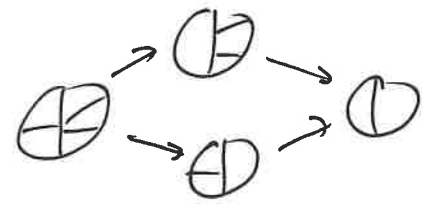
Last time

- A disklike n-category $\mathcal{C}$.
  - $\mathcal{C}^k(X)$ "morphisms of shape X"
  - strictly associative gluing — (generalisation of Moore path space)
  - at top level isotopic morphisms are equal
- Canonical extension from balls to arbitrary mflds, via

$$\underrightarrow{\mathcal{C}_k}(M^k) = \operatorname*{colim}_{\mathcal{D}(M)} \Psi_{M;\mathcal{C}}$$



(But what about the hocolim?)

- First example: TL as a disklike 2-category.

More examples:

- $\Pi_{\leq n}(T)$, $\quad \Pi_{\leq n}(T)^{k<n}(X) := \mathrm{Maps}(X \to T)$

$$\Pi_{\leq n}(T)^n(X) := [X \to T]_\partial$$

- "String diagrams for a traditional n-category with strong duality"

$$\mathcal{C}^{k<n}(X) := \{C\text{-labelled string diagrams in } X\}$$

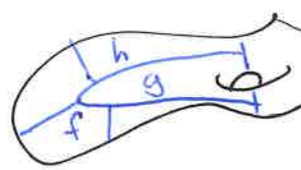
$$\mathcal{C}^n(X) := \bigsqcup \mathcal{C}^n(X;c), \quad \mathcal{C}^n(X;c) = \mathbb{C}\{C\text{-string diagrams}\}/\text{kernel of evaluation.}$$

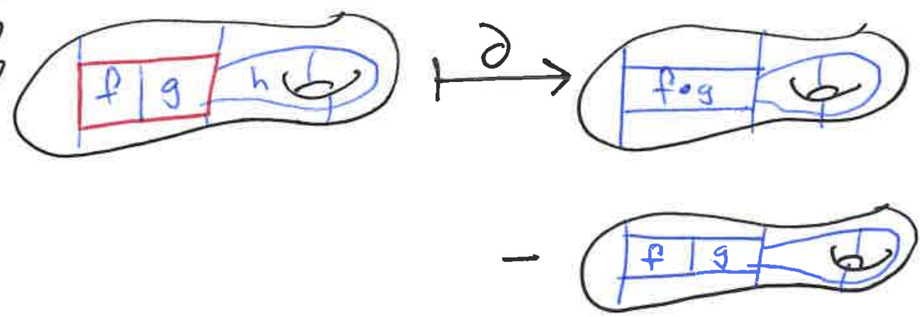
- $\mathrm{Bord}^{n,d}(X^k) = \{(d-n+k)\text{-dim PL submflds } W \text{ of } X \times \mathbb{R}^\infty$ so $\partial W = W \cap (\partial X \times \mathbb{R}^\infty)\}$

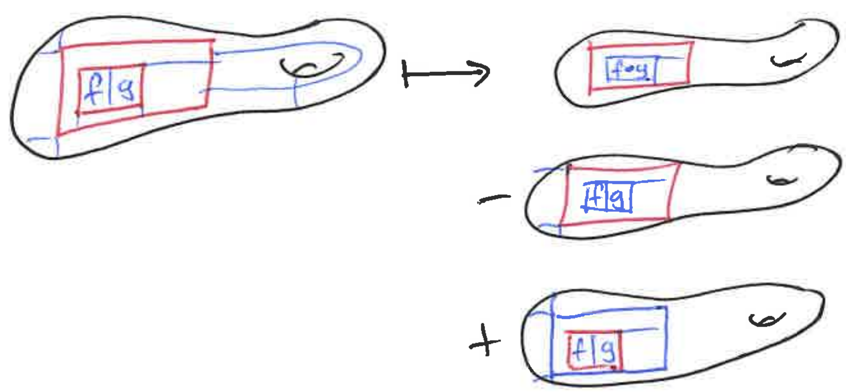
$\mathrm{Bord}^{n,d}(X^n) = \{\text{homeomorphism classes of } d\text{-mflds rel bdy}\}$

e.g. take $n=1$ for the usual 1-category of bordisms
$n=d$ for the higher categorical structure
$n>2d$ for the symmetric monoidal structure.

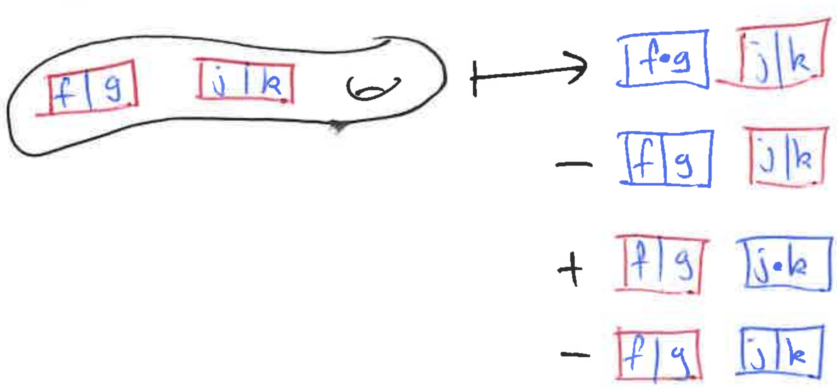
Back to the blob complex

$B_0(M;\mathcal{C}) = \{$  $\}$

$B_1(M;\mathcal{C}) = \{$  $\}$

$B_2(M;\mathcal{C}) = \{$ 

and

 $\}$

$\mathcal{B}_k(M;\mathcal{C}) = \{$ labelled decompositions with k designated "blobs",
a boundary term for each way of
forgetting a blob
an extra boundary term for each way to
"glue up" the labels of an innermost blob $\}$

→ insert ~~discussion of blob complex of balls~~

Certainly diffeomorphisms act on the blob complex:

$$\mathrm{Diff}(X \to Y) \times \mathcal{B}_*(X;\mathcal{C}) \to \mathcal{B}_*(Y;\mathcal{C})$$

just by moving the decompositions and blobs around, and
applying the diffeomorphism "locally" to each ball label.

However at this level isotopic diffeomorphisms do not
have <u>equal</u> actions.

Nevertheless, if $f \underset{\text{isotopic}}{\sim} g$, and $x \in \mathcal{B}_m(X;\mathcal{C})$, then

$$\exists y \in \mathcal{B}_{m+1}(Y;\mathcal{C}) \text{ so } \partial y = f(x) - g(x).$$

In fact, we get

$$C_*(\mathrm{Diff}(X \to Y)) \otimes \mathcal{B}_*(X;\mathcal{C}) \to \mathcal{B}_*(Y;\mathcal{C})$$

- compatible with the naive action
- compatible (up to htpy) with gluing
- compatible (up to htpy) with composition of (families of) diffeomorphisms.

Why?

Let's look at a 1-parameter family f_t of diffeomorphisms, and

$$x \in \mathcal{B}_0(M), \qquad x = \boxed{a \mid b}$$

so $f_0(x) = \boxed{a \mid b}$, $f_{\frac{1}{2}}(x) = \boxed{a \; \} \; b}$, $f_1(x) = \boxed{a \supset b}$

We can define $y \in \mathcal{B}_1(M)$ as

$$y = \boxed{a \mid b} - \boxed{a \supset b}$$

and then $\partial y = \boxed{a \cdot b} - \boxed{a \mid b} - \boxed{a \cdot b} + \boxed{a \supset b}$

cancel because of the compatibility of gluing and diffeos. and isotopy within a single morphism

$$= f_1(x) - f_0(x).$$

General idea for 1-parameter families:

- pick a fine enough open cover,
- and use a partition of unity to control local "pause" and "fast forward" buttons,
- to reduce to the case that in each short subinterval of $[0,1]$, the diffeomorphism is only "moving" inside some small ball.
- this is the previous case.

What about higher blob degrees and higher families?

<u>Lemma</u> (B.0.2) a k-parameter family of diffeomorphisms can be "localised", so that in each small patch of "time" motion is occuring in at most k balls of some open cover.

Then use a similar recipe.

(In the paper we give a much more abstract argument, based on a htpy equivalent "topological blob complex" in which the action is ~~the~~ tautological, to avoid having to do the book-keeping of explicit homotopies.)

The upshot is that replacing $\mathcal{C}^n(X^n)$ with $\mathcal{B}_*(X;\mathcal{C})$ produces sth almost like a disklike n-category

(although now enriched in chain complexes, rather than Vec)

except:

- (good): gluing of n-morphisms ~~is~~ injective (not just $k<n$)
- (bad): diffeomorphisms act strangely.

An "A_∞ disklike n-category" $\mathcal{C}$

should have: • n-morphisms enriched in Chain

• replace the "isotopic diffeos equal" axiom with

$$C_*(\mathrm{Diff}(X \to Y)) \otimes \mathcal{C}^n(X) \to \mathcal{C}^n(Y)$$

compatible with gluing ~~and composing diff~~

and composing diffeomorphisms.

Now: $\hat{\mathcal{C}}$ defined by $\hat{\mathcal{C}}^n(X) := \mathcal{B}_*(X;\mathcal{C})$, $\hat{\mathcal{C}}^{k<n}(X) = \mathcal{C}(X)$

is an A_∞ disklike n-category, the "blob resolution" of $\mathcal{C}$.

Where next? • Modules for disklike categories

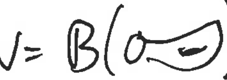
• Modules as labels

• $M \overset{L}{\underset{\mathcal{C}}{\otimes}} N$ as $\mathcal{B}(\underset{M}{\bullet}\overset{\mathcal{C}}{\rule{2em}{0.4pt}}\underset{N}{\bullet})$

• Product/fibration formulas

$\mathcal{B}($  $) = \mathcal{B}(\underset{M}{\bullet}\overset{\mathcal{C}}{\rule{2em}{0.4pt}}\underset{N}{\bullet})$

where $\mathcal{C} = \mathcal{B}(O)$

$M = \mathcal{B}($  $)$, $N = \mathcal{B}($  $)$.