathbb{C} & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

and every representation can be decomposed as a direct sum of these.

axiomatised as certain types of tensor categories

What are quantum symmetries?

axiomatised as certain types of tensor categories

Alternative notions:

- modular tensor categories
- planar algebras
- λ-lattices
- vertex operator algebras
- ....

A **fusion category** is a finitely semisimple, rigid monoidal category.

Some features of $\mathrm{Rep}\,G$ are missing here:

- We don't require $V \otimes W \cong W \otimes V$.
- We don't require $V^{**} \cong V$.
- We don't require a forgetful functor to Vec.

a generalisation of finite groups

Theorem (Deligne)

A symmetric fusion category is $\mathrm{Rep}\,G$ for some finite (super) group.

Symmetric here means we have $V \otimes W \cong W \otimes V$, and these isos give an action of the symmetric group.

"Fusion categories are noncommutative finite groups"

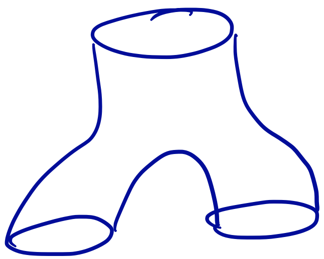
What are quantum symmetries?

the algebraic data
classifying
a topological field theory

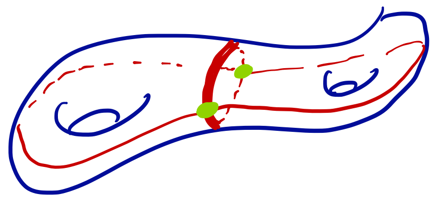
An (n+1)-dimensional topological field theory is a functor

$$\{\text{n-manifolds}\} \longrightarrow \{\text{vector spaces}\}$$

$$\{\text{(n+1)-d cobordisms}\} \longrightarrow \{\text{linear maps}\}$$



A <u>local</u> TFT can be computed "by gluing",



and associates higher algebraic data to lower dimensional manifolds.

the algebraic data
<u>classifying</u>
a topological field theory

Lurie's proof of the Cobordism hypothesis:

A local (n+1)d TFT is determined by its value on a point,

and this value may be any fully dualizable n-category

Douglas–Schommer-Pries–Snyder:

The fully dualizable 2-categories are the (multi-)fusion categories

and hence:

the algebraic data classifying a topological field theory

local 2+1-d TFTs are classified by fusion categories

What are quantum symmetries?

mathematical models of topological phases of matter

What's out there?

↳ finite groups:

- $\mathrm{Rep}\, G$
- $\mathrm{Vec}\, G$
- $\mathrm{Vec}^{\omega} G$, $\omega \in H^3(G; \mathbb{T})$
- "G-extensions":

  $\mathcal{C} = \bigoplus_g \mathcal{C}_g$, $\mathcal{C}_e$ a fusion category

  $\mathcal{C}_g$ a $\mathcal{C}_e$-$\mathcal{C}_e$-bimodule category

G-extension of $\mathcal{C}$ are classified [ENO]

by homotopical yoga: $BG \longrightarrow BBrPic(\mathcal{C})$

What's out there?

- finite groups
- quantum groups at roots of unity

For $\mathfrak{g}$ a cx semisimple Lie algebra,
$U_q\mathfrak{g}$ is the quantised universal enveloping algebra,
and $\mathrm{Rep}\,U_q\mathfrak{g}$ is a braided tensor category.
When q is a root of unity $\mathrm{Rep}\,U_q\mathfrak{g}$ has a natural semisimple quotient, which is a fusion category.

What's out there?

- finite groups
- quantum groups at roots of unity
- Izumi's quadratic categories
  - $\otimes$-categories with a group G of invertible objects, and one other 'orbit'
  - for each group and transitive action, there is a discrete variety parametrising examples
  - overall classification still hard; it look like there's at least one infinite family

What's out there?

- finite groups
- quantum groups at roots of unity
- Izumi's quadratic categories
- the 'extended Haagerup' fusion category

  (more on this in a moment!)

What's out there?

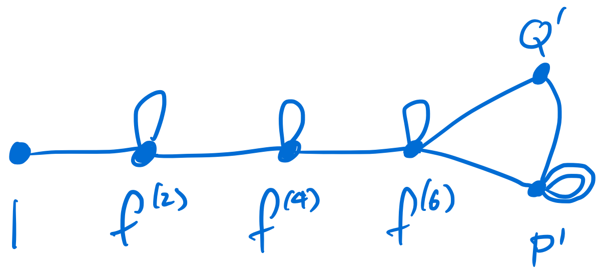
- finite groups
- quantum groups at roots of unity
- Izumi's quadratic categories
- the 'extended Haagerup' fusion category

—— and not much else!?

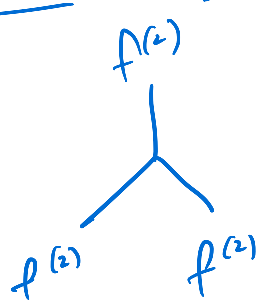
Extended Haagerup is a strange object!

- First discovered as a pair of Morita-equivalent fusion categories

- $\mathcal{EH}_1$ has 6 simple objects



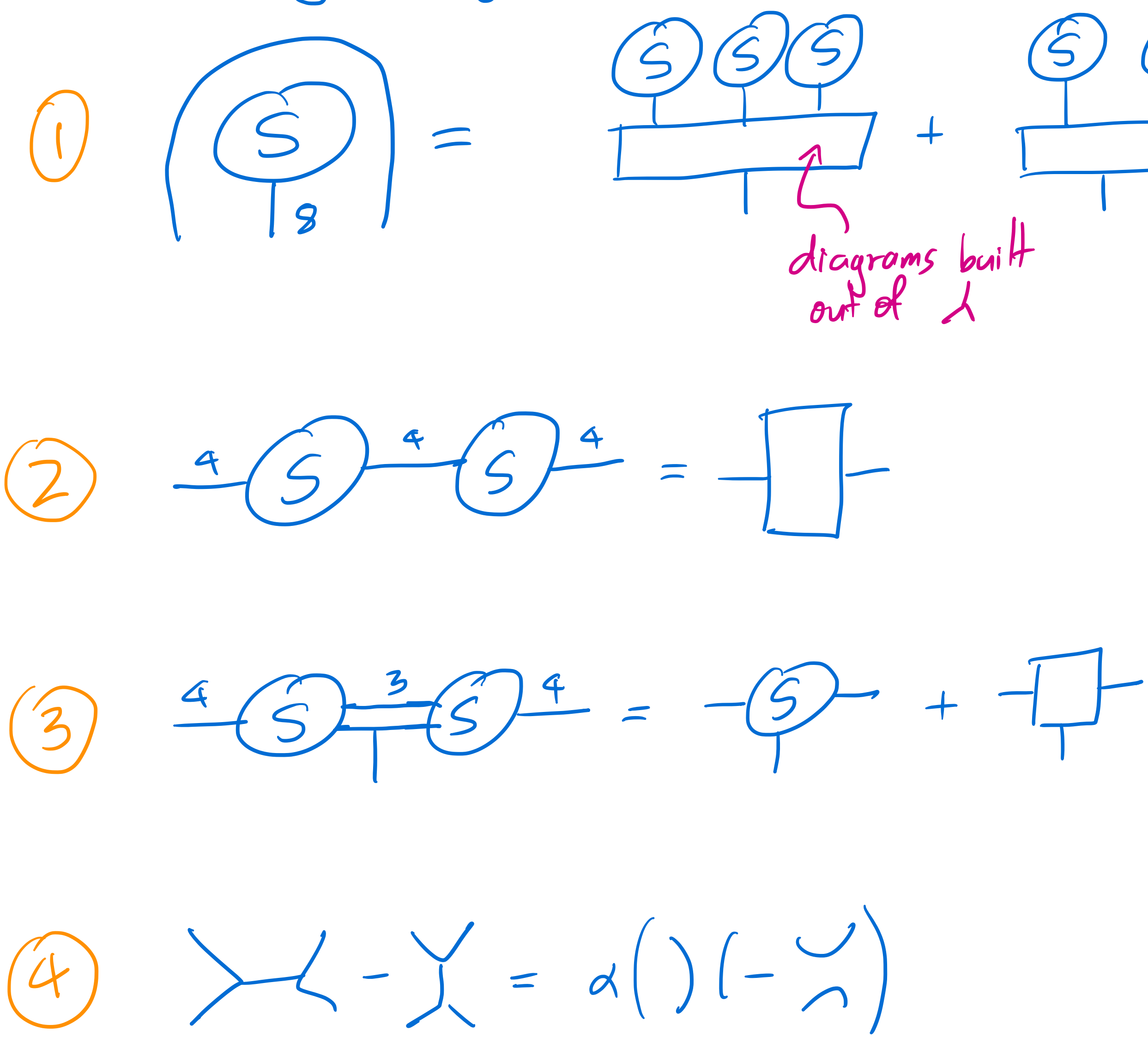
- It is generated as a $\otimes$-category by two morphisms



and

$\in \mathrm{Hom}\left(f^{(2)\otimes 8} \longrightarrow 1\right)$

- satisfying 'jellyfish' relations



To prove $\mathcal{EH}_1$ exists we:

- show these relations suffice to evaluate all closed diagrams
- find a 'faithful representation' of the quotient.

arXiv:0909.4099

We'd really like to understand this example!

- is it truly exotic?
  - 'obstructions'?
- can we find easier constructions?
  - then find more like it? (at present we don't even have candidates)

Recent work on $\mathcal{EH}$ focusses on its

- Drinfeld centre (if $\mathcal{C}$ is a spherical fusion category, $Z(\mathcal{C})$ is a <u>modular tensor category</u>)
- Brauer-Picard groupoid (the collection of all Morita equivalent fusion categories, and the equivalences)

Drinfeld centre

arXiv:1404.3955
arXiv:1606.07165

- $Z(\mathcal{C}) = \{(X,\beta) \mid X \in \mathcal{C},\ \beta_Y : X \otimes Y \xrightarrow{\cong} Y \otimes X\}$
- Most of the TFT invariants are determined by $Z(\mathcal{C})$.
- With Terry Gannon and Kevin Walker, we've pinned down the modular data for $Z(\mathcal{E}\mathcal{H})$.
  - lots of combinatorics, Galois theory, and $SL(2, \mathbb{Z}/N\mathbb{Z})$ rep theory!
  - constrains possible CFT realisations of $Z(\mathcal{E}\mathcal{H})$
  - tantalising hints of a connection to $(F_4)_4$!

Brauer-Picard groupoid

($\mathcal{C} \underset{\text{Morita}}{\simeq} \mathcal{D}$ iff $Z(\mathcal{C}) \underset{\text{br}}{\cong} Z(\mathcal{D})$)

- At most four fusion rings for categories Morita equivalent to $\mathcal{EH}_1$. (combinatorics)
- At most one category for each ring and no Morita auto-equivalences (uniqueness of the $\mathcal{EH}_1$-$\mathcal{EH}_2$ bimodule)
- Then construct everything remaining:

$$\begin{tikzcd}
\mathcal{EH}_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathcal{EH}_2 \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
\mathcal{EH}_3 \arrow[r, leftrightarrow] & \mathcal{EH}_4
\end{tikzcd}$$

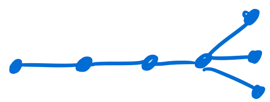
(a generalisation of the original construction, using a beautiful skein theory for the 2-category $\mathcal{EH}_{12}$)

arXiv: ... next week? Grossman, M, Penneys, Peters, Snyder.

Our method generalises the

graph planar algebra embedding theorem

(and unifies distinct perspectives of it from the subfactor and $\otimes$-category literatures)

Given a graph Γ (e.g. ) and a 'weight' $d: V(\Gamma) \to \mathbb{C}$, there is an associated multifusion category $\mathcal{G}(\Gamma)$.

- The original construction of $\mathcal{E}\mathcal{H}_{1,2}$ built a faithful map into such a category.

- It's long been known that every fusion category $\mathcal{C}$ admits such an embedding for Γ one of the 'principal graphs' for $\mathcal{C}$.

- We extend this to multifusion categories, and turn it into a classification:

$$\{\text{embeddings } \mathcal{C} \to \mathcal{G}(\Gamma)\} \longleftrightarrow \{\text{module categories for } \mathcal{C} \text{ with principal graph } \Gamma\}$$

(as well as pivotal/unitary versions)

- we build this from scratch, so it's accessible with soaking first in the subfactor literature.

- Finally, even this is too hard for $\mathcal{C} = \mathcal{EH}_1$ or $\mathcal{EH}_2$, but the beautiful skein theory of the multifusion category $\mathcal{EH}_{12}$ allows us to construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$ as modules over it!

- What next?
  - Sadly, $\mathcal{EH}_3$ and $\mathcal{EH}_4$ did not reveal connections with other fusion categories

    (previously, apparently exotic examples have been 'explained' by their BP groupoid)
  - what's going on with $(F_4)_4$?
    - grafting? twisted equivariantisation?
  - continue the search for more examples?