A "disklike n-category" $C$ consists of:

- a collection of functors $C^k : E_k \text{-balls}^3 \rightarrow \text{Set}$
  $E^\text{"isomorphisms"}_3$)

- for each $(k-1)$-ball $Y^k \rightarrow \mathcal{D} \times \mathcal{X}^{k+1}$, a
  restriction map $C^k(X) \rightarrow C^k(Y)$

- gluing maps $C^{k+1}(X_1) \times C^{k+1}(X_2) \rightarrow C^{k+1}(X_1 \cup X_2)$
  which compose strictly associatively

- identities, for balls modeled on "punctured products" (e.g.,
  $C^k(X) \rightarrow C^{k+1}(X = I)$
  (no claim yet that these behave like identities)

- at level $n$, two isotopic isomorphisms $f : X \cong Y$
  induce the same map $C^n(X) \cong C^n(Y)$
  and collar maps
  
  act by the identity
Isotopy actually we want something stronger:

if \( c \in C^n(X) \), \( f \sim g : X \to Y \) via \( h_t \)

and \( h_t(\partial c) \) is independent of \( t \).

then \( f(c) = g(c) \).

We now define \( G_b : \partial_k \text{ manifolds} \to \text{ set} \)

\( \circ \)

\( \text{ isomorphisms} \)

via the decomposition post \( \partial_k(W^k) \) for any \( k \)-manifold \( W \).

A decomposition of \( W \) is a sequence of manifolds

\[ M_0 \to M_1 \to \cdots \to M_m = V \]

so \( M_0 \) is a disjoint union of balls, and each map \( \partial \) is

the gluing map coming from gluing all two copies of a closed

calm-0 submanifold of the body.

Assume it is a ball decomposition if \( M_0 \) is a disjoint union of balls.

A permissible decomposition of \( W \) is a map

\[ \bigsqcup_{a} X_a \to W \]

which can be completed (in an unpecked way) to a ball decomposition,

which restricts to \( \partial W \) as a decomposition.
There's an poset structure on permissible decompositions, given by realisation as a prefix.

For any n-category C, we have a functor

\[ \Psi_{w,e} : \mathbb{O}(W) \to \text{Set} \]

defined by

\[ \Psi_{w,e}(x) = \bigsqcup_{\beta} \prod_{x_{\alpha}} \mathbb{E}(x_{\alpha}; \beta) \]

where \( \beta \) runs over all possible "boundary data" for \( \mathbb{E}(x) \)

which is compatible (for all ball decompositions completing \( x \))

with the gluing maps,

and \( x_{\alpha} \) runs over the balls.

Alternatively, \( \Psi_{w,e}(x) \) is the subset of \( \text{fet} \prod_{x_{\alpha}} \mathbb{E}(x_{\alpha}) \)

so that for every ball decomposition completing \( x \), the restrictions of \( f \) to the two gluing loci agree at each step.

With these,

\[ C^{k}(w) = \colim_{\mathbb{O}(W)} \Psi_{w,e} \]

with this definition, we get restriction maps

\[ \delta : C^{k}_{\ast}(w) \to C^{k-1}_{\ast}(\delta W) \]
as well as giving maps
\[ \Sigma(W_1) \times \Sigma(W_2) \to \Sigma(W_1 \uplus W_2) \]

(Say something about enriching, and fixing boundary data)

**Examples**

- Maps to \( T \), \( T \in \text{End}(T) \), (more generally, \( C(X) = \text{Maps}(X \otimes F \to T) \))

- "String diagrams" for a "traditional n-category with strong duality"

  \[ C(X^d) = \text{EC-labeled "string-diagram" stratifications of } X^d \]

  \[ C(X) = \text{Formal linear combos/kernel of evaluation.} \]

  (more generally, \( C \) on n-cat and \( F \) on m-algd.

  \[ C(X) = C\text{-string diagrams in } X \otimes F \)

- Dimensional reduction

  \[ (SE)(x) = \Sigma(x \otimes F) \]

- Bordism n-category of d-manifolds

  \[ \text{Bord}^n_{d}(X^k) = \frac{1}{2} (d-n+k)\text{-dim PL submanifolds } W \text{ of } X \times \mathbb{R}^d \]

  \[ \partial W = W \cap (\partial X \times \mathbb{R}^d) \times I \]

  eg take \( n=1 \), \( n=d \), or \( n>2d \) to capture the symmetric monoidal structure.
How do we handle 'homotopical' higher categories?

We call them *A₀*-disklike n-categories (but perhaps disklike (co)algebras would have been better)

Everything is the same, except:
- we enrich in *Chain* (or *Top*, or...)
- instead of asking for homomorphisms to act on n-disks, for \(X, X', X''\) n-disks,
- ask for a map

\[
C_*(\text{Homeo}(X; c \to X'; c')) \otimes E(X; c) \to E(X'; c')
\]

which is associative up to coherent homotopy, i.e.

\[
C_*(\text{Homeo}(X; c \to X'; c')) \otimes C_*(\text{Homeo}(X'; c' \to X''; c'')) \otimes E(X; c)
\]

\[
C_*(\text{Hom}(X; c \to X''; c'')) \otimes E(X; c)
\]

\[
C_*(\text{Hom}(X; c \to X'; c')) \otimes C_*(\text{Hom}(X'; c' \to X''; c'')) \otimes E(X; c)
\]

\[
E(X'; c')
\]

are homotopic, and higher arity actions give higher homotopies.

(Awkward: do we insist this is compatible with the existing 'strictly associative' action on the 0-cells? Or replace it?)
Modules

We don't attempt to define functors between disklike \( n \)-categories, but instead define (higher) bimodules, which we call sphere modules.

Let's begin with just a module.

A marked \( k \)-ball is a pair \((B^k, \{x^i = 1, x^i > 0 \})\) such that

\[
\sum_{i=1}^{k} x_i = 1, \quad x_i > 0
\]

The standard \( k \)-ball is in the northern hemisphere.

A \( C \)-module \( M \) is a collection of functors, for \( 1 \leq k \leq n \)

\[
M^k : \{ \text{marked } k \text{-balls} \} \to \text{Set}
\]

with restriction maps for

\[
\gamma^k : M(\text{marked } k \text{-ball}) \to \text{Set}
\]

with restriction maps for

\[
\gamma^k : M(\text{unmarked } k \text{-ball}) \to \text{Set}
\]

and \( \gamma^k \) giving \( M(\text{unmarked } k \text{-ball}) \to \text{Set} \)

and gluing maps corresponding to

\[
\begin{array}{cc}
\text{and }
\end{array}
\]

\[
\begin{array}{c}
\text{and }
\end{array}
\]