Higher dimensional algebras

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(Kevin Walker)

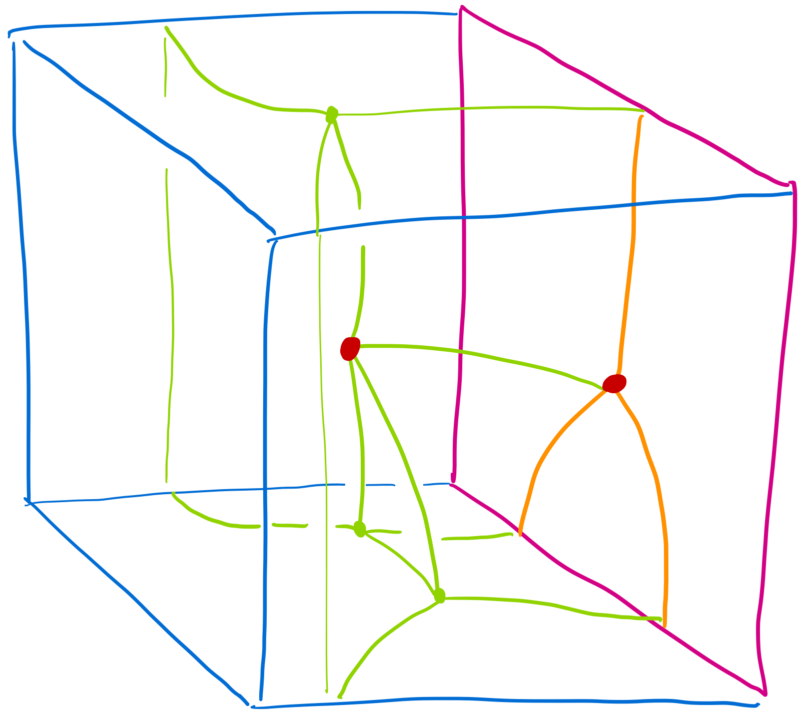
Our goal is to understand

modules for higher categories

particularly in the pivotal/semisimple/unitary settings.

This is the mathematical formulation of studying

domain walls for topological phases



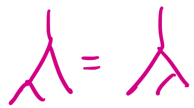
Low-dimensional cases:

- An algebra acting on a vector space consists of
  an algebra map $A \longrightarrow \mathrm{End}(V)$.

  When A is semisimple $/\mathbb{C}$, and V is indecomposable,
  there is a <u>projection</u> $p \in A$, $p^2 = p$
  so that $V = Ap$ and the action is just by multiplication.

- A $\otimes$-category acting on a 1-category consists of
  a $\otimes$-functor $\mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$.

  When $\mathcal{C}$ is a <u>fusion category</u>, and $\mathcal{M}$ is indecomposable,
  there is a special Frobenius algebra $A \in \mathcal{C}$, $m : A \otimes A \to A$,

  so that $\mathcal{M} \cong {}_{\mathcal{C}}\mathrm{mod}\text{-}A$. ("Ostrik's theorem")

• A 3-category $\mathcal{V}$ acting on a 2-category $\mathcal{C}$

$$\mathcal{V}^0 \longrightarrow \{\mathcal{C}\}$$

$$\mathcal{V}^1 \longrightarrow \{\text{2-functors } \mathcal{C}\to\mathcal{C}\}$$

$$\mathcal{V}^2 \longrightarrow \{\text{pseudonatural transformations}\}$$

$$\mathcal{V}^3 \longrightarrow \{\text{modifications}\}$$

When $\mathcal{V}$ is a braided category (i.e. $\mathcal{V}^0 = \{\bullet\}$, $\mathcal{V}^1 = \{\bullet\}$),
and $\mathcal{C}$ is a $\otimes$-category we

- only see the identity 2-functor $\mathcal{C}\to\mathcal{C}$
- pseudonatural transformations of the identity are exactly objects of the Drinfeld centre: $\mathcal{V} \xrightarrow{\text{braided}} Z(\mathcal{C})$

What is the data internal to $\mathcal{V}$ which classifies such modules?

For any pivotal n-category $\mathcal{C}$, we define its "(algebra) completion $\overline{\mathcal{C}}$" with the following properties:

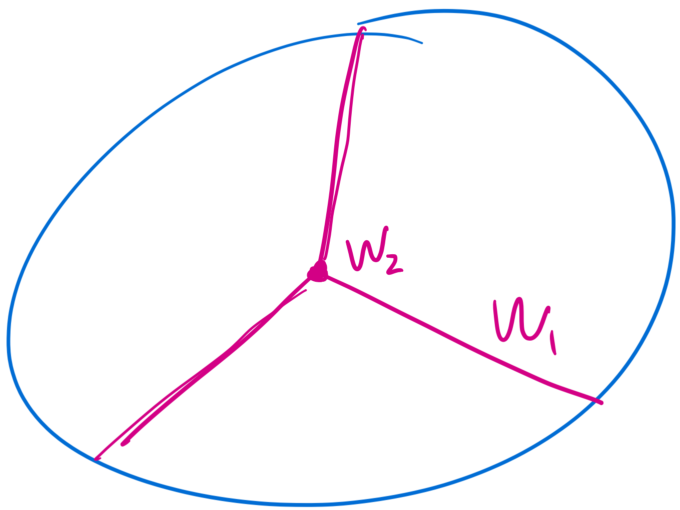
- For $n=1$, $\overline{\mathcal{C}} \simeq \mathrm{Idem}(\mathcal{C})$
- For $n=2$, $\mathcal{C}$ a $\otimes$-category,
  $\overline{\mathcal{C}} \simeq \mathrm{Alg}(\mathcal{C})$ (the 2-category of Frobenius algebras, bimodules, and intertwiners)
- There is a functor $\iota : \mathcal{C} \to \overline{\mathcal{C}}$, inducing a Morita equivalence (and $(\overline{\mathcal{C}}, \iota)$ is terminal amongst such pairs)
  $\Rightarrow$ $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent iff $\overline{\mathcal{C}}$ and $\overline{\mathcal{D}}$ are functorially equivalent.
- For $\mathcal{C}$ 'sufficiently semisimple' an equivalence $\overline{\mathcal{C}} \xrightarrow{\simeq} \mathrm{Mod}(\mathcal{C})$.
  The inverse provides a generalisation of Ostrik's theorem in all dimensions.

What is ... a (full) (string diagram) stratification?

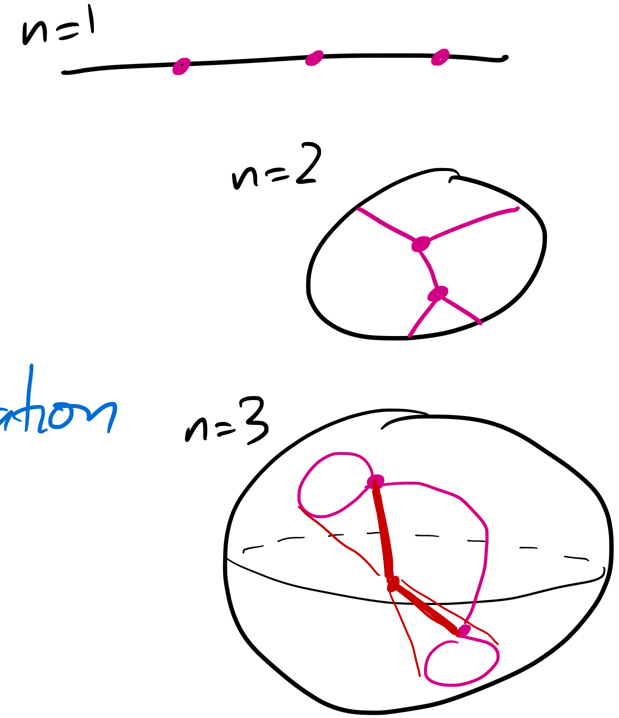
A stratification of an n-manifold W is a sequence

$$W = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n$$

so $W_k \setminus W_{k+1}$ is a codimension k submanifold of W.



It is a string diagram stratification if every $x \in W_k \setminus W_{k+1}$ has a nbhd in W which is a product of its nbhd in W_k and the cone over a string diagram stratification of some $(n-k-1)$-sphere.



Equivalently, if:

- it is the 0-ball with its unique stratification
- it is $W \times I$ for some W with an SD stratification
- it is $\mathrm{cone}(W)$, where W is an SD stratification of a sphere
- it is $W_1 \sqcup W_2$, where W_1 and W_2 have SD stratifications, or
- it is W glued to itself along Y, where W has an SD stratification, and both restrictions to Y give the same stratification.

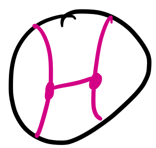
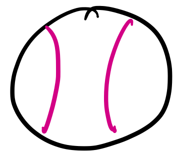
[Siebenmann '72!]

It is a full stratification if

every component of $W^k \setminus W^{k-1}$ is a $(n-k)$-ball,

and any such component meeting ∂W does so in exactly one

$(n-k-1)$-ball.

Example:  but not 

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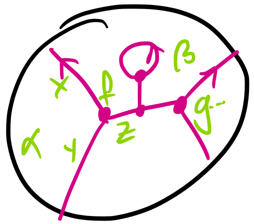
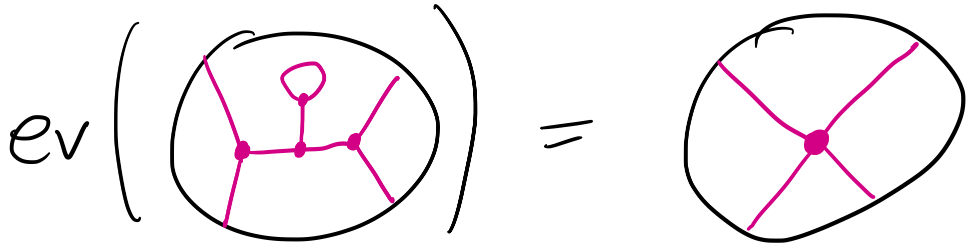
- it is $W_1 \sqcup W_2$, where W_1 and W_2 have full stratifications, or
- it is W glued to itself along Y, where W has a full stratification, and both restrictions to Y give the same stratification.

If $\mathcal{C}$ is an n-category, an m-dimensional $\mathcal{C}$-string diagram is an SD-stratified ball W_1^m with each $(m-k)$ ball in $W_k \setminus W_{k+1}$, labelled by some k-morphism of $\mathcal{C}$.



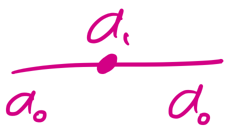
Any reasonable definition of a (pivotal) n-category lets you <u>evaluate</u> a $\mathcal{C}$-string diagram to an m-morphism, so isotopic $\mathcal{C}$-string diagrams have the same evaluation.



(Indeed, a minimal definition of an n-category axiomatises just this evaluation map.)

The 0-morphisms of $\overline{\mathcal{C}}$ are the <u>algebras in $\mathcal{C}$</u>.

An algebra a in $\mathcal{C}$ consists of: $(a_0, a_1, a_2, \dots, a_n, b_0, b_1, \dots, b_{n-1})$

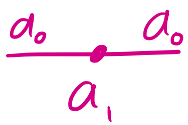
- a choice of a_0, a 0-morphism in $\mathcal{C}$
- a choice of , a 1-morphism
- for every k-sphere S, with a <u>full</u> string diagram stratification labelled by morphisms a_j for $j \le k$, a choice of $a_{k+1,S}$, a $k+1$ morphism with boundary S
- for each $a_{k,S}$, $k<n$, an n-morphism $b_{k,S}$ with boundary

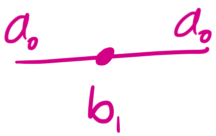
$$(a_{k,S} \times B^{n-k-1}) \cup (a_{k,S} \times B^{n-k-1})$$

- satisfying the 'egalitarian axiom' (to be specified in a moment)

Examples

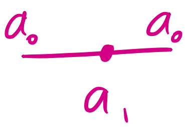
- An algebra in a 1-category $\mathcal{C}$ consists of:

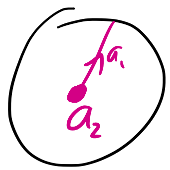
a_0, 

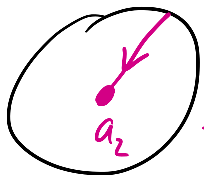
, 

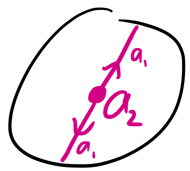
(satisfying some conditions)

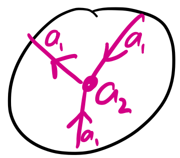
- An algebra in a 2-category consists of

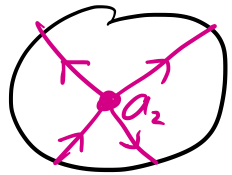
a_0, 

, 

, 

, 

, 

, 

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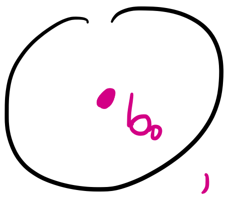
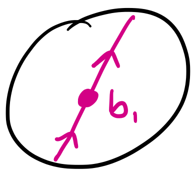


, 

(satisfying some conditions)

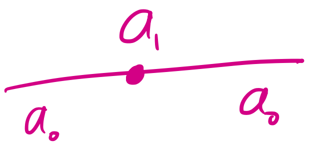
Say a string diagram labelled by a_k, b_k is <u>replete</u>

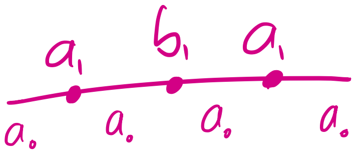
if
- ignoring all the b_k, the stratification is full,
- each codimension k cell in that full stratification satisfies exactly one of:
  - the cell meets the boundary in a single codimension $k+1$ cell
  - the cell contains a single b_k.

The 'egalitarian axiom':

any two replete string diagrams with the same boundary are equal.

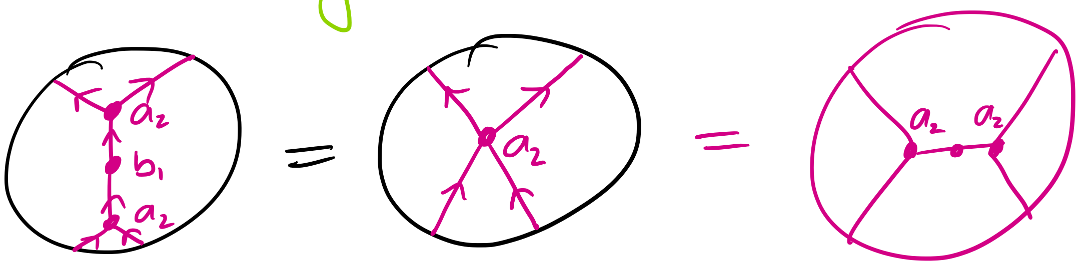
Examples

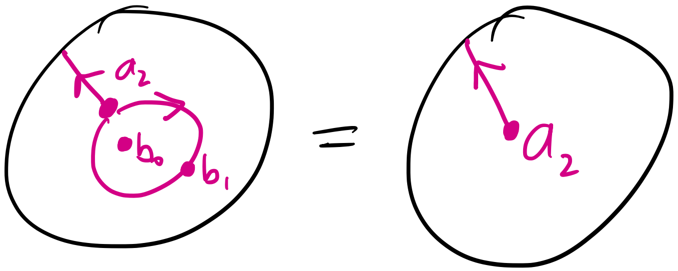
$n=1$: 

and 

are replete, and hence evaluate to the same morphism.

In fact this relation generates everything required by the egalitarian axiom.

$n=2$: 



These two classes of relations generate everything.

Higher morphisms

A j-morphism $m : \bar{e}^j$ of shape X^j (a j-ball)

has a boundary ∂m which is a string diagram on ∂X

labelled by $j' < j$ morphisms of $\bar{e}$. $m : \bar{e}^1$

The data of m is a tuple $(m_j, m_{j+1}, \dots, m_n, b_j, \dots, b_n)$ where

- m_ℓ is a choice, for each ℓ-j ball Y, with
  ∂Y containing a <u>full</u> string diagram labelled by the $m_{\ell'}$ for $\ell' < \ell$,
  of $m_{\ell,Y} : e^\ell(X \times Y)$
- $b_{\ell,Y}$ is an n-morphism with boundary $(m_{\ell,Y} \times B^{n-\ell-1}) \cup (m_{\ell,Y} \times B^{n-\ell-1})$
- all satisfying the 'same' egalitarian axiom.

Examples $n=1$. Say $(a_0, a_1, b_1), (a_0', a_1', b_1') : \bar{\mathcal{E}}^0$. $\quad a_1 b_1 a_1 b_1 = a_1 b_1$

$$\bar{\mathcal{E}}^1\big((a_0,a_1,b_1) \longrightarrow (a_0',a_1',b_1')\big) = \left\{ a_0 \xrightarrow{m_1} a_0' \;\middle|\; a_1 b_1 m_1 = m_1 b_1' a_1' = m_1 \right\}$$

Theorem $\bar{\mathcal{E}} \cong \mathrm{Idem}(\mathcal{E})$ via

$(a_0, a_1, b_1) \longmapsto (a_0, a_1 b_1)$ (recall $a_1 b_1$ is an idempotent)

$(m) \longmapsto m_1 b_1'$ (this is a morphism in $\mathrm{Idem}(\mathcal{E})$: $a_1 b_1 m_1 b_1' = m_1 b_1' = m_1 b_1' a_1' b_1'$)

$(X, p, \mathrm{id}_X) \longleftarrow (X, p)$

$f \longleftarrow f$

Proof The only interesting thing to check is that

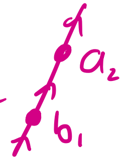
$$(a_0, a_1, b_1) \underset{a_1}{\overset{a_1 b_1}{\rightleftarrows}} (a_0, a_1 b_1, \mathrm{id}_{a_0})$$

is an isomorphism in $\bar{\mathcal{E}}$. □

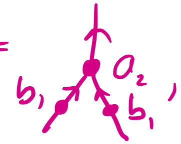
$\bar{\mathcal{E}}$ is a "puffier" version of the usual idempotent completion.

Examples

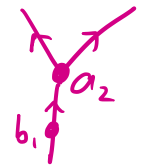
$n=2$: $\mathcal{C} \simeq \mathrm{Alg}\,\mathcal{C}$ (special Frobenius algebras)

$(a_0, a_1, a_2, b_0, b_1) \longmapsto \Big(A = \text{image}$ 

,

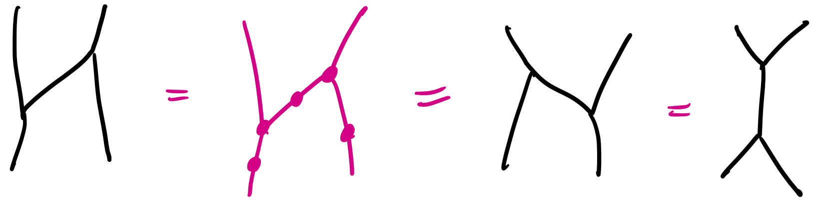
$m: A \otimes A \to A$ = 

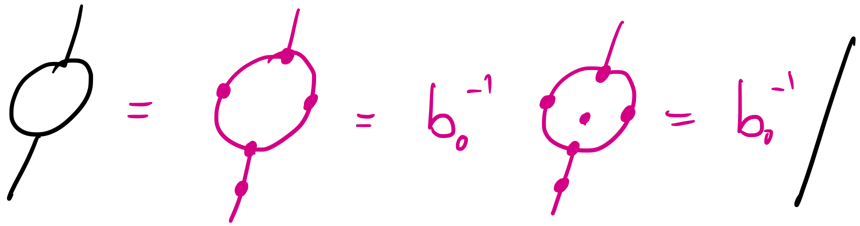
,

$\Delta: A \to A \otimes A$ = 

$\Big)$

We can verify the relations:



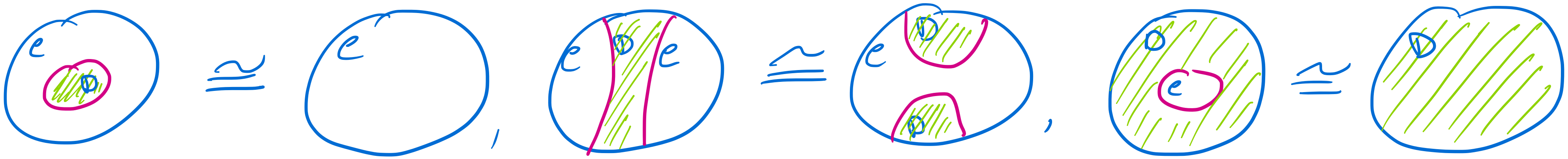


Morita equivalence $\mathcal{C} \to \overline{\mathcal{C}}$

- a k-morphism x of $\mathcal{C}$ gives a k-morphism of $\overline{\mathcal{C}}$, using

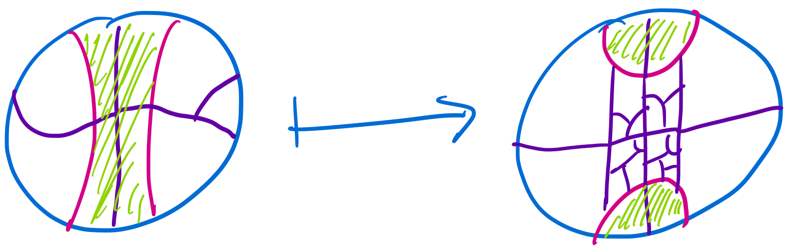
  $a_k = x$, a_j for $j>k$ iterated identities.

- a Morita equivalence ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ is a bimodule category with isomorphisms



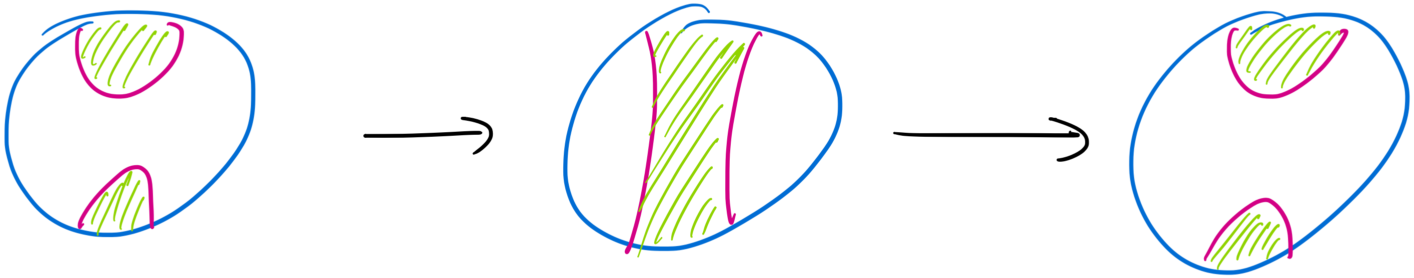
- we need to be able to rewrite $\overline{\mathcal{C}}$ diagrams as $\mathcal{C}$ diagrams

Example

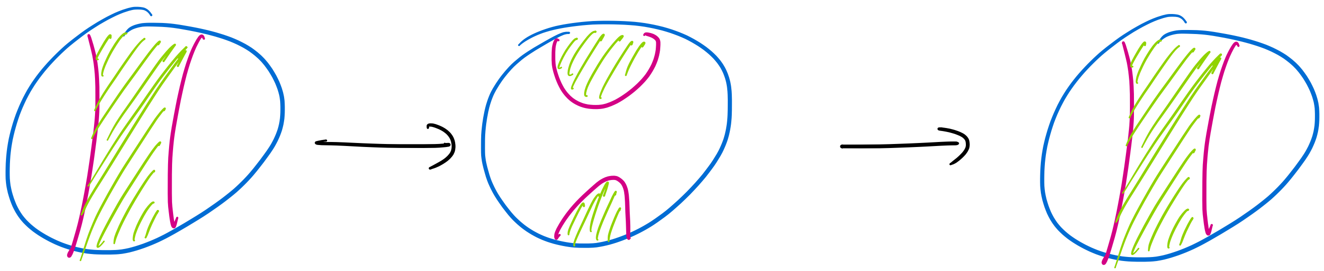


In the $\overline{\mathcal{C}}$ region to be removed, choose an SD stratification refining the $\overline{\mathcal{C}}$ diagram, and label it with the underlying morphisms in $\mathcal{C}$.

are these isomorphisms?

- 

is very close to the identity on the nose:
in the e region we've just refined the stratification, labelling all new strata with identity morphisms

- 

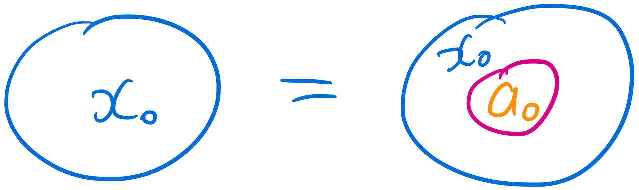
relies on

<u>Lemma</u> In a mixed e-$\bar{e}$ diagram in which the e region retracts to the boundary, you can test equality by converting everything to full e diagrams.

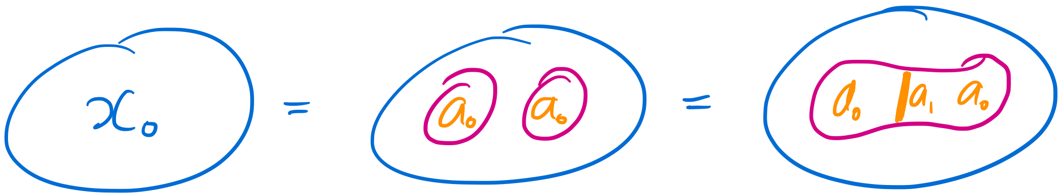
The inclusion $\iota : \mathcal{C} \to \overline{\mathcal{C}}$ is terminal amongst functors $\varepsilon : \mathcal{C} \to \mathbb{D}$ inducing a Morita equivalence:

$$\begin{tikzcd} \mathbb{D} \arrow[rr, dashed, "\exists!\varphi"] & & \overline{\mathcal{C}} \\ & \mathcal{C} \arrow[ul, "\varepsilon"] \arrow[ur, "\iota"'] \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

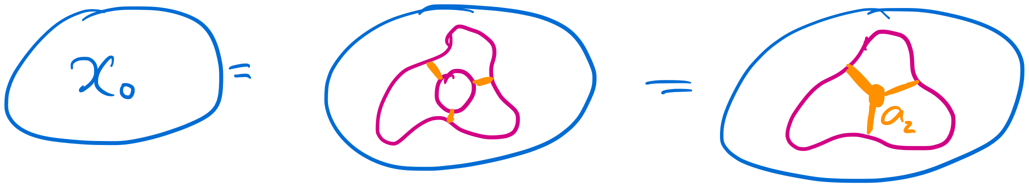
Sketch: given a 0-morphism x in $\mathbb{D}$, we can construct an algebra (i.e. a 0-morphism of $\mathcal{C}$) using the Morita isomorphisms:



for some $a_0 : \mathcal{C}^0$



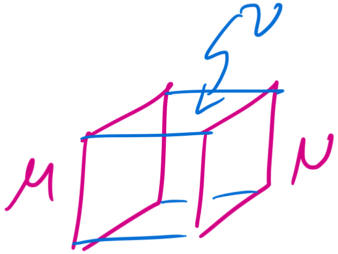
for some $a_1 : \mathcal{C}^1$



for some $a_2 : \mathcal{C}^1$

The handle cancellation laws for Morita isomorphisms imply the egalitarian axiom.

Applications

- Compute bilayers: 

$$\mathcal{M} \underset{\mathcal{V}}{\otimes} \mathcal{N} \simeq \mathcal{V}(A \to B)$$

where $\mathcal{M} = A\text{-mod}$, $\mathcal{N} = \text{mod-}B$,

A, B 3-algebras in $\mathcal{V}$.

- Example: $\mathcal{V} = G^{(3)}$ (G-labelled soap films)

$\mathcal{M} = \mathrm{Vec}$ (with trivial G action)

$\mathcal{N}$ a $\otimes$-category with G acting by $\otimes$-automorphisms

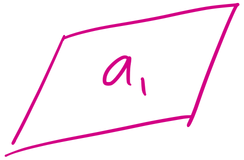
$\mathrm{Vec} \underset{G^{(3)}}{\otimes} \mathcal{N} \simeq \mathcal{N}^G$, the equivariantisation

- example $\mathcal{V} = \mathrm{Rep}G$

$\mathrm{Vec} \underset{\mathrm{Rep}G}{\otimes} \mathcal{N} \simeq \mathcal{N} /\!\!/ G$, the de-equivariantisation.

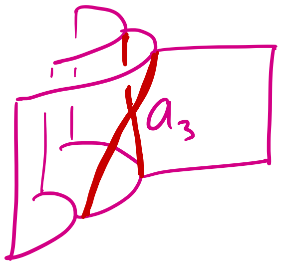
... higher dimensional analogues of (de)equivariantisation?

3-dimensional algebras in Vec are fusion categories:

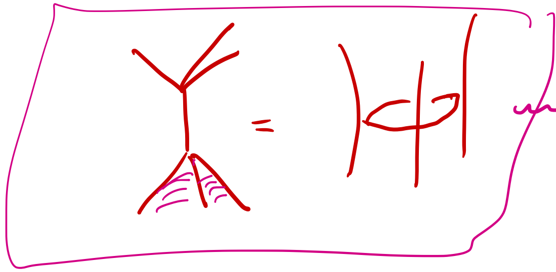


$\rightsquigarrow \mathrm{Fun}(\mathrm{Irr}\,\mathcal{C})$

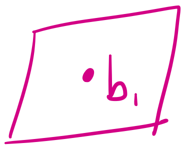
$\rightsquigarrow \bigoplus_{X Y Z : \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(\mathbb{1} \to XYZ)$



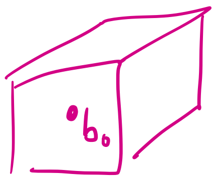
$\rightsquigarrow \alpha : \bigoplus \underline{\mathcal{C}(\mathbb{1} \to XYU)} \otimes \underline{\mathcal{C}(\mathbb{1} \to ZUW)} \to \bigoplus \underline{\mathcal{C}(\mathbb{1} \to XVW)} \otimes \underline{\mathcal{C}(\mathbb{1} \to YZV)}$



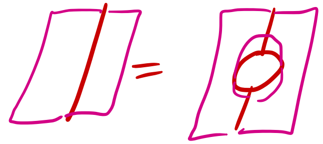
$\rightsquigarrow$ pentagon equation



$\rightsquigarrow (d_i^{-1})_{i : \mathrm{Irr}\,\mathcal{C}}$



$\rightsquigarrow \underline{\mathcal{D}^{-1}}$



$\rightsquigarrow$ splitting of

$$\mathcal{C}(a \to \oplus X) \otimes_{\mathrm{End}(\oplus X)} \mathcal{C}(X \to b)$$
$$\downarrow$$
$$\mathcal{C}(a \to b)$$

More generally, 3-algebras in $\mathcal{V}$, a braided tensor category, are $\mathcal{V}$-enriched fusion categories.