Diagrammatic methods

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Quantum Symmetries
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(Kevin Walker)

Topics

- "Generators mod relations" presentations
  of pivotal categories/planar algebras
  - an example: the Fibonacci category

- Temperley-Lieb-Jones categories
  - Jones-Wenzl idempotents
  - an example: the D_{2n} categories

- algebra objects, and skein theory for module categories

- (... diagrams for Drinfeld centres, subfactor planar algebras,
  string diagrams for higher categories ... ?)

Diagrams and quantum symmetries

Victor's talks have given you an algebraic perspective on fusion categories.

RepG is a 'prototypical' fusion category, and it's often helpful to think of a general fusion category as a 'noncommutative' or 'quantum' generalisation of a finite symmetry group.

Noah's talks this week will give a rather different perspective, via the cobordism hypothesis:

"Local topological field theories are classified by their value on a point, which is a certain sort of nice higher category".

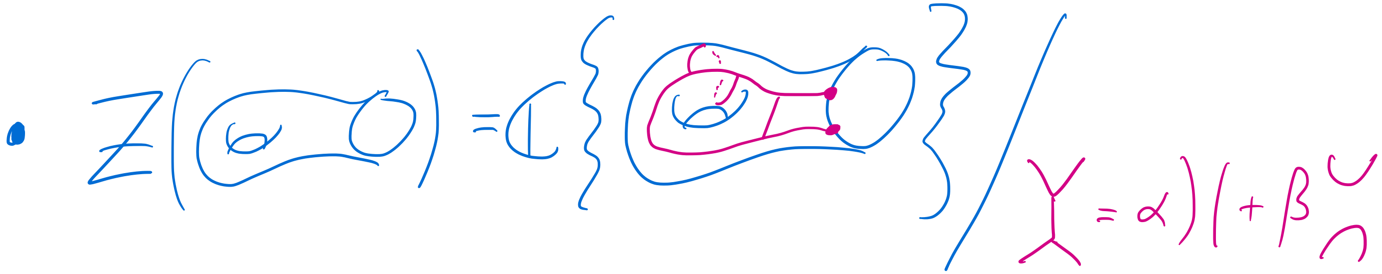
The essential idea is that local TFTs assign a set of "fields" to each manifold, with rules for gluing together fields.

A higher category just describes the local part, when the manifolds are balls.

Examples

- $Z(M^n) = \mathbb{C}\{\text{principal } G\text{-bundles on } M\}$

  $\rightsquigarrow$ the corresponding higher category is $\mathrm{Vec}G$ as an n-category ("Dijkgraaf-Witten theory")

- 

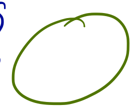
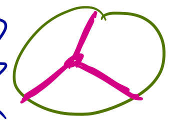
  $\rightsquigarrow$ the corresponding 2-category is the fusion category Fib.

Perhaps suprisingly every local TFT / higher category can be described in terms of "string diagrams" (with perhaps complicated labels and local relations) on manifolds / balls.

My lecture series takes this idea seriously, and develops some of the theory of quantum symmetries (fusion categories, etc) purely diagrammatically.

Let's suppose we have some skein theory (i.e. local linear relations) for planar, unoriented, trivalent graphs. (Later: "skein theory" = "pivotal category" = "planar algebra")

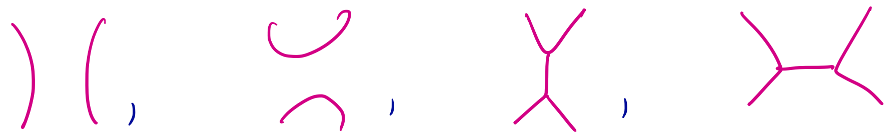
Let's write $\mathcal{P}_n$ for the space of (formal linear combos of) planar trivalent graphs with n bdy points, modulo these relations.

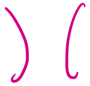
Let's assume $\mathcal{P}_0 = \mathbb{C}\{$  $\}$, $\mathcal{P}_1 = \mathbb{C}\{\ \}$, $\mathcal{P}_2 = \mathbb{C}\{$  $\}$, $\mathcal{P}_3 = \mathbb{C}\{$  $\}$

and $\dim \mathcal{P}_4 \leq 2$

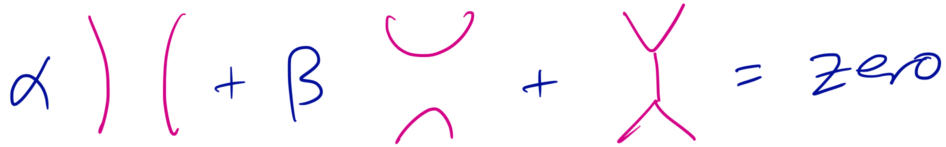
What can we say?

In $\mathcal{P}_4$, we have (at least!) the diagrams

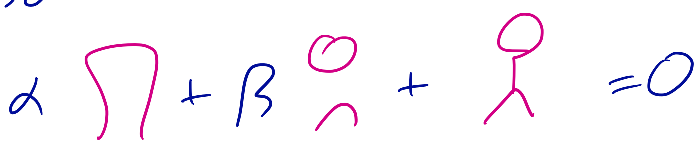


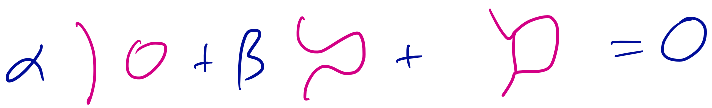
Exercise derive a contradiction if  and  are linearly dependant!

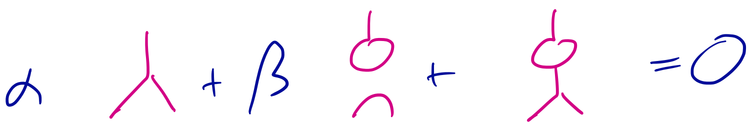
We must have a relation of the form

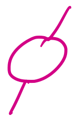


and so:



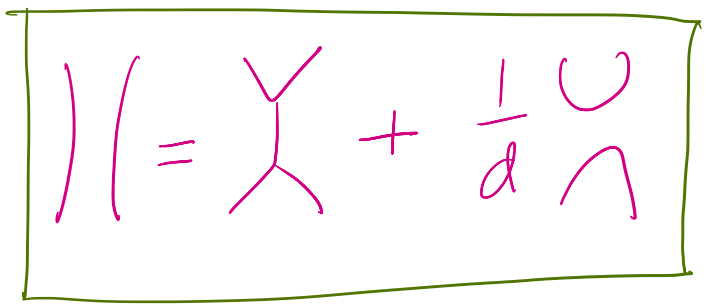




Now  = zero, and  $= b/$ for some b,

so $\quad \alpha + \beta d = \alpha d + \beta + b = \alpha + b = 0.$

Then $d = \frac{1 \pm \sqrt{5}}{2}$, $\alpha = -b$, $\beta = -b \frac{1 \mp \sqrt{5}}{2} = \frac{b}{d}$

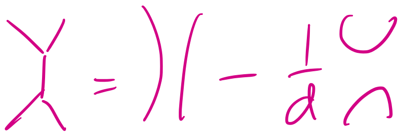


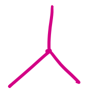
(and we may as well renormalize the vertex so $b=1$.)

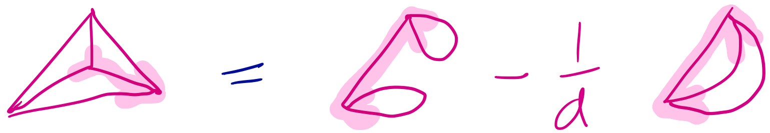
This relation "looks really powerful" and suggests there's perhaps at most one☆ example satisfying these conditions!

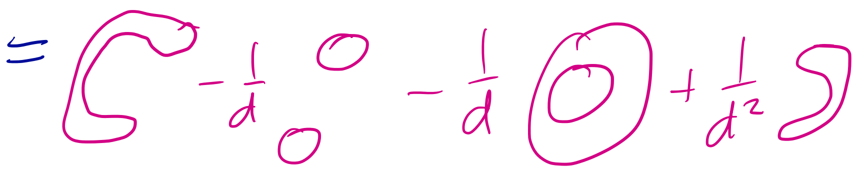
☆(two: $d = \frac{1 \pm \sqrt{5}}{2}$)

Why?

Claim 1  allows us to evaluate all closed diagrams

Proof In a closed diagram, every  must be adjacent to another ! The relation lets us replace that part of the diagram with a linear combination of strictly simpler diagrams.

Example 



$$= d - d - d + d^{-1} = 1$$

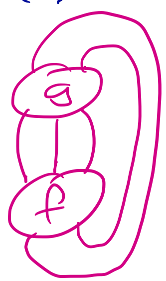
Such a skein theory (i.e. with $\dim P_0 \leq 1$) is called *evaluable*.

(Equivalently, in a semisimple category the tensor unit is simple.)

A general fact: every ideal in an evaluable skein theory is contained in the *negligible* ideal.

Defn  is negligible if every closed diagram built using f is zero.

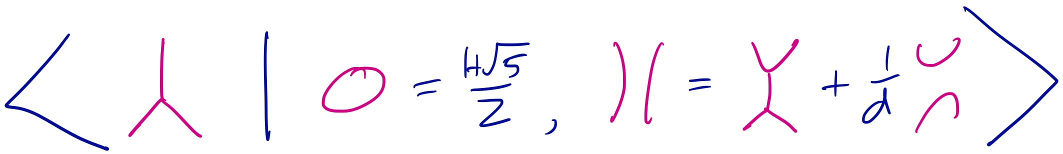
Proof Say  $\in I$, and non-negligible.

Then there is some  so  $\neq 0$,

so the empty diagram is in I, and hence I is the entire skein theory.

If we've discovered some relations that let us prove evaluability, form the free skein theory given by those generators and relations.

In our example, let Q be



It's evaluable, and there's a "functor"

$$F : Q \longrightarrow \mathcal{P}$$

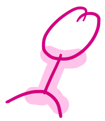
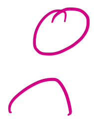
(Later, when we interpret $\mathcal{P}$ and Q as categories, this will be surjective on objects and morphisms)

Thus $\mathcal{P}$ is completely determined by the kernel of this functor, which is an ideal, and hence a sub-ideal of Q's unique maximal ideal, the negligibles.

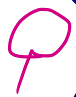
If we add the assumption that $\mathcal{P}$ is nondegenerate, we're done: $\mathcal{P}$ must be $\mathcal{Q}/\mathcal{N}$.

(Although we may not be satisfied with this description if we don't know a good description of $\mathcal{N}$.)

In fact, in our example $\mathcal{N} \neq \{0\}$!

Observe  $=$  $- \frac{1}{d}$  $=$ zero,

and since in every closed diagram  appears next to another vertex,

 $\in \mathcal{N}$. In fact $\mathcal{N} = \langle$  $\rangle$, but we're not quite ready to prove this.

We potentially have a bigger problem...

what if Q is actually zero!?

(This could happen if the relations let us evaluate a closed diagram in two different ways. This would say $\dim Q_0 = 0$, and then $\dim Q_n = 0$ for all n.)

Three solutions

(A) Directly show Q is nonzero, using 'confluence'.

This is usually really hard — I don't even know how to do this case!

(B) Find Q (or a non-zero quotient) "elsewhere in mathematics".

For this example, we're in luck: $Q/\mathcal{N} \cong \operatorname{Rep} U_{\xi_{10}} \mathfrak{so}_3$. (Or via Temperley-Lieb, or the chromatic polynomial.)

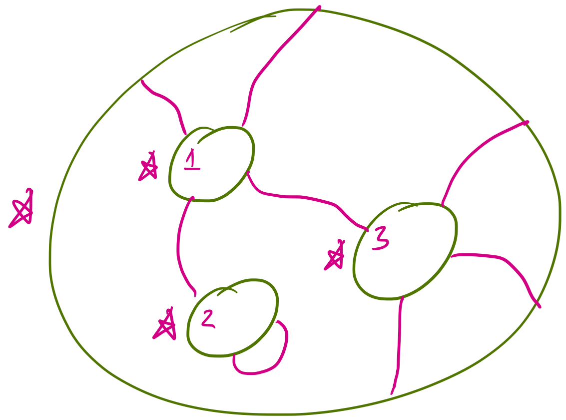
(C) Attempt a "general purpose" construction, using graph planar algebras / the regular representation.

Hopefully we'll have time later in the week.

One useful formalisation of 2-dimensional string diagrams is as a planar algebra. (due to V. Jones, arXiv:9909027)

A planar algebra P consists of:

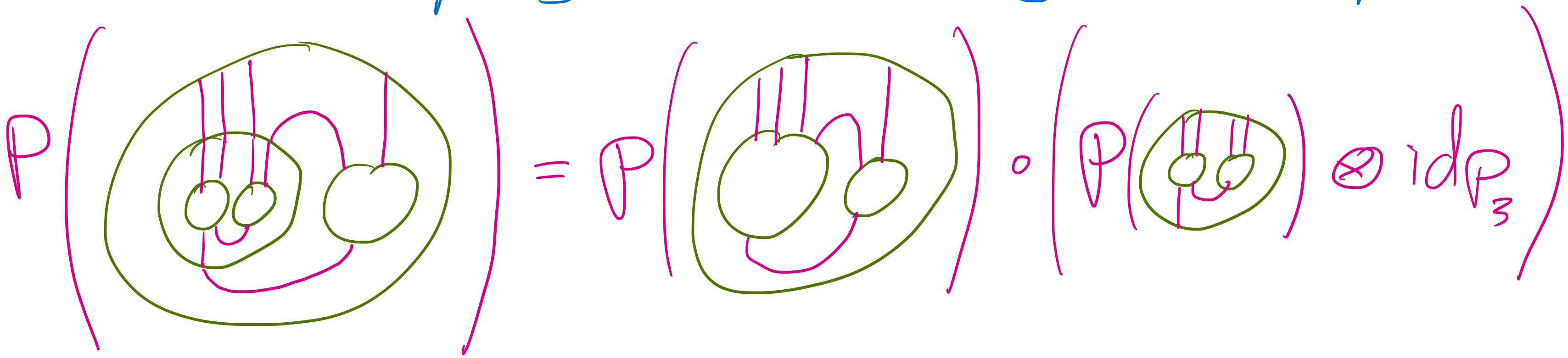
- a collection of vector spaces P_n, for $n:\mathbb{N}$.
- for each "spaghetti and meatballs" diagram



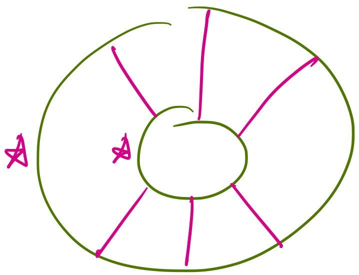
a linear map $P_4 \otimes P_3 \otimes P_4 \longrightarrow P_5$

(from the tensor product of the vector spaces associated to the inner circles to the vector space associated to the outer circle).

- such that
  - gluing one diagram inside another is compatible with composing the corresponding linear maps:



  - the 'radial' diagrams



act by the identity

  - two diagrams which are <u>isotopic</u> rel boundary act by the same linear map.

Example Temperley-Lieb-Jones forms a planar algebra:

$$TLJ_n = \mathbb{C}\{\text{TLJ diagrams with } n \text{ bdy points}\}$$

(the "vegetarian" spaghetti and meatballs diagrams)

$$\bigcirc = \delta \in \mathbb{C}$$

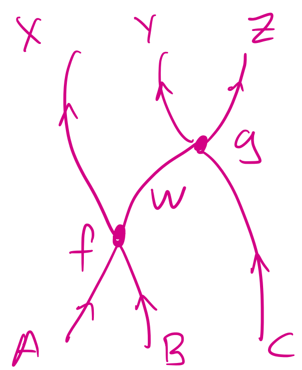
Exercises

- define a morphism of planar algebras
- explain why there is a morphism $TLJ \to P$ for any planar algebra P with $\dim P_0 = 1$.
- discover and prove a formula for $\dim TLJ_n$
- invent some generalisations of this definition!
  - what if I wanted to allow oriented spaghetti?
  - or a Colombian planar algebra:  ?

Another perspective on string diagrams starts with the traditional language of monoidal categories.

A **monoidal string diagram** for a strict monoidal category $\mathcal{C}$ is a planar diagram consisting of

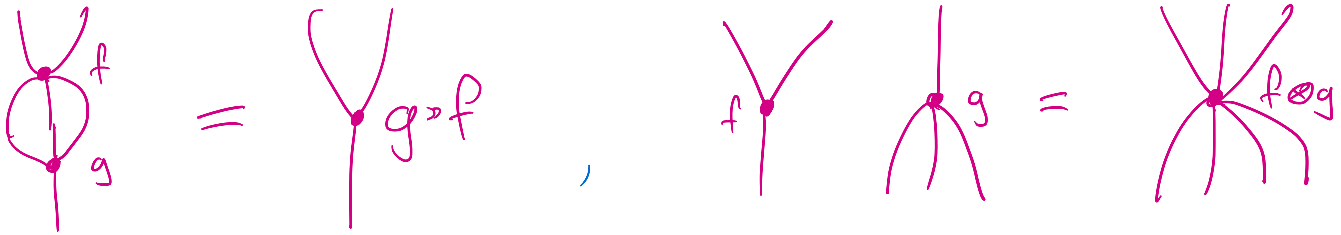
- oriented strings, which point up the page, labelled by objects of $\mathcal{C}$
- "vertices"/"boxes"/"coupons" with incoming strings and outgoing strings, labelled by appropriate morphisms from $\mathcal{C}$



$f : A \otimes B \to X \otimes W$

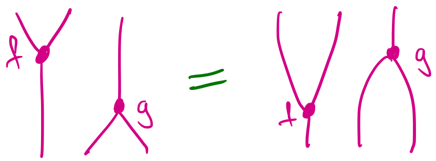
$g : W \otimes C \to Y \otimes Z$

We can represent any morphism in $\mathcal{C}$ as a string diagram, representing composition using "vertical stacking" and tensor product using "horizontal juxtaposition".



Theorem If two monoidal string diagrams are isotopic (through an isotopy which neither introduces critical points in strings nor rotates vertices) then the corresponding morphisms are equal.

Exercise Check



in any monoidal category.

How many axioms did you use?

Definition a monoidal string diagram category $\mathcal{C}$ consists of:

- a set $\mathcal{L}$ of edge labels
- for each pair of words w_{in}, w_{out} in $\mathcal{L}$, a set $\mathcal{L}(w_{in} \to w_{out})$ of vertex labels
- for each pair of words w_{in}, w_{out} in $\mathcal{L}$, a subspace $\mathcal{U}(w_{in} \to w_{out})$ of the (formal linear combos of) monoidal string diagrams labelled using $\mathcal{L}$ with incoming boundary w_{in} and outgoing boundary w_{in}
- such that $\mathcal{U}$ forms an ideal under vertical stacking and horizontal juxtaposition
- and if two string diagrams f and g are monoidally isotopic rel ∂, then $f - g \in \mathcal{U}$.

From this we can build a monoidal category $\hat{\mathcal{C}}$, by

$$\mathrm{Obj}\,\hat{\mathcal{C}} = \mathcal{L}$$

$$\hat{\mathcal{C}}(X \longrightarrow Y) = \frac{\mathbb{C}\{\mathcal{L}\text{-labelled string diagrams from } X \text{ to } Y\}}{\mathcal{U}(X \to Y)}$$

<u>Theorem</u> every monoidal category is monoidally equivalent to one of this form.

<u>Sketch</u> Define $\mathcal{L} = \mathrm{Obj}\,\mathcal{C}$.

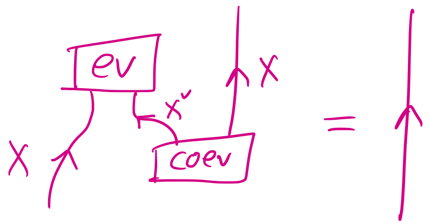
Define $\mathcal{S}(X \to Y) = \mathcal{C}(X \longrightarrow Y)$ (i.e. the vertex label set is just all morphisms)

We have an 'evaluation map' taking string diagrams to morphisms in $\mathcal{C}$.

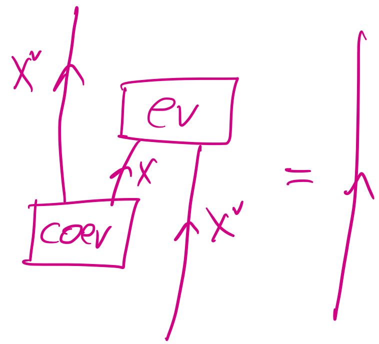
Define $\mathcal{U} = \ker(\mathrm{eval})$.

Recall a monoidal category is <u>rigid</u> if

- for every object X there is <u>dual</u> object $X^\vee$ and maps
$$ev : X \otimes X^\vee \to \mathbb{1}$$
$$coev : \mathbb{1} \to X^\vee \otimes X$$
satisfying



and



- and every X has a pre-dual ${}^\vee X$ so $({}^\vee X)^\vee = X$.

In a rigid string diagram strings can go up or down the page; and must be oriented <u>to the right</u> at any critical points.

Isotopies may introduce or cancel pairs of critical points, but <u>may not</u> rotate vertices.

Now:

(1) we interpret rigid string diagrams in $\mathcal{C}$ using

 $\rightsquigarrow \mathrm{id}_{X^\vee}$

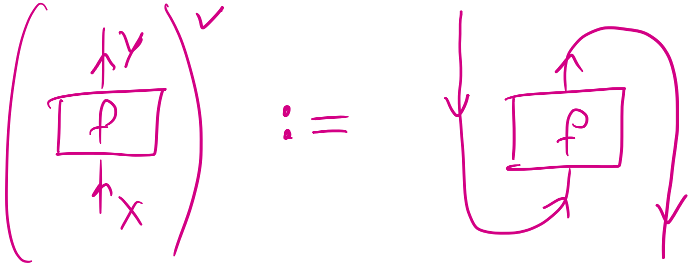
 $\rightsquigarrow ev_X : X \otimes X^\vee \to \mathbb{1}$

 $\rightsquigarrow coev_X : \mathbb{1} \to X^\vee \otimes X$

(2) Theorem two isotopic rigid string diagrams evaluate to the same morphism

(3) Theorem Every rigid monoidal category is equivalent to a category of rigid string diagrams.

Finally recall we can assemble $X \longmapsto X^\vee$ into a monoidal functor $\vee : \mathcal{C} \longrightarrow \mathcal{C}^{op, mop}$, defined on morphisms by



(This is a good exercise!)

and a pivotal structure is a choice of monoidal natural isomorphism $\tau : id_{\mathcal{C}} \xrightarrow{\simeq} \vee\vee$

Now a (pivotal) string diagram has no constraints on the tangencies,

we interpret 

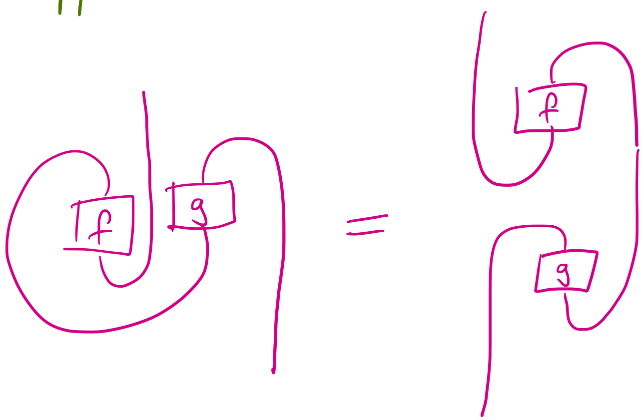
,

a (pivotal) isotopy may freely rotate vertices (i.e. it's just a general isotopy)

Then ① Theorem: isotopic string diagrams have the same evaluation

② Theorem: every pivotal category is pivotal equivalent to a category of pivotal string diagrams.

Exercise How many applications of an 'algebraic' axiom are required to prove:

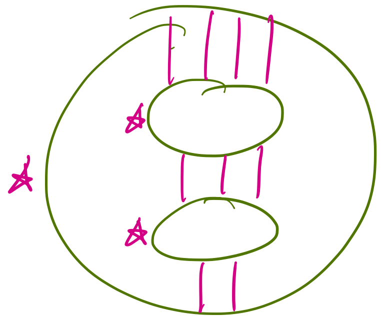


Both planar algebras and pivotal categories are 'just' the theory of planar diagrams up to planar isotopy.

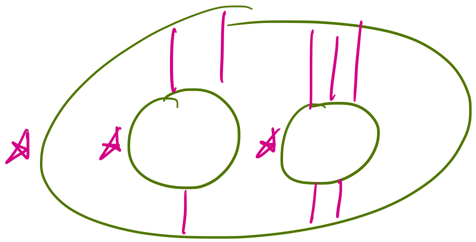
Given a planar algebra $\mathcal{P}$, we define a pivotal category $\mathcal{C}_\mathcal{P}$ by: $\mathrm{Obj}\,\mathcal{C}_\mathcal{P} = \mathbb{N}$

$$\mathcal{C}_\mathcal{P}(n \to m) = \mathcal{P}_{n+m}$$

Composition is provided by the tangle



and tensor product is $+$ on objects and provided by



on morphisms.

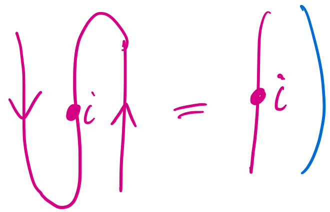
Exercises:

- check this is rigid
- write down the 'obvious' pivotal structure.

Given a pivotal category $\mathcal{C}$ and a choice of an object X

which is symmetrically self-dual

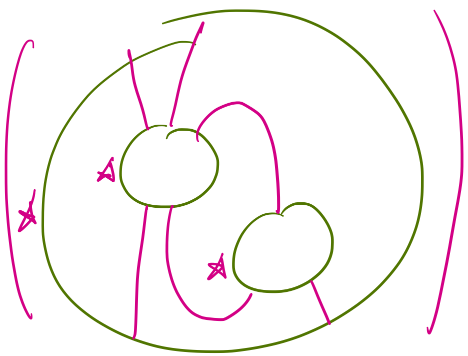
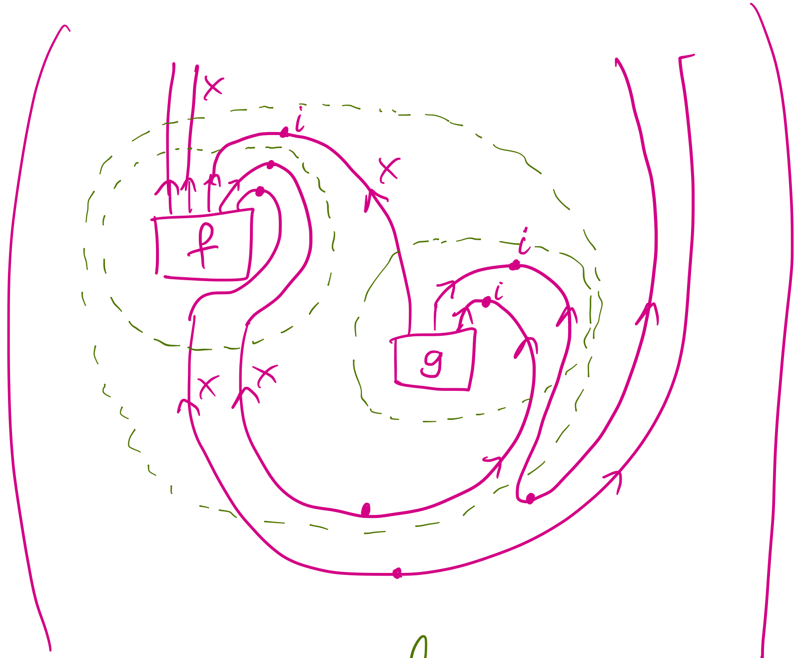
(i.e. we have 

and 

)

we define a planar algebra $P_{\mathcal{C}}$ by

$$(P_{\mathcal{C}})_n := \mathcal{C}(\mathbb{1} \to X^{\otimes n})$$ and

$P_{\mathcal{C}}$ () $(f \otimes g) := \mathrm{eval}$ (

)

<u>Exercise</u> explain carefully why this satisfies the axioms of a planar algebra!

(In fact this is much easier if we work with the <u>oriented</u> planar algebras you defined earlier; then we don't need to ask that X is symmetrically self-dual.)

<u>Theorem</u> $\mathcal{C}_{\mathcal{P}_{\mathcal{C},X}} \underset{\text{pivotal}}{\cong} \langle X \rangle_{\mathcal{C}}$ (the full subcategory of $\mathcal{C}$ on objects $X^{\otimes n}$)

<u>Theorem</u> $\mathcal{P}_{\mathcal{C}_{\mathcal{P}},1} \underset{\text{P.A.}}{\cong} \mathcal{P}$

(see e.g. arXiv:0810.4186)

At first it feels disconcerting to hear

"$\mathrm{Obj}\,\mathcal{C}_p = \mathbb{N}$"

What about examples like $\mathrm{Rep}\,G$, where the objects are representations, and decompose uniquely as direct sums of irreps?

Slogan "objects don't matter"

We recover the objects we expect to see by taking the idempotent completion / Karoubi envelope.

For any 1-category $\mathcal{C}$,

$$\mathrm{Obj}\,\mathrm{Kar}\,\mathcal{C} := \{(X:\mathcal{C},\ p:\mathcal{C}(X \to X),\ p^2 = p)\}$$

$$\mathrm{Kar}\,\mathcal{C}((X,p) \to (Y,q)) := \{(f:\mathcal{C}(X \to Y),\ p \circ f = f = f \circ q)\}$$

If $\mathcal{C}$ is monoidal/rigid/pivotal/..., so is $\mathrm{Kar}\,\mathcal{C}$ in a natural way.

We also need to formally adjoin direct sums.

$\mathrm{Mat}(\mathcal{C})$ has objects words in $\mathcal{C}$

and $\mathrm{Mat}(\mathcal{C})(\oplus X_i \to \oplus Y_j) = \left\{ \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} : \mathcal{C}(X_i \to Y_j) & \cdots \\ & \vdots & \end{pmatrix} \right\}$

with composition by matrix multiplication.

(again, when $\mathcal{C}$ is monoidal/rigid/pivotal/etc $\mathrm{Mat}(\mathcal{C})$ inherits this)

Exercise
- show $\mathrm{Kar}(\mathcal{C}) \cong \mathcal{C}$
- if $\mathcal{C}$ already has direct sums, show $\mathrm{Mat}(\mathcal{C}) \cong \mathcal{C}$.

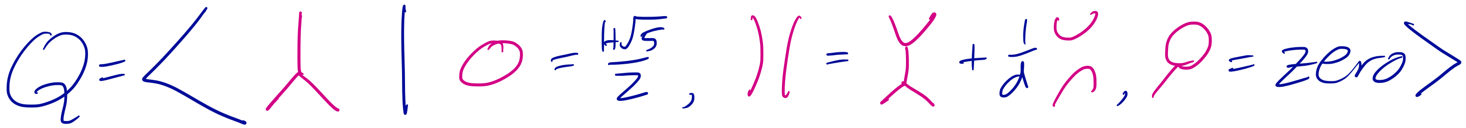
$\mathrm{Mat}(\mathrm{Kar}(\mathcal{C}))$ is the "Cauchy completion"

Finally, without having to use abelian categories,
we can say $\mathcal{C}$ is semisimple if
there exist objects X_i so

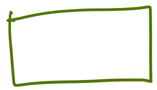
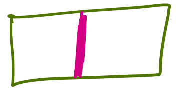
$$\dim \mathcal{C}(X_i \to X_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

and every object is isomorphic to a direct sum of the X_i's
(or, more generally, if $\mathrm{Mat}(\mathrm{Kar}(\mathcal{C}))$ satisfies this condition)

Let's return to our skein theory



and study the idempotents.

We have $\mathbb{1} =$ , and $X =$ .

<u>Claim</u> every idempotent is a direct sum of copies of $\mathbb{1}$ and X,
and in particular $X \otimes X \cong \mathbb{1} \oplus X$.

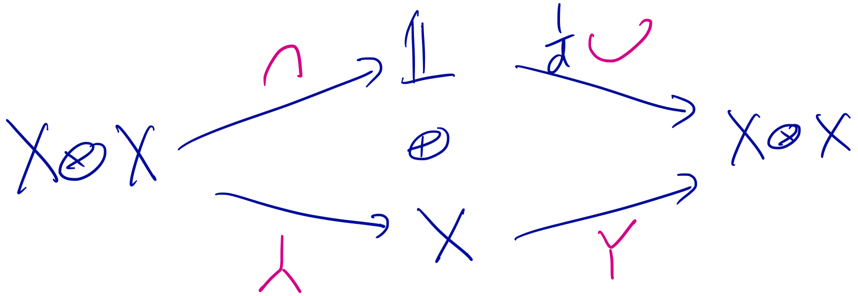
What does $A \cong B \oplus C$ even mean? We have morphisms

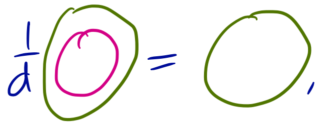
$$\begin{tikzcd}
 & B \arrow[dr, "\iota_1"] & \\
A \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \oplus & A, \\
 & C \arrow[ur, "\iota_2"'] &
\end{tikzcd}$$

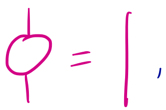
so $\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \gg (\iota_1 \quad \iota_2) = (\pi_1 \gg \iota_1 + \pi_2 \gg \iota_2) = id_A,$

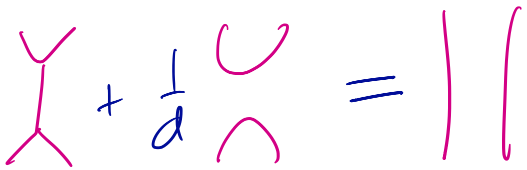
and $(\iota_1 \quad \iota_2) \gg \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \iota_1 \gg \pi_1 & \iota_2 \gg \pi_2 \\ \iota_1 \gg \pi_2 & \iota_2 \gg \pi_2 \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ 0 & id_C \end{pmatrix}.$

We can construct an explicit isomorphism $X \otimes X \cong \mathbb{1} \oplus X$ as follows:



and check:  , the identity on $\mathbb{1}$.

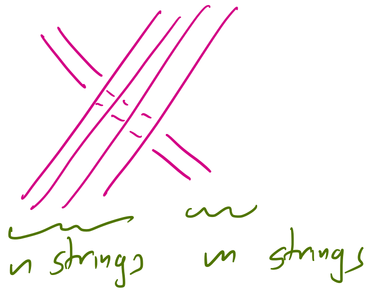
 , the identity on X,

and  is exactly the relation we just discovered!

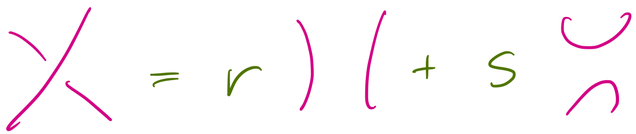
Exercises

- explain why there are no maps between $\mathbb{1}$ and X.
- prove $X^{\otimes n} \cong F_{n-1} \mathbb{1} \oplus F_n X$, where the F_n are Fibonacci numbers
- argue that every object in the idempotent completion is a subobject of $X^{\otimes n}$ for some n, and hence show the idempotent completion is semisimple.

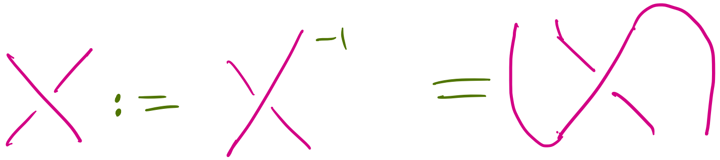
Exercises

- Explain which axiom for a braiding ensures that if $\beta_{1,1} =$ , then $\beta_{n,m} =$ 

Thus in Fib a braiding is determined by $r, s : \mathbb{C}$,



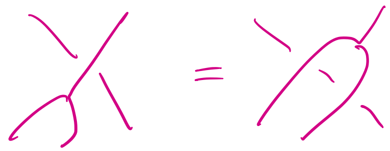
- Explain, from the axioms for a braiding, why



and solve for r and s

- Explain why the only remaining conditions are

and

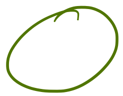


and verify these.

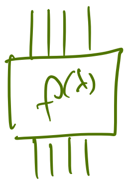
D_{2n} categories

(following arXiv:0808.0764)

We've seen already that the Temperley-Lieb-Jones category is the "simplest possible" pivotal category from the "generators and relations" point of view —

- generators: an unoriented arc 
- relations:  $= q + q^{-1}$

(and, when q is a primitive $(2\lambda+2)$th root of unity,

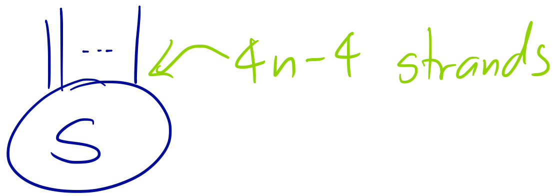


= zero)

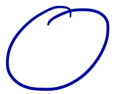
We'll now study the "next simplest", with a single generator.

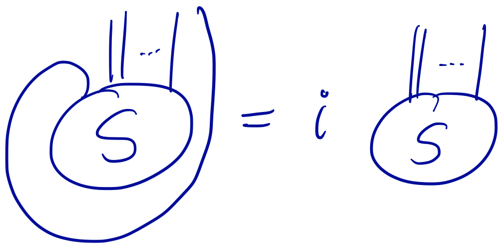
Defn Fix $n \geq 1$, and let $q = \exp\left(\frac{\pi i}{4n-2}\right)$.

Consider the pivotal category/planar algebra/skein theory with generator



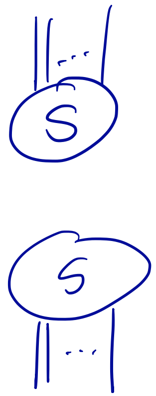
modulo the relations

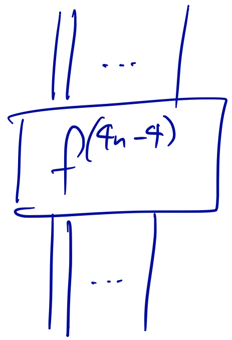
①  $= q + q^{-1}$

② 

③ 

$=$ zero

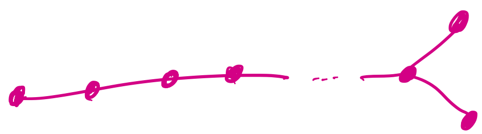
④ 

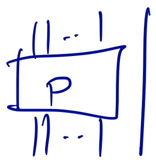
$= [2n-1] \cdot$ 

Theorem

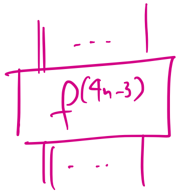
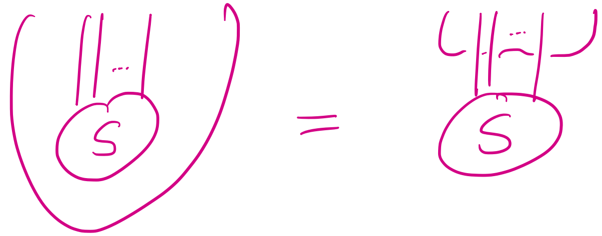
(A) every closed diagram is a scalar multiple of the identity (the tensor unit is simple)

(B) the isomorphism classes of simple idempotents are indexed by vertices of the D_{2n} Dynkin diagram

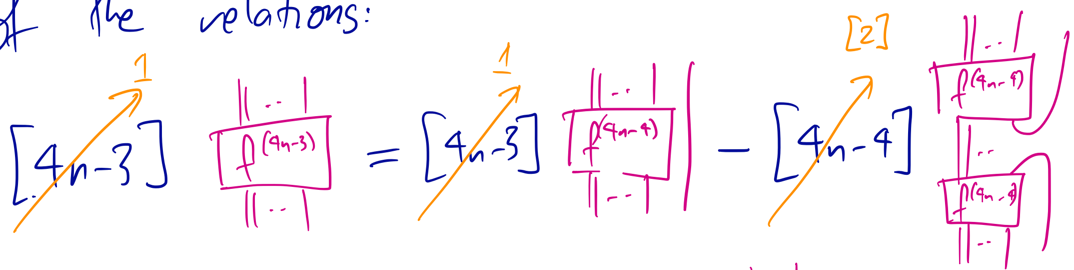


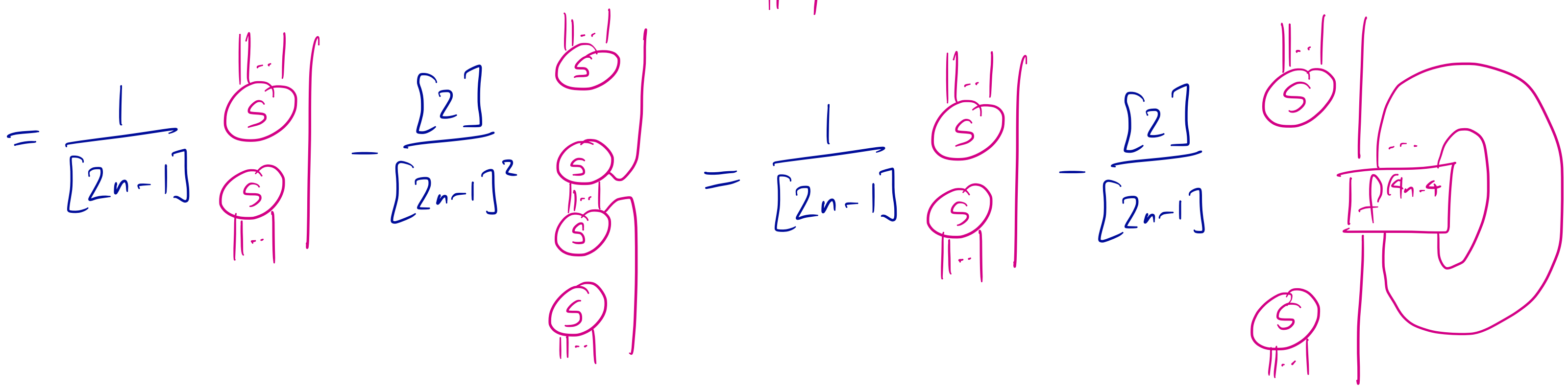
and  is isomorphic to the direct sum of the idempotents associated to the vertices adjacent to P in this graph.

To prove either of these theorems we need

 = zero and 

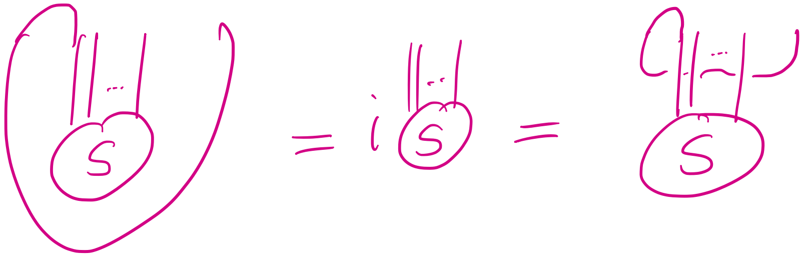
Note $f^{(4n-3)} \in \mathcal{N}$ at $q = \exp\left(\frac{\pi i}{4n-2}\right)$. In fact it is zero as a consequence of the relations:



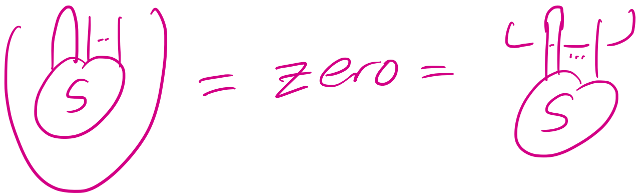


= zero

Then we check each way of adding a cap to the 2nd equation gives zero:

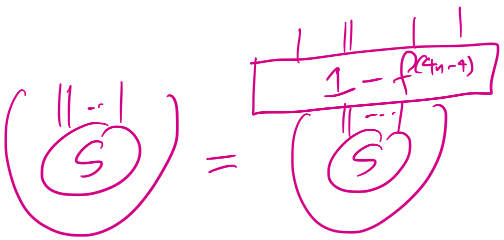


(by expanding out the TLJ braiding!)

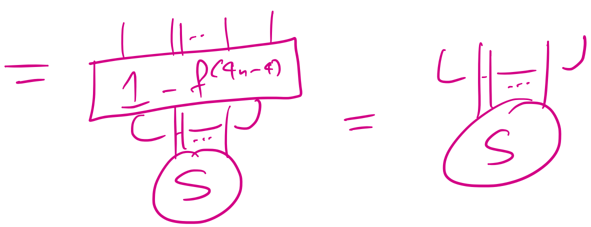


etc

Now



every term here has a cap

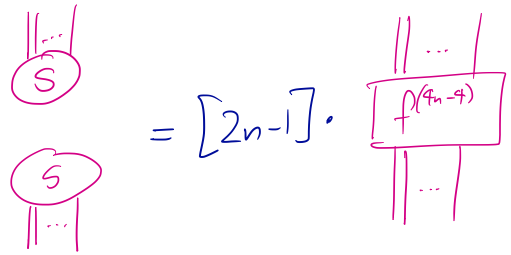


Note, however, that

Now prove theorem (A) — every closed diagram can be evaluated using these relations:

1. Pick a pair of S's, and slide them towards each other using the 'half-braiding'

2. Once they're adjacent, remove them using

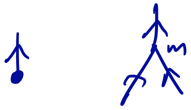


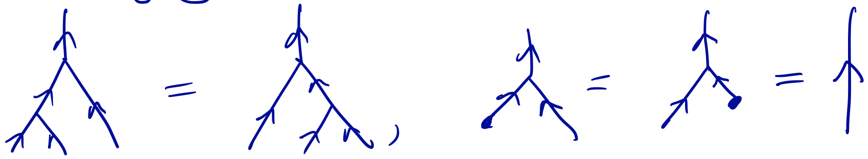
3. Repeating (1), we get down to 0 or 1 S's.
4. If there's 1, it must have a cap, so return zero
5. Otherwise, use $\bigcirc = q+q^{-1}$ to evaluate to a number.

In fact, one can prove this algorithm is independent of choices and respects all the defining relations, giving a direct proof of the consistency of this presentation! (See arXiv:0808.0764)

Algebras

An <u>algebra</u> (in a monoidal category $\mathcal{C}$) consists of:

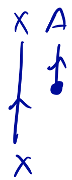


satisfying


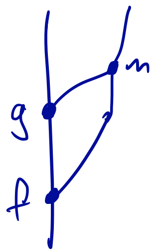
We can define the <u>projective modules</u> for an algebra A entirely diagrammatically.

First define fmod-A, a category, with

- $\mathrm{Obj}(\text{fmod-}A) = \mathrm{Obj}(\mathcal{C})$
- $\text{fmod-}A(X \to Y) = \mathcal{C}(X \to YA)$

with identity $1_X :=$


and

composition $f \gg g :=$


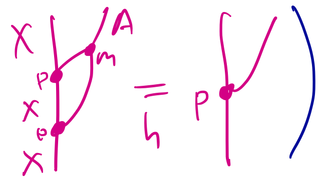
<u>Exercise</u> verify composition is associative!

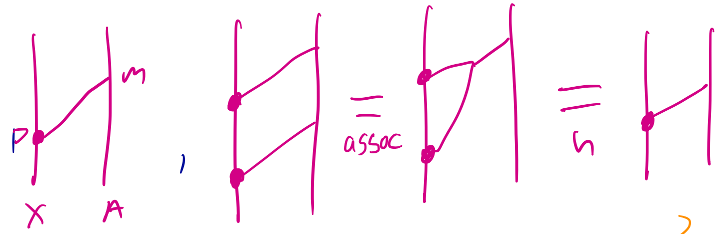
We then define $\mathrm{pmod}\text{-}A := \mathrm{Kar}(\mathrm{fmod}\text{-}A)$.

What does this have to do with the usual $\mathrm{mod}\text{-}A$, which is defined to be the category of A-module objects internal to $\mathcal{C}$?

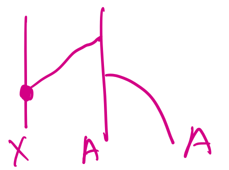
In general, even in Vec, they disagree — plenty of (honest) algebras have modules which are not projective. We might expect to need A semisimple.

There is a functor, however, $i : {}_{\mathcal{C}}\mathrm{pmod}\text{-}A \longrightarrow {}_{\mathrm{Kar}\,\mathcal{C}}\mathrm{mod}\text{-}A$

sending $(X:\mathcal{C},\ p:\mathcal{C}(X\to XA),$  $)$

to $((XA,$  $))$,

the underlying object in Kar $\mathcal{C}$

$\triangleleft :=\Big($  , Exercise: this is a map in Kar $\mathcal{C}$ $\Big)$, Exercise: this map satisfies the axioms for a module object action $\Big)$

Exercise define the functor $i: {}_e\mathrm{pmod}\text{-}A \xrightarrow[\text{Kore}]{} \mathrm{mod}\text{-}A$ at the level of morphisms, and verify functoriality.

Exercise the functor i is fully-faithful.

Theorem If A is a special Frobenius algebra in $\mathcal{C}$, then the functor $i: \mathrm{pmod}\text{-}A \longrightarrow \mathrm{mod}\text{-}A$ is an equivalence of categories.

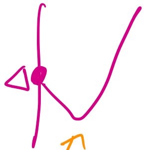
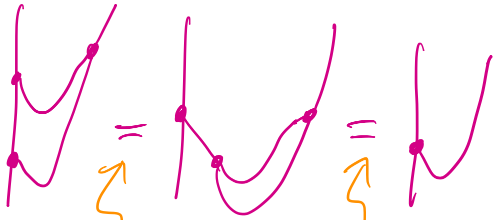
Definition a special Frobenius algebra in a pivotal category $\mathcal{C}$ is a symmetrically self-dual algebra further satisfying

Exercise: how does this relate to the usual definition?

Proof We just need to verify i is essentially surjective.

Given $(m, \triangleleft)$: mod-A, let's define a projective module

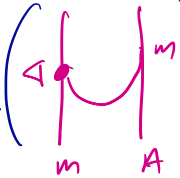
(for simplicity, let's assume $\mathcal{C}$ is idempotent complete, so we don't need to worry about Kar $\mathcal{C}$)

$($X := m$,$ p :=  , h := 

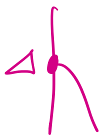
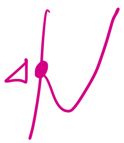
the coevaluation

axiom for $\triangleleft$

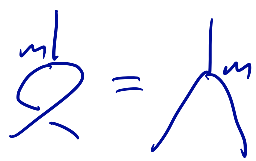
specialness

Now $i(X, p, h) =$ image()

which is isomorphic (in mod-A) to the original m,

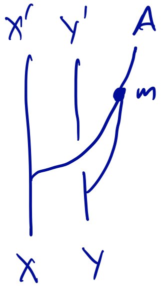
via the maps  and  .

Exercise: verify this isomorphism, carefully noting when you need to use specialness.

Now, if $\mathcal{C}$ is braided, and the algebra A is commutative, so 

we can define a monoidal structure on fmod-A:

on objects, the tensor product is the same as in $\mathcal{C}$

on morphisms, $f \otimes g =$ 

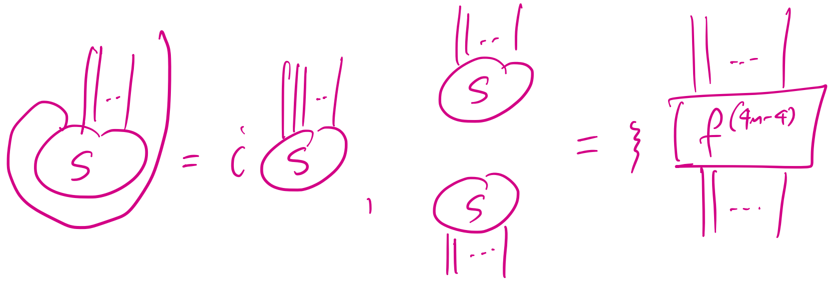
Exercise which axiom for a monoidal category do we need commutativity for?

This then gives a monoidal structure on pmod-A = Kar(fmod-A).

We'd like to describe an intrinsic generators and relations presentation of fmod-A.

The story so far:

- the D_{2n} categories, with relations



are evaluable, but we haven't yet computed the minimal idempotents and their tensor products

- the categories fmod-A, pmod-A $:=$ Kar(pmod-A), and mod-A, and a diagrammatic proof that

$$ {}_e\mathrm{Pmod}\text{-}A \cong_{\mathrm{Kare}} \mathrm{mod}\text{-}A $$

when A is a special Frobenius algebra.

<u>Theorem</u> Suppose $\mathcal{C} = \langle X_\alpha, r_\beta \rangle$ is a presentation of a braided monoidal category.

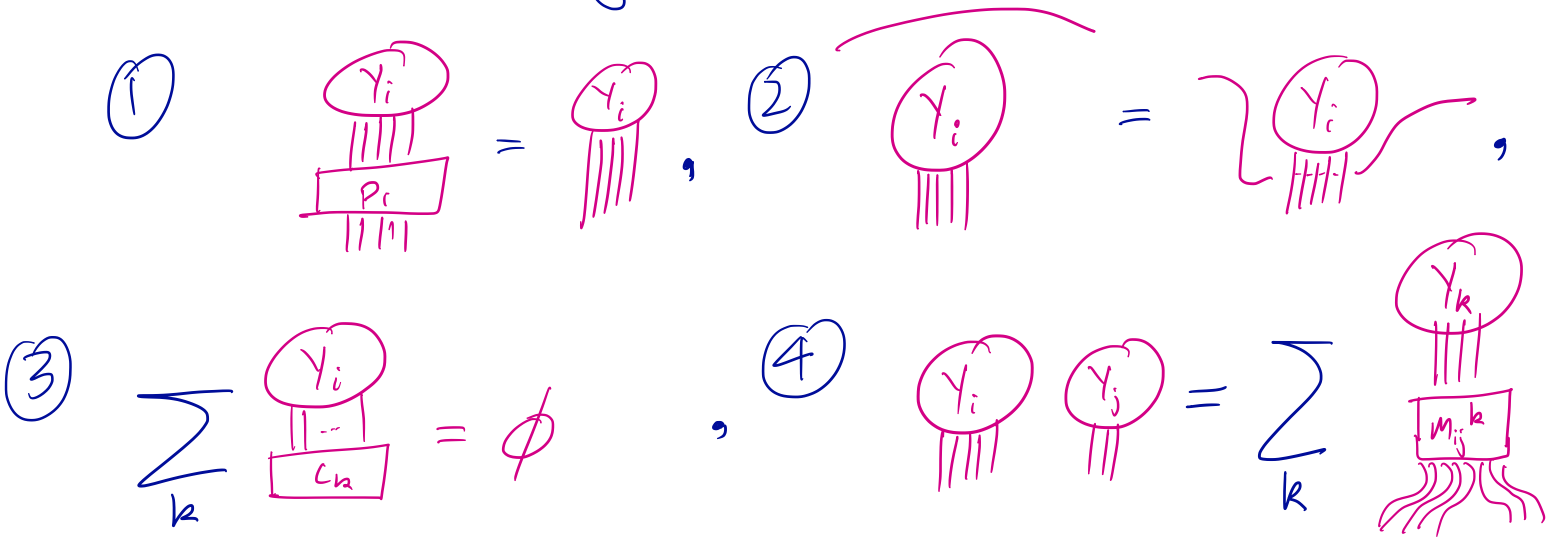
Say A is an algebra in $\mathcal{C}$, and $A \cong \bigoplus P_i$,

where the P_i are minimal projections in $\mathcal{C}$.

We have $m = \sum_{i,j,k} m_{ij}^k$, where $m_{ij}^k : P_i \otimes P_j \to P_k$,

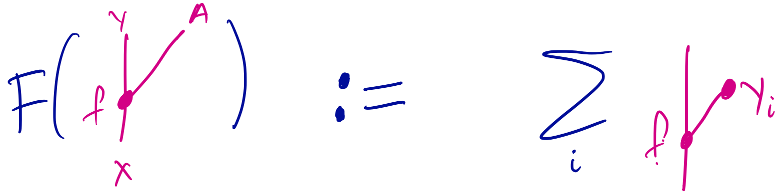
and $\iota = \sum \iota_k$, where $\iota_k : \mathbb{1}_{\mathcal{C}} \to P_i$

Then $\mathrm{fmod}\text{-}A$ has a presentation P, with generators $\{X_\alpha, Y_i\}$ with one Y_i for each P_i, and relations $\{r_\beta\}$ along with

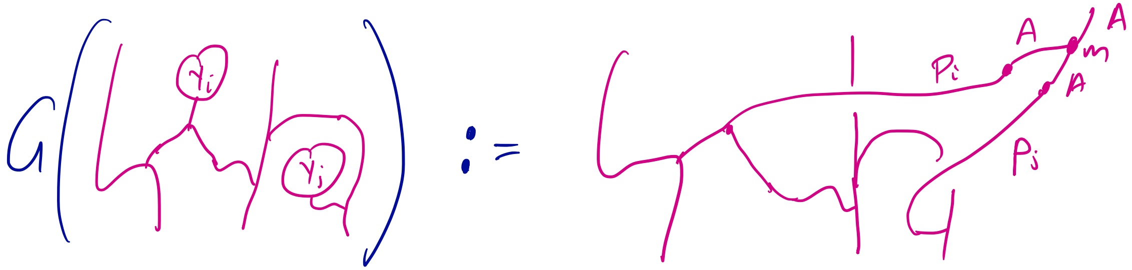


Proof sketch We'll define functors $F : \text{fmod-}A \rightleftarrows P : G$

(where P is the category given by the presentation).



(here,  denotes the projection from A to the summand P_i followed by Y_i)

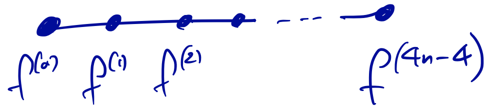


Exercises

- make this as precise as you like!
- why are F and G monoidal functors?
- why are F and G inverses?
- see that it all works if $\mathcal{C}$ is not braided but A is a commutative algebra in $Z(\mathcal{C})$.
- when is fmod-A pivotal?

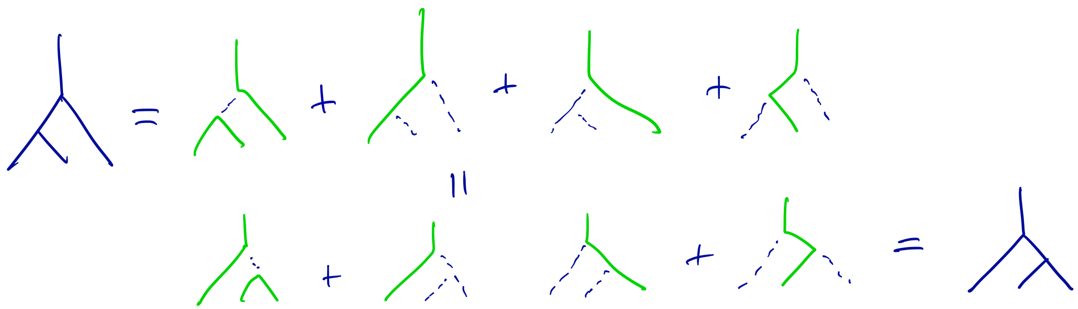
How does this explain the D_{2n} skein theory?

What are the algebras in TL at $q = \exp\left(\frac{\pi i}{4n-4}\right)$?

This is A_{4n-3}, with principal graph 

Claim $A := f^{(0)} \oplus f^{(4n-4)}$ has an algebra structure given by:

$A \otimes A =$	$f^{(0)}$ $\oplus$	$f^{(4n-4)}$ $= A$
$f^{(0)} \otimes f^{(0)}$		0
$\oplus$ $f^{(0)} \otimes f^{(4n-4)}$	0	
$\oplus$ $f^{(4n-4)} \otimes f^{(0)}$	0	
$\oplus$ $f^{(4n-4)} \otimes f^{(4n-4)}$		0

Sketch: 

where the interesting calculation is

which only holds at $q = \exp\left(\frac{\pi i}{4n-4}\right)$

Aside: in fact, we know all the algebra objects in TLJ at different roots of unity (Ostrik, arXiv:0111139, but also the earlier subfactor literature.)

Algebra objects. The simple algebra objects A in the monoidal categories $C_{k,m}$ are as follows.

TLJ at $q = \frac{m\pi i}{k+2}$, with principal graph A_{k+1}.

A	when	A-mod	commutative in $C^{\mathrm{br}}_{k,\ell}$
$\mathbf{1} \oplus f^{(k)}$	k even	$D_{\frac{k}{2}+2}$	if $k \equiv 0 \pmod 4$
$\mathbf{1} \oplus f^{(k)}$	k, m both odd	$T_{\frac{k+1}{2}}$	if $k\ell \equiv 3 \pmod 4$
$\mathbf{1} \oplus f^{(6)}$	$k = 10$	E_6	yes
$\mathbf{1} \oplus f^{(8)} \oplus f^{(16)}$	$k = 16$	E_7	no
$\mathbf{1} \oplus f^{(10)} \oplus f^{(18)} \oplus f^{(28)}$	$k = 28$	E_8	yes

See [Ost03a] for details. For details on the categories A-mod (in subfactor language) see [BN91; Izu91; Izu94; Jon01; Kaw95; Xu98]. When A is commutative the category A-mod has the structure of a fusion category. For details on the skein theory of these fusion categories see [Big10; MPS10]

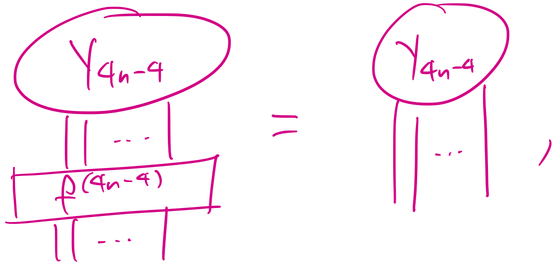
(from "A field guide to categories with A_n fusion rules", arXiv:1710.07362)

What is the skein theory for ${}_{TLJ}\mathrm{fmod}\text{-}(f^{(0)} \oplus f^{(4n-4)})$?

There are <u>two</u> new generators, corresponding to the simple summands of the algebra.

They satisfy:

①:

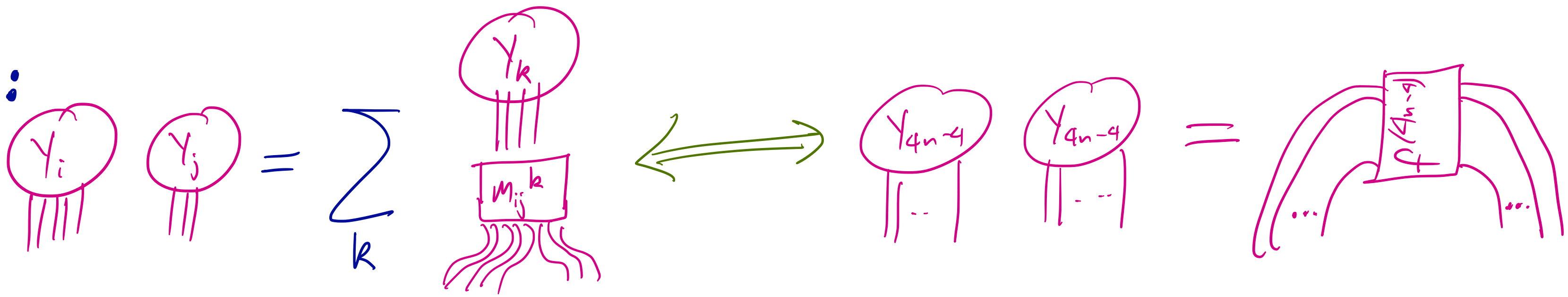


②:

③:

so we can leave out Y_0

④:



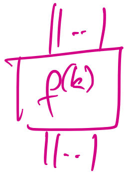
This is <u>exactly</u> our skein theoretic presentation for D_{2n}!

Exercise

- calculate the structure coefficients for the algebras in A_{11} and in A_{29}
- write down the corresponding skein theories for the free module categories
- in A_{29}, with $A = f^{(0)} \oplus f^{(10)} \oplus f^{(18)} \oplus f^{(28)}$, derive formulas in the skein theory for Y_{18} and Y_{28} in terms of Y_{10}, thereby showing the skein theory is singly-generated
- observe that this gives Bigelow's skein theory for E_8 from arXiv:0903.0144

(no one has ever done this exercise!)

Let's return to D_{2n}, and compute the minimal idempotents, and fusion rules.

- We have all the idempotents 

for $0 \le k \le 4n-4$

Are they still minimal, and non-isomorphic?

$$D_{2n}\left(f^{(k)} \to f^{(l)}\right) = \left\{ \right.$$

$\Bigg|$ g a planar diagram built out of S's, with $k+l$ boundary points $\left. \right\}$

Using the evaluation algorithm, we can ensure g has at <u>most one</u> 

.

If $k+l < 4n-4$, that 

has a cap, and so must be zero!

Therefore for $0 \le k \le 2n-1$, the $\boxed{f^{(k)}}$ are still minimal idempotents, and pairwise non-isomorphic.

Next: $\boxed{f^{(2n-2)}} = \frac{1}{2}\left(\boxed{f^{(2n-2)}} + S\right) + \frac{1}{2}\left(\boxed{f^{(2n-2)}} - S\right) =: P_+ + P_-$

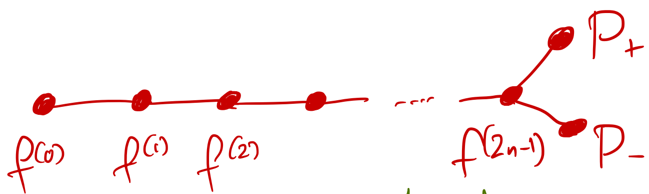
and the two terms are minimal idempotents which are non-isomorphic.

Exercise!

Finally: $\boxed{P_\pm}\,\Big| \;\simeq\; \boxed{f^{(2n-1)}}$

Exercise: in both directions, the morphism is just S.

Putting this all together, the minimal idempotents are



Exercise: we know how to tensor each of these with a strand. Explain why this shows every idempotent is a direct sum of these ones.

We can also do these calculations from the perspective of

$$ {}_{TLJ}\mathrm{fmod}-\left(f^{(0)} \oplus f^{(4n-4)}\right) $$

$$
\begin{aligned}
D_{2n}\left(f^{(k)} \to f^{(\ell)}\right) &\equiv TLJ\left(f^{(k)} \longrightarrow f^{(\ell)} \otimes \left(f^{(0)} \oplus f^{(4n-4)}\right)\right) \\
&\cong TLJ\left(f^{(k)} \longrightarrow f^{(\ell)} \oplus f^{(4n-4-\ell)}\right) \\
&\cong TLJ\left(f^{(k)} \to f^{(\ell)}\right) \oplus TLJ\left(f^{(k)} \to f^{(4n-4-\ell)}\right) \\
&= \begin{cases} \mathbb{C}^2 & \text{if } k=\ell=2n-2 \\ \mathbb{C}^1 & \text{if } k=\ell \text{ or } k=4n-4-\ell, \text{ but } k\neq \ell \\ \mathbb{C}^0 & \text{otherwise} \end{cases}
\end{aligned}
$$

This tells you how $f^{(k)}$ decomposes, but to compute the tensor product rules we'd need to use the algebra structure...

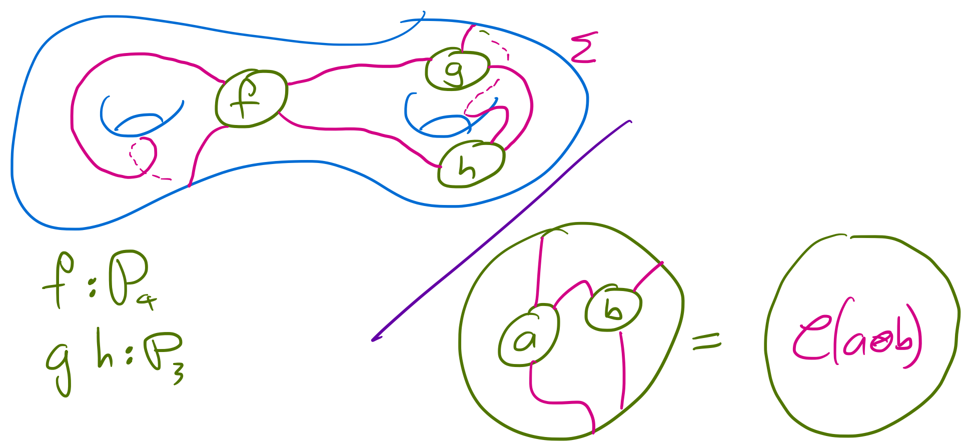
Possible next topics

- annular TL, lowest weight decompositions
  - triple point obstructions
- classification of small examples
  - by rank, index, global dimension, ...
- constructing exotic categories using graph planar algebras
- annular categories, Drinfeld centres, tube algebras
- string diagrams for n-categories

Drinfeld centres

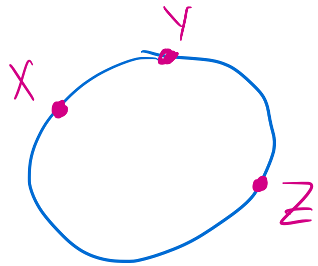
Given a planar algebra/pivotal category/skein theory $\mathcal{C}$, we immediately get a vector space

$$\int_\Sigma \mathcal{C} := \left\{ \quad \right\}$$

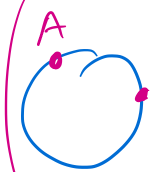


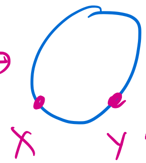
for any oriented surface Σ.

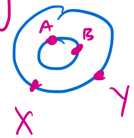
Going down just one dimsion, $\int_{S^1} \mathcal{C}$ is a 1-category:

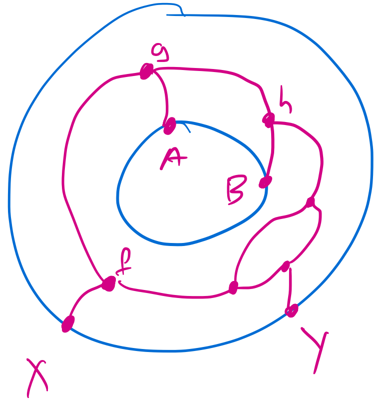
$\mathrm{Obj} \int_{S^1} \mathcal{C} := \Bigg\{$ 

$\Bigg| \; X, Y, Z : \mathrm{Obj}\, \mathcal{C} \Bigg\}$

$\left(\int_{S^1} \mathcal{C}\right)\Bigg($ 

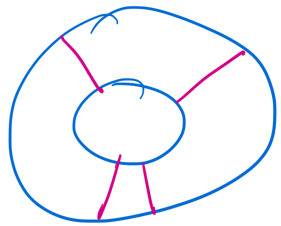
$\to$ 

$\Bigg) := \int$ 

$\mathcal{C} = \Bigg\{$ 

$\Big/ \sim \Bigg\}$

Composition is just by gluing annuli concentrically.

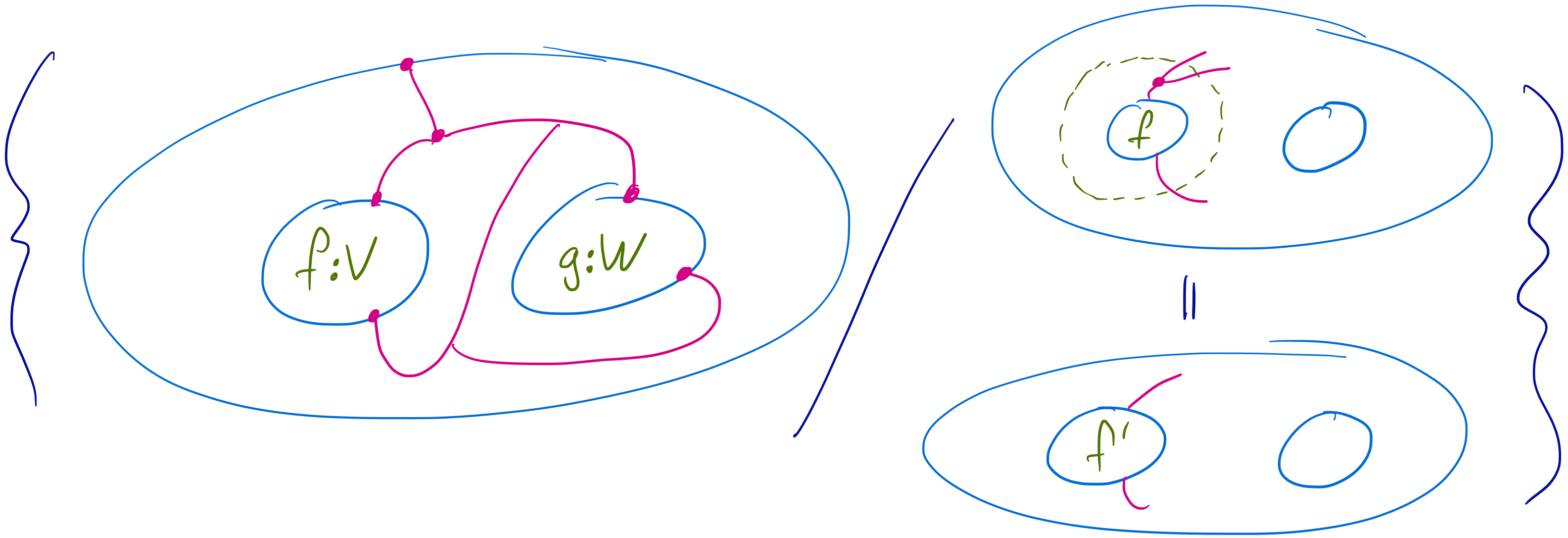
Identities are the morphisms  .

Exercise explain why checking associativity is the same as checking associativity in $\pi_1(X)$.

While $\int_{S^1}\mathcal{C}$ is only a 1-category, $\mathrm{Rep}\int_{S^1}\mathcal{C}$ naturally has the structure of a braided monoidal category. (Recall $\mathrm{Rep}\,\mathcal{D} := \mathrm{Fun}(\mathcal{D}\to\mathrm{Vec})$)

Given two representations $V, W : \mathrm{Rep}\int_{S^1}\mathcal{C}$,

define $V\otimes W :=$



where $f' := V\Big($  $\Big)(f)$

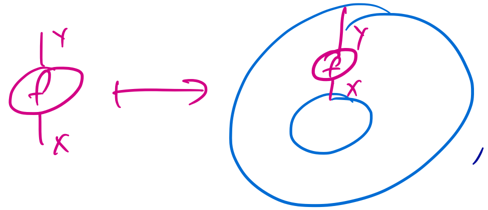
This gives a new representation of the category $\int_{S^1}\mathcal{C}$, by gluing annular diagrams on the outside.

The braiding $V\otimes W \longrightarrow W\otimes V$ is given by dragging the pictures around!

Let's understand why (and when) this agrees with the usual Drinfeld centre of a monoidal category:

$$\mathrm{Obj}\, Z(\mathcal{C}) := \{(X : \mathcal{C},\ \beta : X \otimes - \xrightarrow[\otimes]{\cong} - \otimes X\}$$

Because there is a functor $\mathcal{C} \longrightarrow \int_{S^1} \mathcal{C}$,



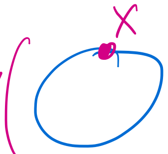
any $V : \mathrm{Rep} \int_{S^1} \mathcal{C}$ gives a representation of $\mathcal{C}$ (not a module category for $\mathcal{C}$ as a $\otimes$-category, just a representation of the underlying 1-category)

and if $\mathcal{C}$ is <u>semisimple</u> then every representation is representable, i.e. isomorphic to one of the form $Y \longmapsto \mathcal{C}(X \to Y)$ for some object $X : \mathcal{C}$.

We'll take that X as the underlying object of the object in $Z(\mathcal{C})$ that we're building.

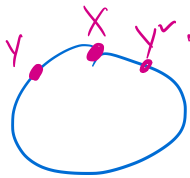
Now we need $\beta_Y : X \otimes Y \longrightarrow Y \otimes X$.

We have

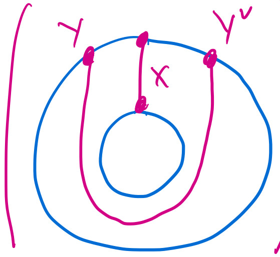
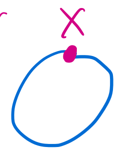
$$\mathrm{id} \in \mathcal{C}(X \longrightarrow X) \cong V\Big($$  $$\Big)$$

and, using pivotality,

$$\mathcal{C}(X \otimes Y \longrightarrow Y \otimes X) \cong \mathcal{C}(X \to Y \otimes X \otimes Y^\vee)$$

$$\cong V\Big($$  $$\Big)$$

$$\rotatebox{90}{$\in$}$$

$$V\Big($$  $$\Big)\Big(\text{“id”} \in V\big($$  $$\big)\Big)$$

Exercises

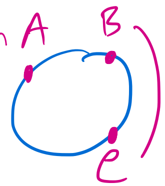
- check β_Y is an isomorphism
- check β_Y is natural in Y
- check β_Y is monoidal in Y

Exercise extend this to a functor $\mathrm{Rep}\int_{S^1}\mathcal{C} \to Z(\mathcal{C})$

(hint: use Yoneda. bonus: is this functor co- or contra-variant?)

In fact, this functor was 'the hard direction' — we needed to use semisimplicity of $\mathcal{C}$ to construct it.

The 'easy direction' $Z(\mathcal{C}) \to \mathrm{Rep}\int_{S^1}\mathcal{C}$ should send an object with a half-braiding (X, β) to the representation

$V($ 

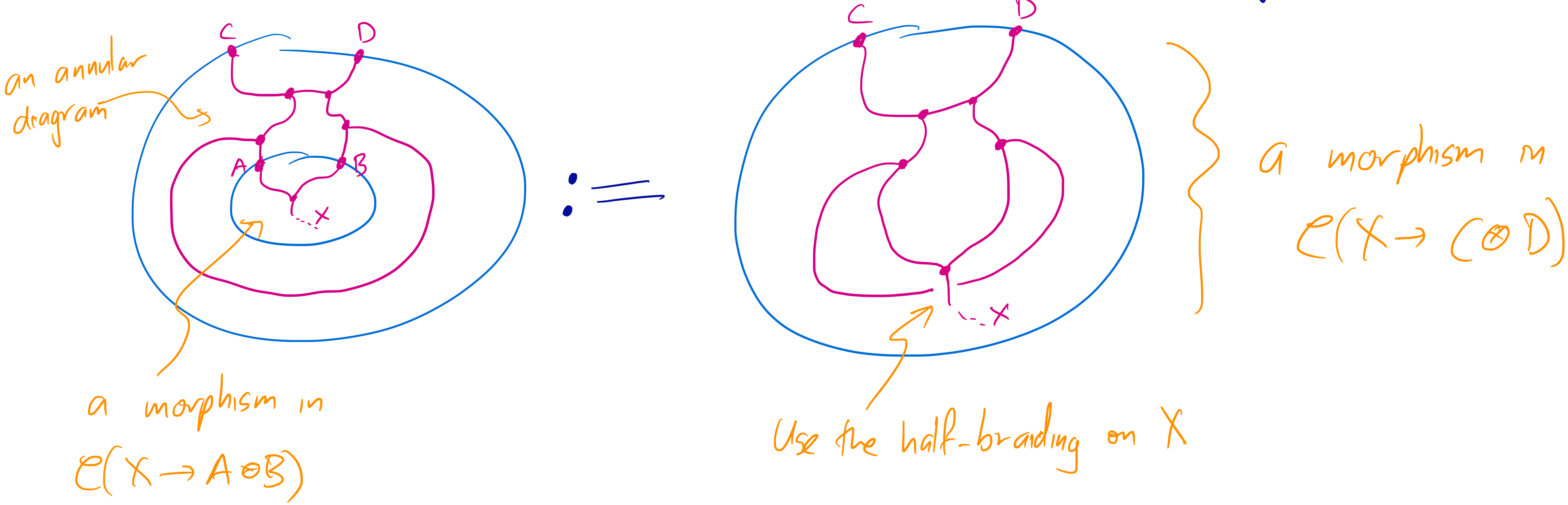
$) := \mathcal{C}(X \to A \otimes B \otimes C)$

but this isn't well-defined! We only have a cyclic ordering on the points around the circle, but need a linear ordering to take homs to.

Of course, the half-braiding saves us, giving isomorphisms

$$\mathcal{C}(X \to A \otimes B \otimes C) \to \mathcal{C}(X \otimes C^\vee \to A \otimes B) \to \mathcal{C}(C^\vee \otimes X \to A \otimes B) \to \mathcal{C}(X \to C \otimes A \otimes B)$$

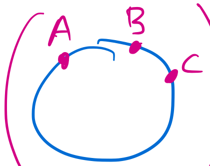
We also need to define the action of annular diagrams:



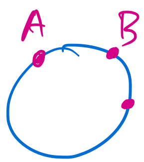
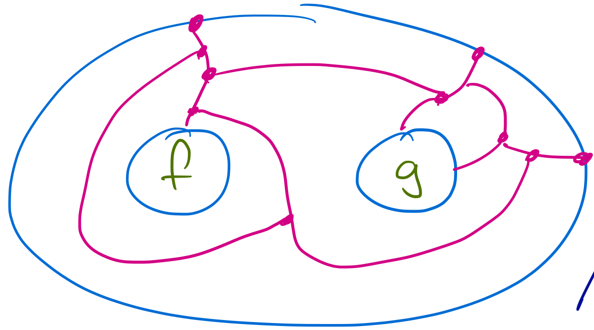
We still have many proof obligations:

- the functors $Z(\mathcal{C}) \rightleftarrows \mathrm{Rep} \int_{S^1} \mathcal{C}$ form an equivalence
- at least one of the functors is monoidal (Exercise: why just one?)
- that functor is braided
- (we may also want to define and compare pivotal structures on either side)

Let's check monoidality of the functor $\mathcal{V}: Z(\mathcal{C}) \to \mathrm{Rep}\int_{S^1}\mathcal{C}$.

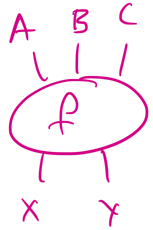
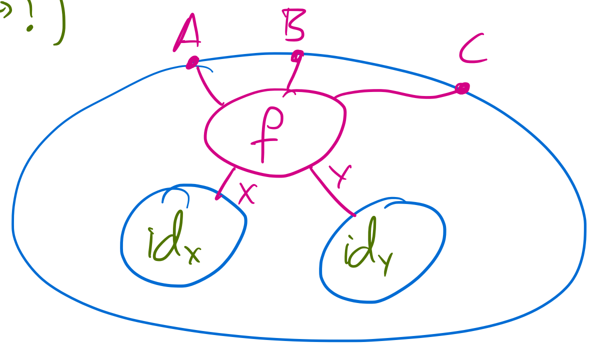
$$\mathcal{V}\Big(\underbrace{(X,\beta)\otimes(Y,\gamma)}\Big)\Big($$  $$\Big) = \mathcal{C}(X\otimes Y \longrightarrow A\otimes B\otimes C)$$

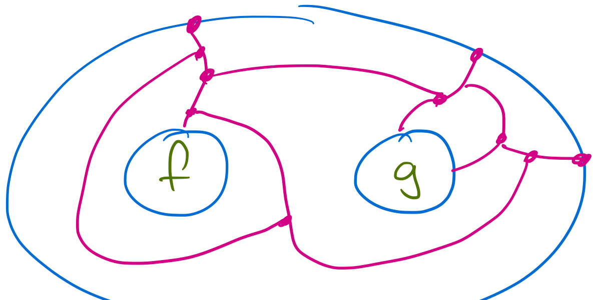
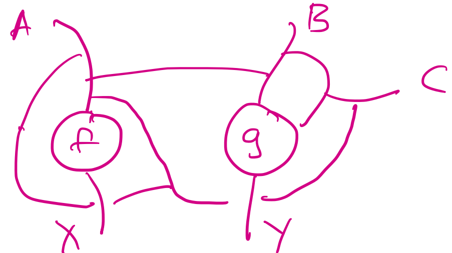
$$= \Big(X\otimes Y,\ X\otimes Y\otimes Z \xrightarrow{\gamma} X\otimes Z\otimes Y \xrightarrow{\beta} Z\otimes X\otimes Y\Big)$$

$$\Big(\mathcal{V}(X,\beta)\otimes\mathcal{V}(Y,\gamma)\Big)\Big($$  $$\Big) = \Big\{$$  $$\Big/ \sim \Big\}$$

$f : \mathcal{C}(X \to ?)$

$g : \mathcal{C}(Y \to ?)$

We can map one way via  $\longmapsto$ 

and the other via  $\longmapsto$ 

This interpretation of $Z(\mathcal{C})$ as $\mathrm{Rep} \int_{S^1} \mathcal{C}$ explains why physicists think of $Z(\mathcal{C})$ as classifying the 'point defects' or 'pointlike excitations' of the 2d lattice model realising $\mathcal{C}$.

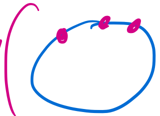
An object of $Z(\mathcal{C})$ is a collection of vector spaces describing the possible quantum states of an excitation, along with rules (the annular action) for how these states interact with the surrounding state of the bulk theory.

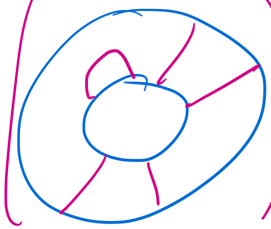
Let's describe the simplest example of a Drinfeld centre.

(to physicists, this example is "the toric code")

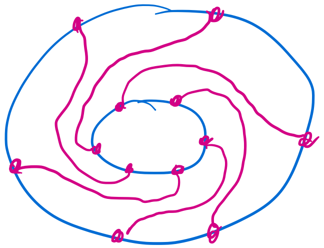
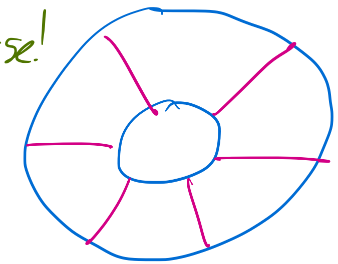
Let $\mathcal{C} = \mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z} = TLJ_{\delta=1} = \left\langle \text{no generators} \;\middle|\; \bigcirc = 1,\ \substack{\cup \\ \cap} =)(\right\rangle$.

A representation V of $\int_{S^1}\mathcal{C}$ consists of:

- for each $n:\mathbb{N}$, a vector space $V_n = V\Big($  $\Big)$

- however, $V\Big($  $\Big)$ gives an isomorphism (Exercise: what is the inverse?)

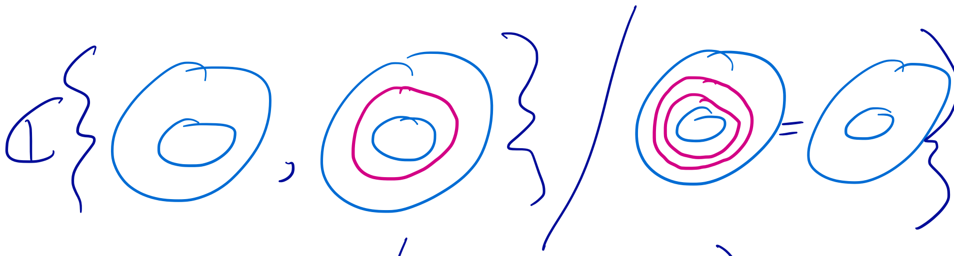
$$V_{n+2} \cong V_n$$

and since  $=$ (Exercise!)  wherever we put the cap we get the <u>same</u> isomorphism, and so the representation V is entirely determined by V_0 and V_1, and the annular tangles between these.

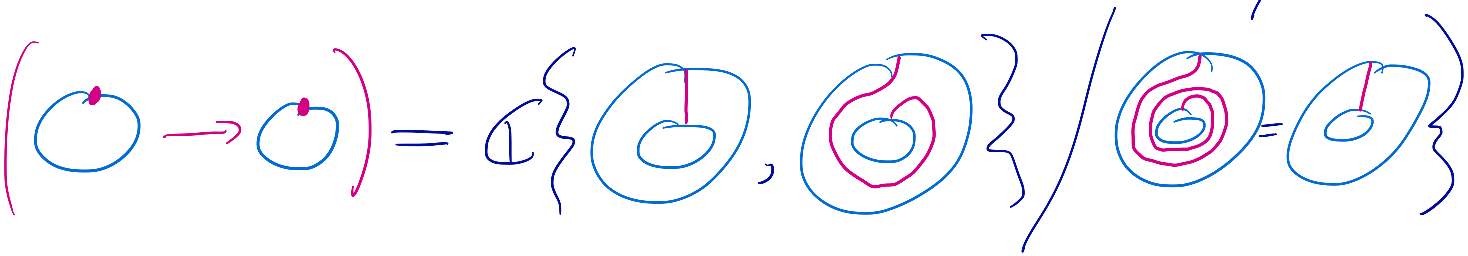
In fact, there are no maps between V_0 and V_1, so we get a direct sum decomposition into two pieces.

(equivalently, in any irrep $V_0 = 0$ or $V_1 = 0$)

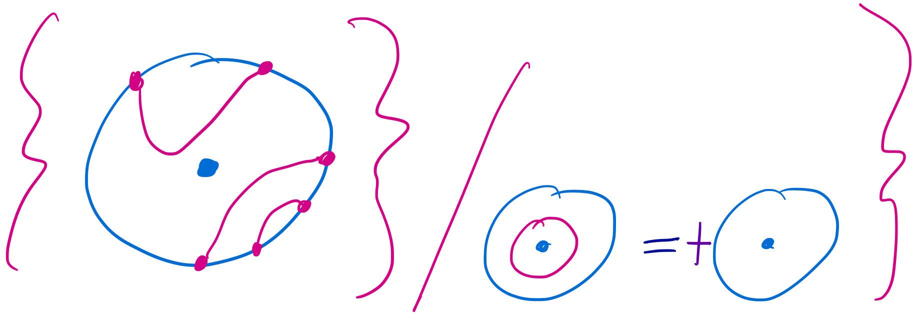
What is $\left(\int_{S^1} e\right)(\bigcirc \to \bigcirc)$? Just $\mathbb{C}$



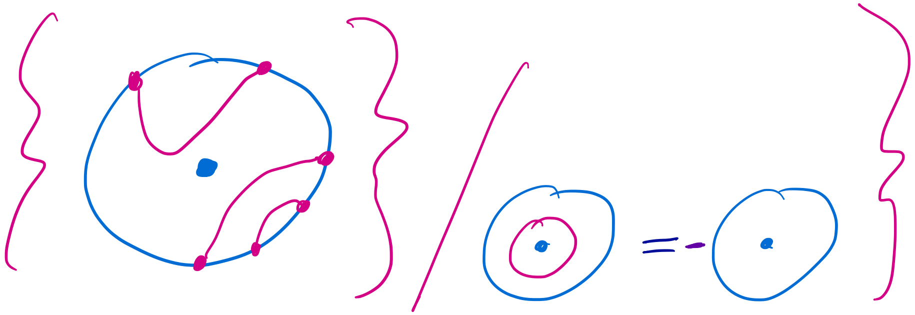
Similarly $\left(\int_{S^1} e\right)$



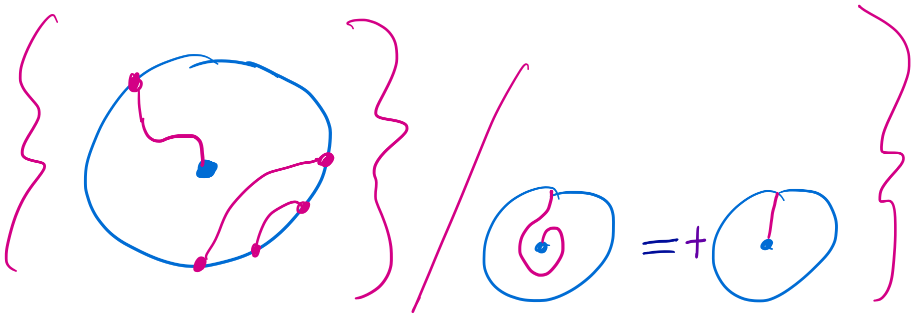
From these facts we have the classification of irreps:

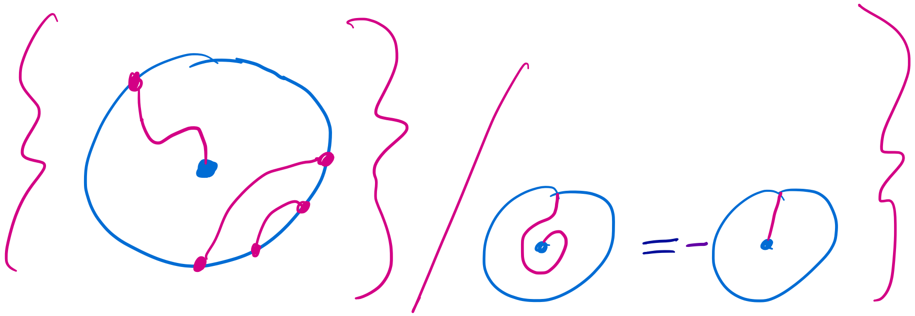
$V^{triv} =$ 

"magnetic"

$V^{m} =$ 

"electric"

$V^{e} =$ 

$V^{em} =$ 

Exercises

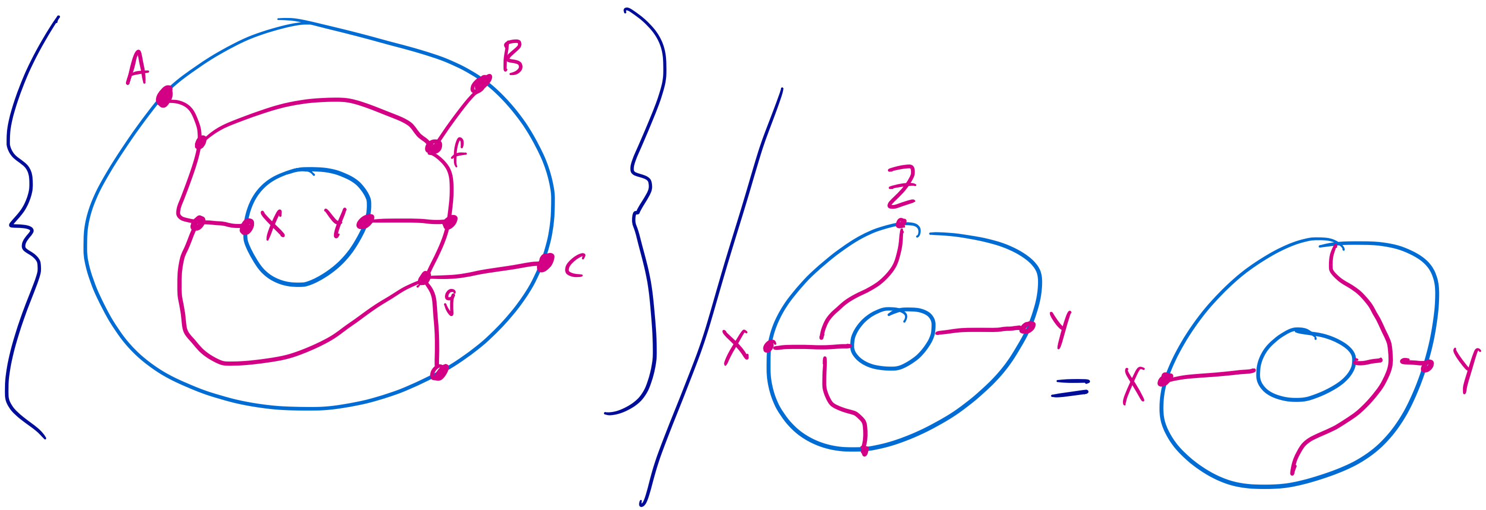
- calculate the tensor products
- calculate the braiding maps
- compute the S & T matrices
- verify $Z(\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z})$ is modular.

From this perspective we can describe the functor

$$\mathcal{G} : \mathcal{C} \otimes \mathcal{C}^{\text{bop}} \longrightarrow Z(\mathcal{C}) = \mathrm{Rep} \int_{S^1} \mathcal{C}$$

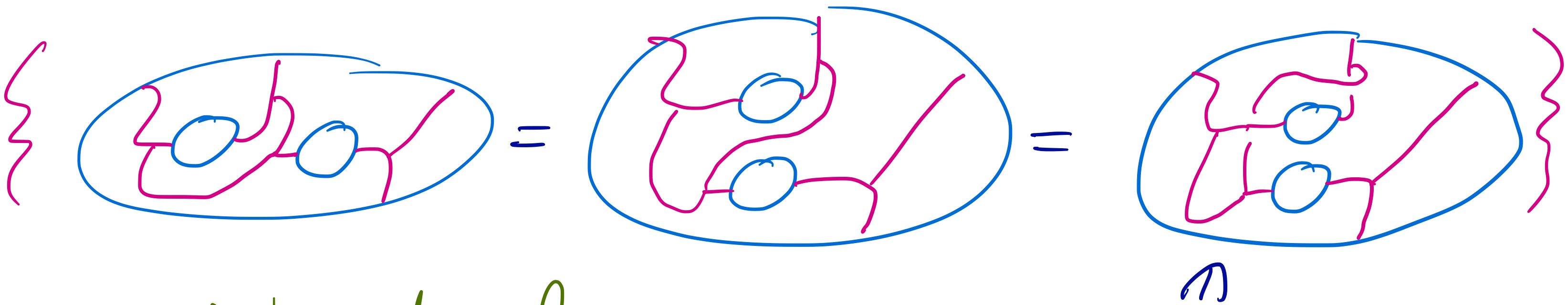
whenever $\mathcal{C}$ is braided.

Define $\mathcal{G}(X,Y) =$



Observe this is monoidal:

$\mathcal{G}(X,Y) \otimes \mathcal{G}(W,Z) =$



$\mathcal{G}(X \otimes W, W \otimes Z)$

Exercise check the details, and confirm the braiding is preserved.

Exercises

- When $\mathcal{C}$ is symmetric monoidal, show this functor factors through

$$\mathcal{C} \otimes \mathcal{C}^{\mathrm{bop}} \xrightarrow{\otimes} \mathcal{C} \longrightarrow Z(\mathcal{C})$$

- When $\mathcal{C}$ is modular, show this functor is an equivalence.

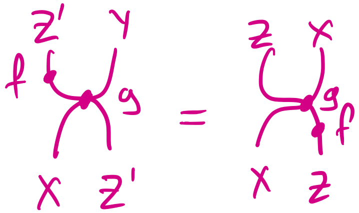
How do we concretely calculate $\mathcal{Z}(\mathcal{C})$?

When $\mathcal{C}$ is finitely semisimple, $\int_{S^1} \mathcal{C}$ is equivalent (after Cauchy completion) to a certain finite-dimensional associative algebra, the tube algebra.

$$\mathrm{Tube}(\mathcal{C}) := \bigoplus_{X,Y,Z:\mathrm{Irr}\,\mathcal{C}} \mathcal{C}(X\otimes Z \to Z\otimes Y) \Big/$$

for all $f : \mathcal{C}(Z \to Z')$

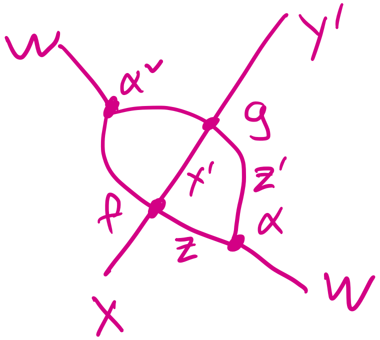
$g : \mathcal{C}(X\otimes Z' \to Z\otimes Y)$



Composition sends $f : \mathcal{C}(XZ \to ZY)$

$g : \mathcal{C}(X'Z' \to Z'Y')$

to $\delta_{X'=Y} \sum_{W:\mathrm{Irr}\,\mathcal{C}} \sum_{\alpha : \text{a basis of } \mathcal{C}(W \to ZZ')}$

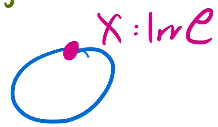


(Here $\alpha^\vee$ is the dual basis element under the pairing

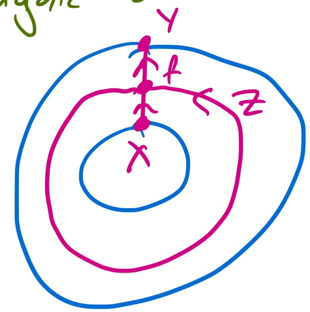
$\mathcal{C}(W \to ZZ') \otimes \mathcal{C}(ZZ' \to W) \xrightarrow{\circ} \mathcal{C}(W \to W) = \mathbb{C}$)

Exercise

- every object in $\int_{S^1}\mathcal{C}$ is a direct sum of objects



- every morphism in $\int_{S^1}\mathcal{C}$ is conjugate to a matrix of morphisms



$X, Y, Z : \mathrm{Irr}\,\mathcal{C}$

$f : \mathcal{C}(XZ \to ZY)$

- use this to prove the equivalence

$$\mathrm{Mat}(\mathrm{Kar}(\mathrm{Tube}\,\mathcal{C})) \simeq \mathrm{Mat}\left(\mathrm{Kar}\left(\int_{S^1}\mathcal{C}\right)\right)$$

(hint: to define a functor out of $\mathrm{Mat}(\mathrm{Kar}(\mathbb{D}))$ it suffices to define a functor out of $\mathbb{D}$.)

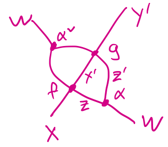
Let's calculate $\mathrm{Tube}(\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z})$.

$$\mathcal{C}(XZ \to ZY) \cong \begin{cases} \mathbb{C} & \text{if } X = Y \\ 0 & \text{otherwise} \end{cases}$$

So $\mathrm{Tube}(\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C}^4$. But what is the algebra structure?

We could compute it using the formula!

$$fg = \delta_{X'=Y} \sum_{W:\text{Irre}} \sum_{\substack{\alpha:\text{ a basis} \\ \text{of} \\ \mathcal{C}(W \to ZZ')}}$$



But there's a shortcut: the S factor shows the algebra has a direct sum decomposition, into $X = Y = \mathbb{C}^{\text{triv}}$ and $X = Y = \mathbb{C}^{\text{sign}}$ pieces.

Since $\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z}$ is unitary, $\mathrm{Tube}(\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z})$ must be semisimple,

so $\mathrm{Tube}(\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

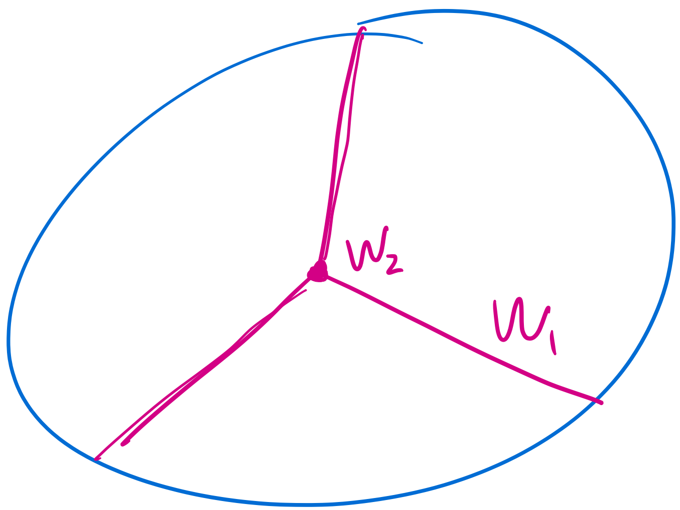
Exercise compute $\mathrm{Tube}(\mathrm{Rep}\,\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Tube}(\mathrm{Fib})$ explicitly.

What is ... a string diagram n-category?

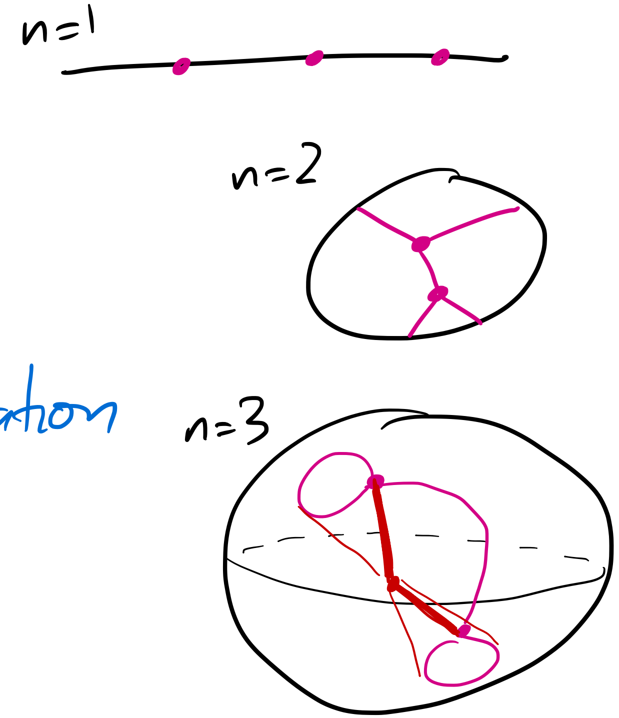
A stratification of an n-manifold W is a sequence

$$W = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n$$

so $W_k \setminus W_{k+1}$ is a codimension k submanifold of W.



It is a string diagram stratification if every $x \in W_k \setminus W_{k+1}$ has a nbhd in W which is a product of its nbhd in W_k and the cone over a string diagram stratification of some $(n-k-1)$-sphere.

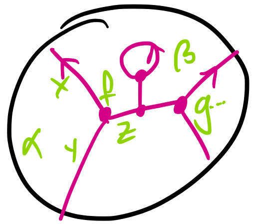


Equivalently, if:

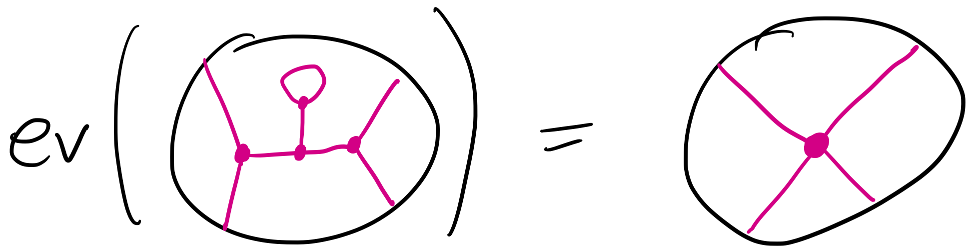
- it is the 0-ball with its unique stratification
- it is $W \times I$ for some W with an SD stratification
- it is $\mathrm{cone}(W)$, where W is an SD stratification of a sphere
- it is $W_1 \sqcup W_2$, where W_1 and W_2 have SD stratifications, or
- it is W glued to itself along Y, where W has an SD stratification, and both restrictions to Y give the same stratification.

[Siebenmann '72!]

If "$\mathcal{C}$ is an n-category", an m-dimensional $\mathcal{C}$-string diagram is an SD-stratified ball W_0^m with each $(m-k)$ ball in $W_k \setminus W_{k+1}$, labelled by some k-morphism of $\mathcal{C}$.



Any reasonable definition of a (pivotal) n-category lets you <u>evaluate</u> a $\mathcal{C}$-string diagram to an m-morphism, so isotopic $\mathcal{C}$-string diagrams have the same evaluation.



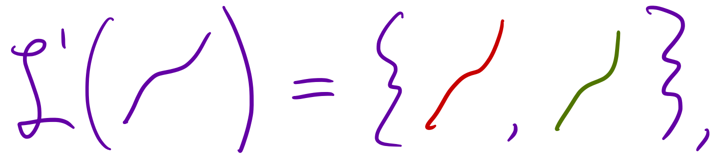
We will <u>define</u> a n-category by axiomatising just this observation.

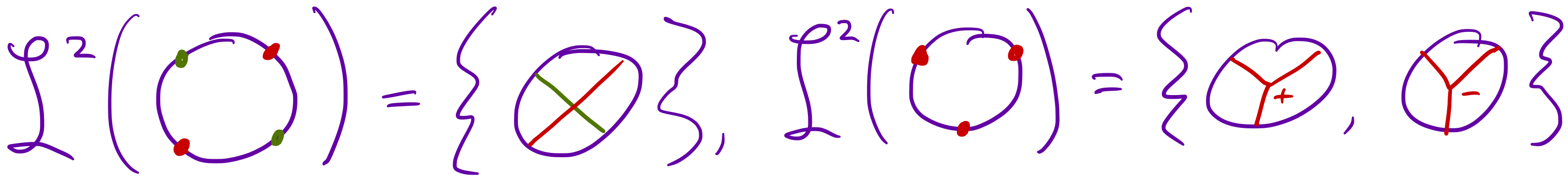
An n-dimensional signature $\mathcal{L}$ consists of

- for each $0 \le k \le n$
- for each oriented* k-ball X,
- for each string diagram stratification α of ∂X
- for each labelling c of each stratum $s \in \alpha$ by an element of $\mathcal{L}^{k-1}$(normal disc of s)

a set $\mathcal{L}^k(X; c)$.

*oriented, unoriented, spin, conformal, equipped with a map to T, etc

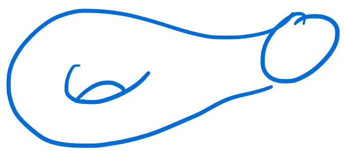
Example: $\mathcal{L}^0(\cdot) = \{\cdot\}$, 



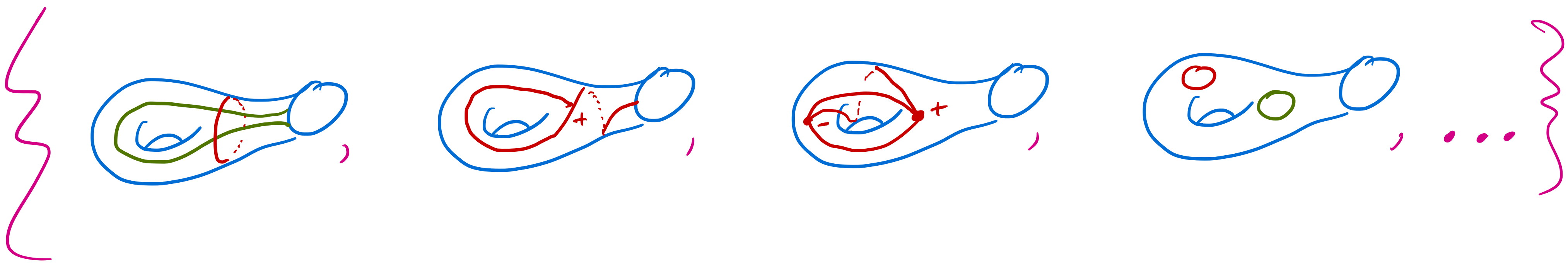
all other $\mathcal{L}^2$ empty.

For a fixed signature $\mathcal{L}$, a string diagram on M^n is a string diagram stratification of M, with each stratum labelled by an element of $\mathcal{L}(\text{normal disc})$

<u>Example</u> with $\mathcal{L}$ as before,

$\mathcal{L}$ string diagrams on 

$\parallel$



A *string diagram n-category* $\mathcal{C}$ consists of:

- a n-signature $\mathcal{L}$
- for each n-ball X with a $\mathcal{L}$-string diagram c in ∂X

  a subspace

$$\mathcal{U}(X;c) \subset \mathbb{C}\left\{\begin{array}{l}\mathcal{L}\text{-string diagrams on } X \\ \text{with boundary } c\end{array}\right\}$$

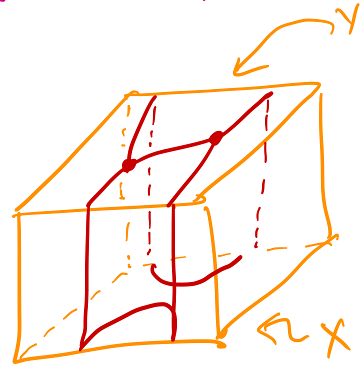
— such that $\mathcal{U}$ forms an ideal with respect to gluing balls

— if f and g are $\mathcal{L}$-string diagrams on X with the same boundary c, such that f and g are *isotopic* rel ∂, then $f-g \in \mathcal{U}(X;c)$.

Without pinning ourselves down to a particular 'traditional' definition of (weak!) n-category, let's sketch the construction of one.

$$\hat{\mathcal{C}}^0 = \mathcal{L}^0(\cdot), \qquad \hat{\mathcal{C}}^1(X \to Y) = \left\{ \begin{array}{l} \text{string diagrams on } [0,1] \text{ with} \\ X \text{ at } 0 \text{ and } Y \text{ at } 1 \end{array} \right\}$$

$$\hat{\mathcal{C}}^{k<n}(X \to Y) = \left\{ \begin{array}{l} \text{string diagrams on } [0,1]^k \text{ with} \\ X \text{ on } [0,1]^{k-1} \times \{0\}, \; Y \text{ on } [0,1]^{k-1} \times \{1\} \\ (\partial X = \partial Y) \times I \text{ on } \partial([0,1])^{k-1} \times I \end{array} \right\}$$



$$\hat{\mathcal{C}}^n(X \to Y) = \mathbb{C}\{\text{string diagrams as above}\} \Big/ \mathcal{U}([0,1]^k; X, Y)$$

Conjecture/hypothesis

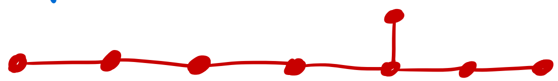
Every pivotal weak n-category is equivalent to a string diagram n-category.

Payoffs:

- straightforward definitions of 'sphere modules'
- Hochschild cohomology for n-categories (arXiv:1009.5025)
- higher dimensional analogues of idempotent completion/algebra completion (coming soon, see also arXiv:1905.09566)
- construction of a 4-category from Khovanov homology (coming soon)
- a proof of stabilisation (k-boring n-categories $=$ $(k+1)$-boring n-categories when $k \geq \frac{n}{2}+1$) using tranversality (someone should do it!)

How do we construct potential new tensor categories?

Suppose we suspect the existence of some tensor category $\mathcal{C}$ with a particular fusion ring $K_0(\mathcal{C})$ in particular suppose we know the principal graph for some object, e.g.

$\Gamma =$ 

Any monoidal category $\mathcal{C}$ embeds via a monoidal functor into the endofunctors of the underlying 1-category $\mathcal{M} = \mathcal{C}$.

$$\begin{aligned} G : \mathcal{C} &\xrightarrow{\otimes} \mathrm{End}(\mathcal{M}) \\ X &\longmapsto X \otimes - \end{aligned}$$

If $\mathcal{C}$ is semisimple, with n simple objects, then $\mathcal{M} \cong \mathrm{Vec}^{\oplus n}$, and $\mathrm{End}(\mathcal{M})$ is semisimple, with simple object E_{ij}, defined by the rule

$$E_{ij}(\mathbb{C}_k) = \begin{cases} \mathbb{C}_i & \text{if } j=k \\ 0 & \text{otherwise} \end{cases}$$

the additive generator of the k-th summand of $\mathrm{Vec}^{\oplus n}$

A general object of $\mathrm{End}(\mathcal{M})$ is (up to iso) a direct sum of these, i.e. a matrix in $M_n(\mathbb{N})$, which we can interpret as the adjacency matrix of a directed graph.

Now $\mathcal{G}(X:\mathcal{C})(\mathbb{C}_k) = X \otimes X_k = \bigoplus n_k^{\ell} X_\ell$

the k-th simple of $\mathcal{C}$

where n_k^{ℓ} is the adjacency matrix for the principal graph, so

$$\mathcal{G}(X) = \Gamma$$

Next we would like to calculate

$$\mathrm{End}(\mathcal{M})(\Theta \to \Lambda) = \bigoplus_{1 \le k \le n} \mathrm{Vec}^{\oplus n}\big(\Theta(\mathbb{C}_k) \to \Lambda(\mathbb{C}_k)\big)$$

graphs, interpreted as functors

$$= \bigoplus_{k} \mathrm{Vec}^{\oplus n}\Big(\bigoplus \theta_k^{\,l}\,\mathbb{C}_l \to \bigoplus \lambda_k^{\,l}\,\mathbb{C}_l\Big)$$

$$= \bigoplus_{k,l} \theta_k^{\,l}\,\lambda_k^{\,l}$$

This has as a basis pairs of edges, one in Θ, one in Λ with the same source and target.

Composition of basis elements is easy:

$$(e \in \Theta,\ f \in \Lambda) \gg (g \in \Lambda,\ h \in \Delta) = \begin{cases} (e,h) & \text{if } f = g \\ \text{zero} & \text{otherwise} \end{cases}$$

How do we take tensor products of graphs?

$\Theta \otimes \Lambda$ = length two paths in K_n, first edge in Θ, second edge in Λ

$\Theta^{\otimes n}$ = length n paths in Θ

How do we tensor morphisms?

$$(e \in \Theta, f \in \Lambda) \otimes (g \in \Delta, h \in \Omega)$$

$$= \begin{cases} (e \gg g \in \Theta \otimes \Delta,\ f \gg h \in \Lambda \otimes \Omega) & \text{if } (e,g) \text{ and } (f,h) \text{ are composable pairs} \\ \text{zero} & \text{otherwise} \end{cases}$$

Now: $\mathrm{End}(\mathcal{M})(\Gamma^{\otimes n} \longrightarrow \Gamma^{\otimes m}) =$

$\mathbb{C}\{$ pairs of paths (γ, γ'), of length n and m, with the same source and target $\}$

Composition is just $(\gamma, \gamma') \circ (\gamma'', \gamma''') = \delta_{\gamma', \gamma''} (\gamma, \gamma''')$

Tensor is concatenation of paths, if possible.

At this point we know there must exist an embedding

$$\langle X : e \rangle \longrightarrow \langle \Gamma : \mathrm{End}(\mathcal{M}) \rangle$$

with a completely combinatorial description of the target monoidal category!

In fact, it gets better. $\mathrm{End}(\mathcal{M})$ is a pivotal category, with duality given by taking right adjoints (which exist as $\mathcal{M}$ is semisimple, and in fact are biadjoint) which at the level of the graphs is orientation reversal.

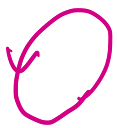
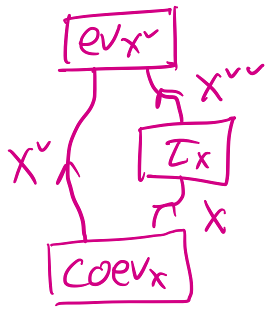
In fact $\Gamma^{\vee\vee} = \Gamma$, so we could take the pivotal isomorphism

$$I : \mathbb{1}_{\mathrm{End}(\mathcal{M})} \longrightarrow \vee\vee$$

to be the identity.

This is a bad idea!

If $\mathcal{C}$ is pivotal, each simple X_i has a categorical dimension

$\dim(X) :=$  $=$ 

$\in \mathbb{C}$

Theorem If we give $\mathrm{End}(\mathcal{M})$ the pivotal structure

$$I_{E_{ij}} = \frac{\dim(X_i)}{\dim(X_j)} \mathbb{1}_{E_{ij}}$$

then $\mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$ is a pivotal embedding.

Exercise verify this (replacing the ratio with its inverse if needed!!)

(c.f. arXiv:1007.3173, arXiv:1810.06076, arXiv:1808.00323)

Idea: ① from the principal graph Γ, deduce the existence of an element $S : \mathcal{C}(1 \to X^{\otimes n})$ satisfying certain planar equations $\{r_i\}$

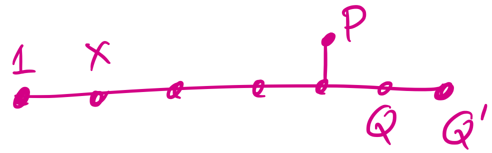
② solve these equations in $\mathrm{End}(\mathcal{M})(1 \to \Gamma^{\otimes n})$

③ this gives an embedding $F : \langle S \mid r_i \rangle \hookrightarrow \mathrm{End}(\mathcal{M})$

④ find more equations if necessary, perhaps by observing equations satisfied by the solution in $\mathrm{End}(\mathcal{M})$

⑤ use this equations to show $\langle S \mid r_i \rangle$ is evaluable and has the desired simple objects and fusion rules.

Let's try to implement this for

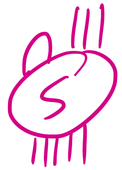
$\Gamma =$ 

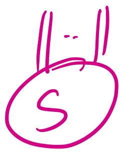
We see $\dim \mathcal{C}(1 \to X^{\otimes n}) =$ # of loops of length n based at 1

n	0	2	4	6	8	10
$\dim \mathcal{C}$	1	1	2	5	14	43
$\dim TL$	1	1	2	5	14	42

Therefore $\mathcal{C}(1 \to X^{\otimes n}) = TL(1 \to X^{\otimes n})$ for $n \leq 8$

$\mathcal{C}(1 \to X^{\otimes 10}) = TL(1 \to X^{\otimes 10}) \oplus \mathbb{C}\{$  $\}$

We can pick this S so  $=$ zero

and  $= \omega$  for some 10-th root of unity ω.

In $\mathcal{C}(X^{\otimes 5} \to X^{\otimes 5})$ we see $\boxed{f^{(5)}}$ must break up as a sum of two non-isomorphic projections

$$\boxed{f^{(5)}} = \left(\alpha \boxed{f^{(5)}} + S\right) + \left((1-\alpha)\boxed{f^{(5)}} - S\right)$$

(we use our remaining freedom to renormalize S here)

In fact, since $\dim P \underset{\substack{\uparrow \\ \text{assume unitarity!}}}{=} \operatorname{FPdim} P = P\text{-entry of FP-eigenvector of } \Gamma$

$$= \frac{1}{4}(1+\sqrt{5})\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}} \sim 3.21834$$

so we know α.

$$\left(\alpha \boxed{f} + S\right)^2 = \left(\alpha \boxed{f} + S\right)$$ gives us a quadratic in S

and we're done!

Exercise Implement 'the idea'

- Show there are finitely many solutions to these equations in $\mathrm{End}(\mathrm{Vec}^{\oplus 8})(1 \longrightarrow E_8^{\otimes 10})$
- Show these equations are enough to evaluate and identify all the idempotents
- Check the principal graph really is E_8
- Conclude there exists a fusion category with principal graph E_8
- In fact, classify such categories using your solutions.

See arXiv:0909.4099 for the construction of "the most exotic known fusion categories".