We will describe physical systems by their observables.

An algebra of observables A is

- an algebra over $\mathbb{C}$, with a
- star structure $\dagger : A \to A$,

satisfying:

- $(z \cdot a)^\dagger = \bar{z} \cdot a^\dagger$
- $(a+b)^\dagger = a^\dagger + b^\dagger$
- $(ab)^\dagger = b^\dagger a^\dagger$

such that there merely exists a norm $\|-\| : A \to \mathbb{R}_{\geq 0}$

(so $\|a+b\| \leq \|a\| \|b\|$, $\|ab\| \leq \|a\| \|b\|$,

and $\|z \cdot a\| = |z| \|a\|$)

satisfying $\|a^\dagger a\| = \|a^\dagger\| \|a\|$.

(Further, if the algebra is infinite dimensional, we require that there merely exists another normed space $(B, \|-\|)$ so $A = B^*$.)

Let's begin with the fundamental definitions.

Suppose we have a system described by a $*$-algebra A.

- An observable is an element $a : A$.
- An event is a projection in A, i.e. a $p : A$ so $p^2 = p = p^\dagger$.
- An element $b : A$ is positive if $b = a^\dagger a$ for some $a : A$.
- A state φ is a linear functional $A \to \mathbb{C}$ so $\varphi(1) = 1$ ("φ is normalised") and $\varphi(a^\dagger a) \geq 0$ ("φ is positive")

Note that any convex linear combination $\sum x_i \varphi_i$ of states is a state.

$(x_i \geq 0, \ \sum x_i = 1)$

- The expected value of an observable a in a state φ is $\varphi(a)$.
- The probability of an event p in a state φ is its expected value, $\varphi(p)$.
  - ☆ $1 : A$ is the "certain" event: $\varphi(1) = 1$ for every state φ.
  - ☆ $0 : A$ is the "impossible" event $\varphi(0) = 0$ for every state φ.
- After observing an event p in the state φ, the new state is

$$\hat{\varphi}(a) = \frac{\varphi(pap)}{\varphi(p)} .$$

Just for now, try not to attach any familiar meanings to these words!

Our next task is understand what all these definitions mean for particular examples of algebras of observables A.

$\Rightarrow$ We'll find that for commutative algebras, all these definitions coincide with the informal interpretation of the words!

$\Rightarrow$ This suggests we should accept that however these definitions play out in the noncommutative case, we should accept them as reasonable descriptions of

events, states, probabilities,

and "updating knowledge on the basis of observations"

in the quantum world.

• What's the simplest example?

$$A = \mathbb{C}^n \quad \text{(with pointwise multiplication)}$$

Events: $p^2 = p = p^+ \iff$ all entries of p are 0 or 1.

Thinking of $\mathbb{C}^n$ as $\mathrm{Fun}(\{1, \dots, n\}) \to \mathbb{C}$, the events are the characteristic functions of subsets of $\{1, \dots, n\}$.

Every event is uniquely the sum of some subset of the "minimal events"

$$\delta_i = (0, \dots, 0, 1, 0, \dots, 0).$$

The support of pq for p and q events is the intersection of the supports of p and q.

$\Rightarrow$ "pq is the event 'p and q'"

Positive observables

$p = a^+ a \iff$ all entries of p are non-negative.

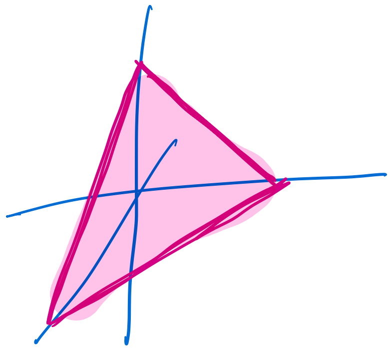
States The linear functionals on $\mathbb{C}^n$ just look like $\mathbb{C}^n$: $\varphi = (\varphi_1, \ldots, \varphi_n)$

$$\varphi(a) = \sum \varphi_i a_i$$

A linear functional is positive iff $\varphi_i \geq 0$

A linear functional is normalised $\varphi(1) = 1$ iff $\sum \varphi_i = 1$

$\mathrm{States}(\mathbb{C}^n) = \Delta^{n-1}$ (the $n-1$ simplex)



These are the probability measures on the finite set $\{1, \ldots, n\}$.

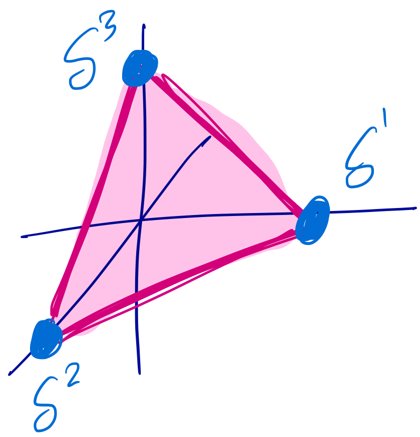
In a state φ the probability of the minimal event $p = \delta_i$ is $\varphi(p) = \varphi_i$.

For a general event $p = \sum_{i \in S \subseteq \{1, \ldots, n\}} \delta_i$, it is $\sum p_i$.

An extremal state is a state which is not a convex combination of other states.

The extremal states for $\mathbb{C}^n$ are the

$$\delta^i = (0, \_, 1, \_, 0)$$



In the state δ^1, the event p_1 is "certain":

$$\delta^1(p_1) = 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 1$$

while the events p_2 and p_3 are "impossible".

In this example,

Every state is uniquely a convex linear combination of extremal states.

Suppose we are in the state

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$$

and we observe the event

$$p = (1, 1, 0, 0).$$

Our update rule says the new state is

$$\hat{\varphi}(a) = \frac{\varphi(pap)}{\varphi(p)} = \frac{\varphi(pa)}{\varphi(p)} \quad \text{(since } A \text{ is commutative)}$$

so
$$\hat{\varphi}(a_1, a_2, a_3, a_4) = \frac{\varphi(a_1, a_2, 0, 0)}{\varphi(1, 1, 0, 0)}$$

$$= \frac{\varphi_1 a_1 + \varphi_2 a_2}{\varphi_1 + \varphi_2}$$

Observe that in the new state, p is <u>certain</u>

and $1 - p$ is impossible.

For a and p both events,

$$\hat{\varphi}(a) = \frac{\varphi(pap)}{\varphi(p)} \overset{\text{assuming commutativity}}{=} \frac{\varphi(pa)}{\varphi(p)}$$

is Bayes' law:

$$P(a \text{ given } p) = \frac{P(a \text{ and } p)}{P(p)}.$$

(Notice that as every observable is a $\mathbb{C}$-linear combination of events, in fact it suffices to define $\hat{\varphi}$ on events.)

That's "all of classical probability".

Every finite-dimensional commutative $*$-algebra is of the form $\mathbb{C}^n$ for some n.

Infinite dimensional commutative von Neumann algebras are $L^\infty(X,\mu)$ for some measure space (X,μ), and the states are exactly the probability measures.

The noncommutative world

In $A = M_2(\mathbb{C})$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

an event $p^2 = p = p^{\dagger}$

must be a self-adjoint matrix, with eigenvalues 0 and 1.

These come in three flavours:

- $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the certain event
- $p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the impossible event.
- otherwise, p is a <u>rank 1 projection</u>.

Solving the equations

① $\operatorname{tr} p = \sum \lambda_i = 0 + 1 = 1$

② $\det p = \prod \lambda_i = 0 \cdot 1 = 0$

③ $p = p^{\dagger}$

we find

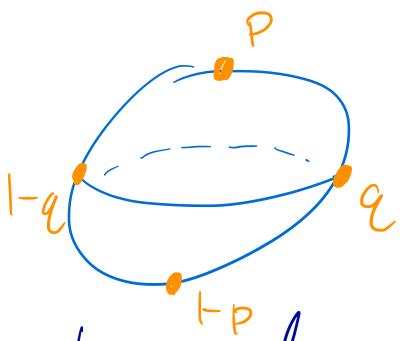
$$p = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

with $x, y, z \in \mathbb{R}$.
and $x^2 + y^2 + z^2 = \underline{1}$

Thus rank 1 projections are indexed by points in the unit sphere in $\mathbb{R}^3$.

$\text{Events}(M_2(\mathbb{C})) =$

$\underline{1}$ $\bullet$ $\sqcup$



$\sqcup$ $\bullet$ 0

Notice events no longer decompose uniquely as sums of indecomposable events:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$p \;+\; 1-p \qquad\qquad q \;+\; (1-q)$

States

By a direct calculation using

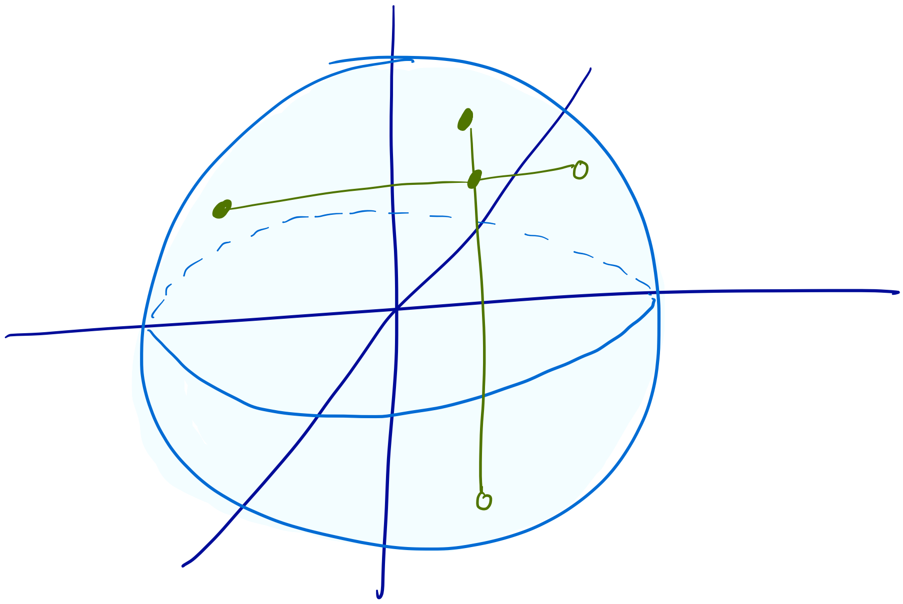
φ linear, $\varphi(1) = 1$, $\varphi(a^+ a) \geq 0$,

we can compute the states:

$$\varphi_{x,y,z}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1+z}{2} \cdot a + \frac{x-iy}{2} \cdot b + \frac{x+iy}{2} \cdot c + \frac{1-z}{2} \cdot d$$

for $x, y, z \in \mathbb{R}$,

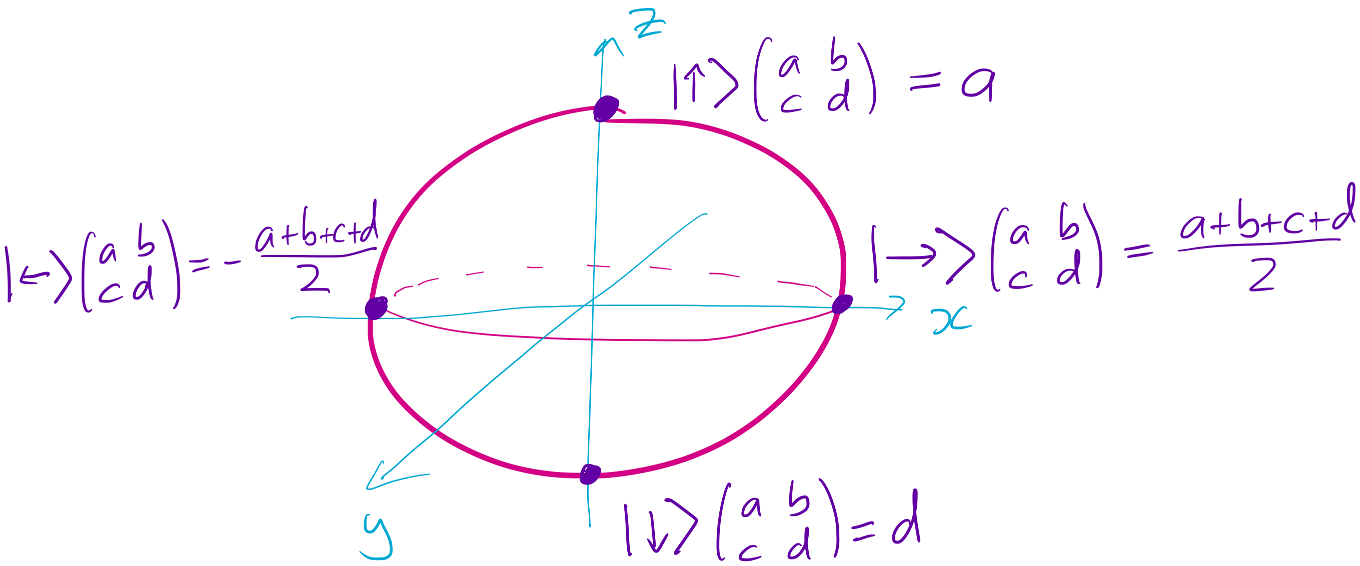
$x^2 + y^2 + z^2 \leq 1$.



Notice
- the space of states is convex.
- but states are not uniquely expressible as convex linear combinations of the extremal states
- states are not probability measures on some set.

Recall the space of states for $M_2(\mathbb{C})$ is the unit ball

$$\varphi_{x,y,z}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1+z}{2}\cdot a + \frac{x-iy}{2}\cdot b + \frac{x+iy}{2}\cdot c + \frac{1-z}{2}\cdot d$$



Let's measure the observable $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on the state $|\uparrow\rangle$.

$|\uparrow\rangle\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1$, so the event E certainly occurs.

When we observe this, the state doesn't change:

$$\hat{\mu}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{|\uparrow\rangle\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)}{|\uparrow\rangle\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)} = \frac{|\uparrow\rangle\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)}{|\uparrow\rangle\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)} = \frac{a}{1} = a$$

So $\hat{\mu} = |\uparrow\rangle$. (This is a good sign: if we observe something we're certain occurs, we don't need to update our knowledge.)

What if we measure the event $F = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$?

$|\uparrow\rangle\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \frac{1}{2}$, so we have a 50% chance of observing F (and hence also a 50% of "not F" $= I - F$)

If we observe F, the new state is

$$\hat{\mu}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{|\uparrow\rangle\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)}{|\uparrow\rangle\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)} = \frac{|\uparrow\rangle\left(\begin{bmatrix} \frac{a+b+c+d}{4} & " \\ " & " \end{bmatrix}\right)}{|\uparrow\rangle\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)} = \frac{a+b+c+d}{2}$$

So $\hat{\mu} = |\rightarrow\rangle$. If, on the other hand, we observe $I - F$, then the new state is $|\leftarrow\rangle$.

We can interpret $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as "measuring in the z-direction" and $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ as "measuring in the x-direction".

After measuring $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the state is either $|\uparrow\rangle$ or $|\downarrow\rangle$.

After measuring $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, the state is either $|\rightarrow\rangle$ or $|\leftarrow\rangle$.

In the classical setting, extremal states are <u>definite</u>
(that is, $\mu(E) = 0$ or 1 for every event E)
and do <u>not change</u> after any measurement.

In the quantum setting, e.g. $M_2(\mathbb{C})$ there may be
<u>no definite states at all</u>!

It gets even stranger!

Bell's inequality

If A, B, C, D are classical observables taking values in $\{\pm 1\}$

(i.e. $\left(\frac{A+1}{2}\right)^2 = \frac{A+1}{2}$)

then for any state μ,

$$\mu(AB + BC + CD - AD) \leq 2$$

Proof: Think of these in a square

$$\begin{tikzcd} A \arrow[r, dash] \arrow[d, dash] & D \arrow[d, dash] \\ B \arrow[r, dash] & C \end{tikzcd}$$

If $A = D$, then $AB + BC + CD - 1 \leq 3 - 1 = 2$

If $A \neq D$, then either $A \neq B$, $B \neq C$, or $C \neq D$.

In any case $AB + BC + CD + 1 \leq 2 - 1 + 1 = 2$.

However in the quantum world it is easy to violate this inequality

$\boxed{\text{and this violation has been experimentally observed!}}$

Example: $\mathcal{A} = M_4(\mathbb{C})$, $\mu(X) = \mathrm{tr}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} X\right)$

$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad B = -\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{bmatrix}, \quad D = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & & \\ -1 & -1 & & \\ & & 1 & -1 \\ & & -1 & -1 \end{bmatrix}$$

and (exercise!) $\mu(AB + BC + CD - AD) = \mu(AB) + \mu(BC) + \mu(CD) - \mu(AD)$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - -\frac{1}{\sqrt{2}} = 2\sqrt{2} > 2.$$

- States and events

State operator correspondence:

$$\left\{\begin{array}{c}\text{positive elements } r : A \\ \text{so } \operatorname{trace}(r) = 1\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{(normal)} \\ \text{states on } A\end{array}\right\}$$

$$r \longmapsto (a \mapsto \operatorname{tr}(ra))$$

Observe for $\mathbb{C}^2$ and $M_2\mathbb{C}$, there was a correspondence

$$\{\text{rank } 1 \text{ projections}\} \longleftrightarrow \{\text{extremal states}\}$$

(both are the unit sphere)

Vector states

Every finite dimensional algebra of observables is <u>semisimple</u> and in fact by Wedderburn's theorem

$$A \cong M_{n_1}\mathbb{C} \oplus M_{n_2}\mathbb{C} \oplus \cdots \oplus M_{n_k}\mathbb{C}$$

More abstractly, every algebra of observables admits a (non-unique! non-physical!) faithful action on some Hilbert space.

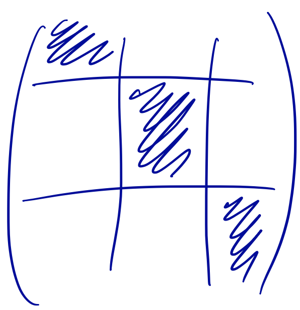
In particular,

$$\bigoplus M_{n_i}\mathbb{C} \quad \curvearrowright \quad \bigoplus \mathbb{C}^{n_i}$$

but matrix multiplication is a faithful action.

We have

$$\bigoplus M_{n_i}\mathbb{C} \xhookrightarrow[\text{morphism}]{*\text{-algebra}} M_{\Sigma n_i}\mathbb{C}$$



Or in infinite dimensions

$$A \hookrightarrow \mathcal{B}(\mathcal{H})$$

Eg. $M_2\mathbb{C} \curvearrowright \mathbb{C}^2$.

Given a unit vector $\xi \in \mathcal{H}$, we can define a state by

$$\varphi_\xi(a) = (a\xi, \xi).$$

This really is a state:

$$\varphi_\xi(\mathbb{1}) = (\xi, \xi) = \|\xi\|^2 = 1.$$

$$\varphi_\xi(b^+b) = (b^+b\xi, \xi) = (b\xi, b\xi) \geq 0.$$

States realisable in this way are called **vector states.**

Notice that the vector itself (indeed, the vector space!) has no physical meaning:

$$\varphi_\xi = \varphi_{e^{i\theta}\xi}$$

and if we embed $A \hookrightarrow \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right)$$

Then $\varphi_{\xi+\eta}(a) = \left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \left(\begin{pmatrix} a\xi \\ 0 \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) =$

$$(a\xi, \xi) = \varphi_\xi(a).$$

What are the vector states of $M_2\mathbb{C}$?

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$= a\|x_1\|^2 + b\bar{x}_1 x_2 + c x_1 \bar{x}_2 + d\|x_2\|^2$$

Matching this up with: $\varphi_{x,y,z}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1+z}{2}\cdot a + \frac{x-iy}{2}\cdot b + \frac{x+iy}{2}\cdot c + \frac{1-z}{2}\cdot d$

we see
$$z = 2\|x_1\|^2 - 1$$
$$x = \bar{x}_1 x_2 + x_1 \bar{x}_2$$
$$y = -i(x_1\bar{x}_2 - \bar{x}_1 x_2)$$

and we can calculate

$$x^2 + y^2 + z^2 = \cdots = 1$$

The vector states are precisely the extremal states.

How does the update rule work for vector states?

Say E is an event, with eigenspaces

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1.$$

Say $\xi = \xi_0 + \xi_1$.

If we observe E, the updated state is

$$\widehat{\varphi_\xi}(a) := \frac{\varphi_\xi(EaE)}{\varphi_\xi(E)}$$

$$= \frac{\langle EaE\xi, \xi \rangle}{\langle E\xi, \xi \rangle}$$

$$= \frac{\langle aE\xi, E\xi \rangle}{\|E\xi\|^2}$$

$$= \left\langle a \frac{E\xi}{\|E\xi\|}, \frac{E\xi}{\|E\xi\|} \right\rangle$$

$$= \varphi_{\frac{E\xi}{\|E\xi\|}}(a).$$

The vector 'collapses' into the relevant eigenspace.

Appendix: Direct derivation of the space of states for $M_2\mathbb{C}$.

(11)

(2) $A = M_2(\mathbb{C})$, with $\begin{pmatrix} w & x \\ y & z \end{pmatrix}^* = \begin{pmatrix} \bar{w} & \bar{y} \\ \bar{x} & \bar{z} \end{pmatrix}$.

The identity of A is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A state $\rho : M_2(\mathbb{C}) \to \mathbb{C}$ must be of the form

$$\rho\begin{pmatrix} w & x \\ y & z \end{pmatrix} = aw + bx + cy + dz$$

for some $a, b, c, d \in \mathbb{C}$.

$$\rho\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow a + d = 1.$$

Let's introduce α so $a = \frac{1+\alpha}{2}$, $d = \frac{1-\alpha}{2}$, with $\alpha \in [-1, 1]$.

(Exercise:) A general positive element of A is of the form $\begin{pmatrix} \omega & \chi \\ \bar{\chi} & \xi \end{pmatrix}$ with $\omega, \xi \in \mathbb{R}_{\geq 0}$, $\chi \in \mathbb{C}$,

and $\omega\xi \geq \chi\bar{\chi}$.

Then $\rho\begin{pmatrix} \omega & \chi \\ \bar{\chi} & \xi \end{pmatrix} = \frac{1+\alpha}{2}\omega + \frac{1-\alpha}{2}\xi + b\chi + c\bar{\chi}$.

This must be real for all $\chi \in \mathbb{C}$, so we must have

$c = \bar{b}$.

Now $\displaystyle\min_{\chi : |\chi|^2 \leq \omega\xi} b\chi + \overline{b\chi} = -2|b|\sqrt{\omega\xi}$,

so we must have $\frac{1+\alpha}{2}\omega + \frac{1-\alpha}{2}\xi - 2|b|\sqrt{\omega\xi} \geq 0$

Minimising this with respect to $s = \omega^{\frac{1}{2}} \xi^{-\frac{1}{2}}$, we find the minimum occurs at $s = \frac{2|b|}{1+\alpha}$.

The minimum still has to be positive, and this says

$$1 \geq \alpha^2 + 4|b|^2.$$

Writing $b = \frac{\beta + i\gamma}{2}$, $c = \frac{\beta - i\gamma}{2}$, we have that a general state is of the form

$$\rho\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \frac{1+\alpha}{2} w + \frac{\beta+i\gamma}{2} x + \frac{\beta - i\gamma}{2} y + \frac{1-\alpha}{2} z$$

with $\alpha^2 + \beta^2 + \gamma^2 \leq 1$, $\alpha, \beta, \gamma \in \mathbb{R}$.

We can draw the set of states (or "Bloch region") as the solid ball