Last time

- A discrete n-category $C$
  - $C^n_\mathbb{h}(X)$ "morphisms of shape $X"
  - strictly associative gluing — (generalisation of Moore path space)
  - at top level isotopic morphisms are equal

- Canonical extension from balls to arbitrary manifolds, via
  \[ C^b_\mathbb{h}(M^n) = \operatorname{colim}_{O(M)} \mathcal{U}_{\text{mic}} \]

(But what about the homotopy?)

- First example: $\text{TL}$ as a discrete 2-category.

More examples:

- $\Pi_{\leq n}(T)$, $\Pi_{\leq n}(T)_\mathbb{h}(X)$ := $\operatorname{Maps}(X \rightarrow T)$
  \[ \Pi_{\leq n}(T)_\mathbb{h}(X) := \left[ X \rightarrow T \right]_\mathbb{h} \]

- "String diagrams for a traditional n-category with strong duality"
  \[ C^{\boxtimes n}(X) := \mathcal{E}$C$-labelled string diagrams in $X \boxtimes \]
  \[ C^n(X) := \bigcup_c C^n(X;c), \quad C^n(X;c) = C^{\boxtimes c}$C$-string diagrams\]

- $\text{Bord}^{m,d}(X^h) := \mathcal{E}(d-n+k)$-dim PL submanifolds $W$ of $X \times \mathbb{R}^d$
  so $\delta W = W \cap (\partial X \times \mathbb{R}^d)$
  \[ \text{Bord}^{m,d}(X^n) = \text{homeomorphism classes of } \text{d-manifolds} \text{ nd bdg} \]
e.g. take $n=1$ for the usual 1-category of bordisms
$n=d$ for the higher categorical structure
$n>2d$ for the symmetric monoidal structure.

Back to the blob complex

$B_0(M;C) =$

$B_1(M;C) =$

$B_2(M;C) =$

and
\[ B_r(M; \mathbb{C}) = \sum \text{labelled decompositions with } k \text{ designated "blobs",} \]

a boundary term for each way of forgetting a blob

an extra boundary term for each way to "glue up" the labels of an innermost blob

\[ \Gamma \rightarrow \text{insect discussion of blob complex of balls} \]

Certainly diffeomorphisms act on the blob complex:

\[ \text{Diff}(X \rightarrow Y) \times B_*(X; \mathbb{C}) \rightarrow B_*(Y; \mathbb{C}) \]

just by moving the decompositions and blobs around, and applying the diffeomorphism "locally" to each ball label.

However at this level isotopic diffeomorphisms do not have equal actions.

Nevertheless, if \( f \) is isotopic \( g \), and \( x \in B_m(X; \mathbb{C}) \), then \( \forall y \in B_{m+1}(Y; \mathbb{C}) \) so \( dy = f(x) - g(x) \).

In fact, we get

\[ C_* (\text{Diff}(X \rightarrow Y)) \otimes B_*(X; \mathbb{C}) \rightarrow B_*(Y; \mathbb{C}) \]

- compatible with the naive action
- compatible (up to htpy) with gluing
- compatible (up to htpy) with composition of (families of) diffeomorphisms
Let's look at a 1-parameter family \( f_t \) of diffeomorphisms, and \( x \in B_0(M), \ x = \begin{pmatrix} a & b \\ \end{pmatrix} \) 

so \( f_0(x) = \begin{pmatrix} a & b \\ \end{pmatrix}, \ f_1(x) = \begin{pmatrix} a & b \\ \end{pmatrix}, \ f_1(x) = \begin{pmatrix} a & b \\ \end{pmatrix} \)

We can define \( y \in B_1(M) \) as

\[
y = \begin{pmatrix} a & b \\ \end{pmatrix} - \begin{pmatrix} a & b \\ \end{pmatrix}
\]

and then \( dy = \begin{pmatrix} a & b \\ \end{pmatrix} - \begin{pmatrix} a & b \\ \end{pmatrix} = - \begin{pmatrix} a & b \\ \end{pmatrix} + \begin{pmatrix} a & b \\ \end{pmatrix} \)

\( = f_1(x) - f_0(x) \).

cancel because of the compatibility of gluing and diffeos, and isotopy within a single morphism

General idea for 1-parameter families:

- pick a fine enough open cover,
- and use a partition of unity to control local "pause" and "fast forward" buttons,
- to reduce to the case that in each short subinterval of \([0,1]\),
  the diffeomorphism is only "moving" inside some small ball.
- and this is the previous case.
What about higher blob degrees and higher families?

Lemma (B.0.2) a k-parameter family of diffeomorphisms can be "localised", so that in each small patch of "time" motion is occurring in at most k balls of some open cover.

Then use a similar recipe.

(In the paper we give a much more abstract argument, based on a lifted equivalent "topological blob complex" in which the action is tautological, to avoid having to do the book-keeping of explicit homotopies.)

The upshot is that replacing $C^n(X)$ with $B^*_n(X;E)$ produces something almost like a disklike n-category (although now enriched in chain complexes, rather than Vec)

except:

* (good): gluing of n-morphisms is injective (not just bi-

* (bad): diffeomorphisms act strangely.
An "A\infty disklike n-category" \mathcal{C} should have:
- n-morphisms enriched in Chain
- replace the "isotopic diffeos equal" axiom with
  \[ C_*(\text{Diff}(X \to Y)) \otimes C^n(X) \to C^n(Y) \]
compatible with gluing and composing diffeomorphisms.

Now: \mathcal{E} defined by \mathcal{E}_n(X) := \text{B}_n(X; \mathcal{C}), \mathcal{E}_k^{\leq n}(X) := \mathcal{E}_n(X)
is an A\infty disklike n-category, the "blob resolution" of \mathcal{C}.

Where next?

- Modules for disklike categories

- Modules as labels

- M \otimes N as \text{B}(\begin{array}{c} e \\ M \end{array})

- Product/fibration formulas

  \text{B}(\begin{array}{cc} e \\ M \\ N \end{array}) = \text{B}(\begin{array}{c} e \\ M \\ N \end{array})

  where \quad e = \text{B}(0), \quad M = \text{B}(0 \to), \quad N = \text{B}(0 \to).