Khovanov homology as a 4-category

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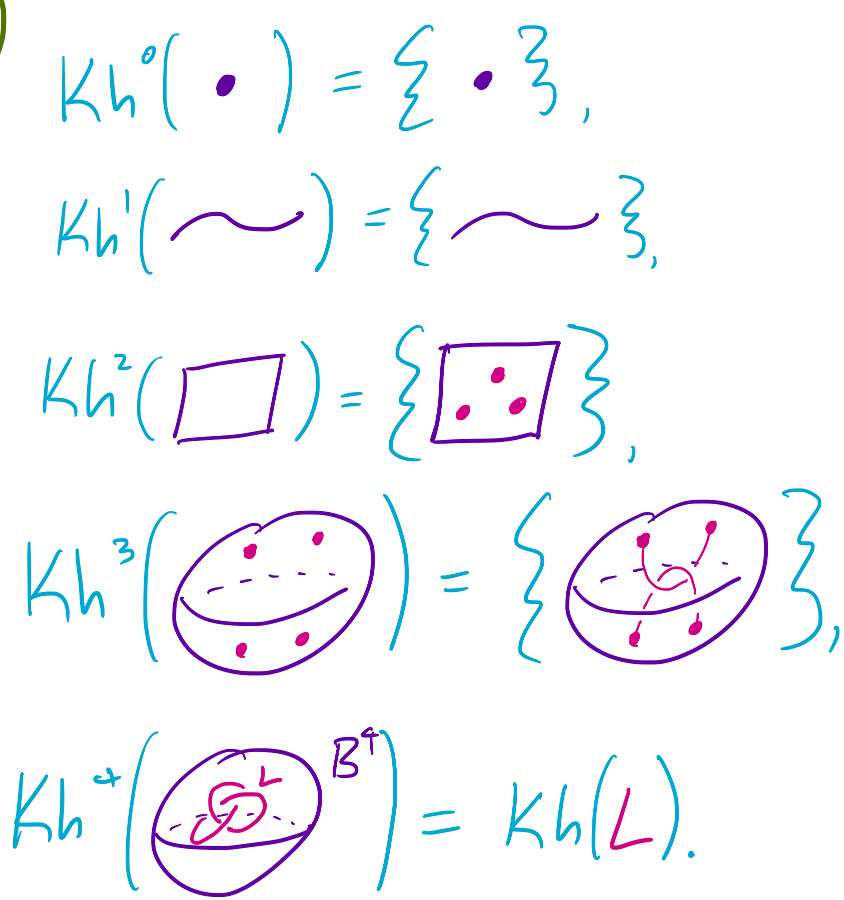

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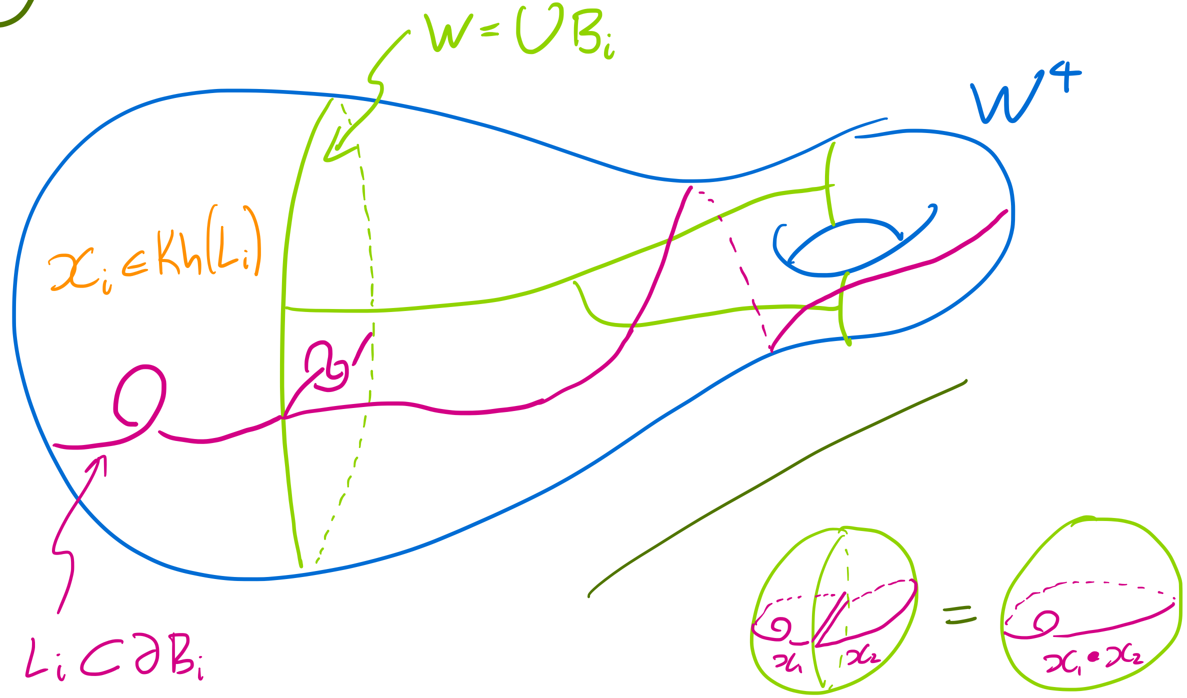

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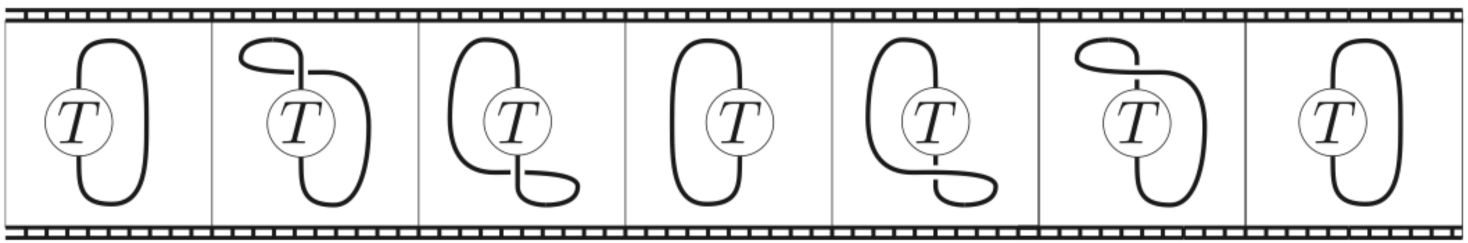
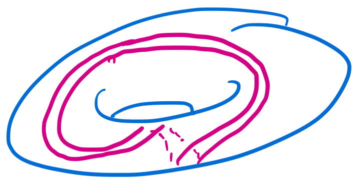


From an n-category $\mathcal{C}$ with "enough duality" we obtain a vector space valued invariant $\int_M \mathcal{C}$ of oriented n-manifolds M.

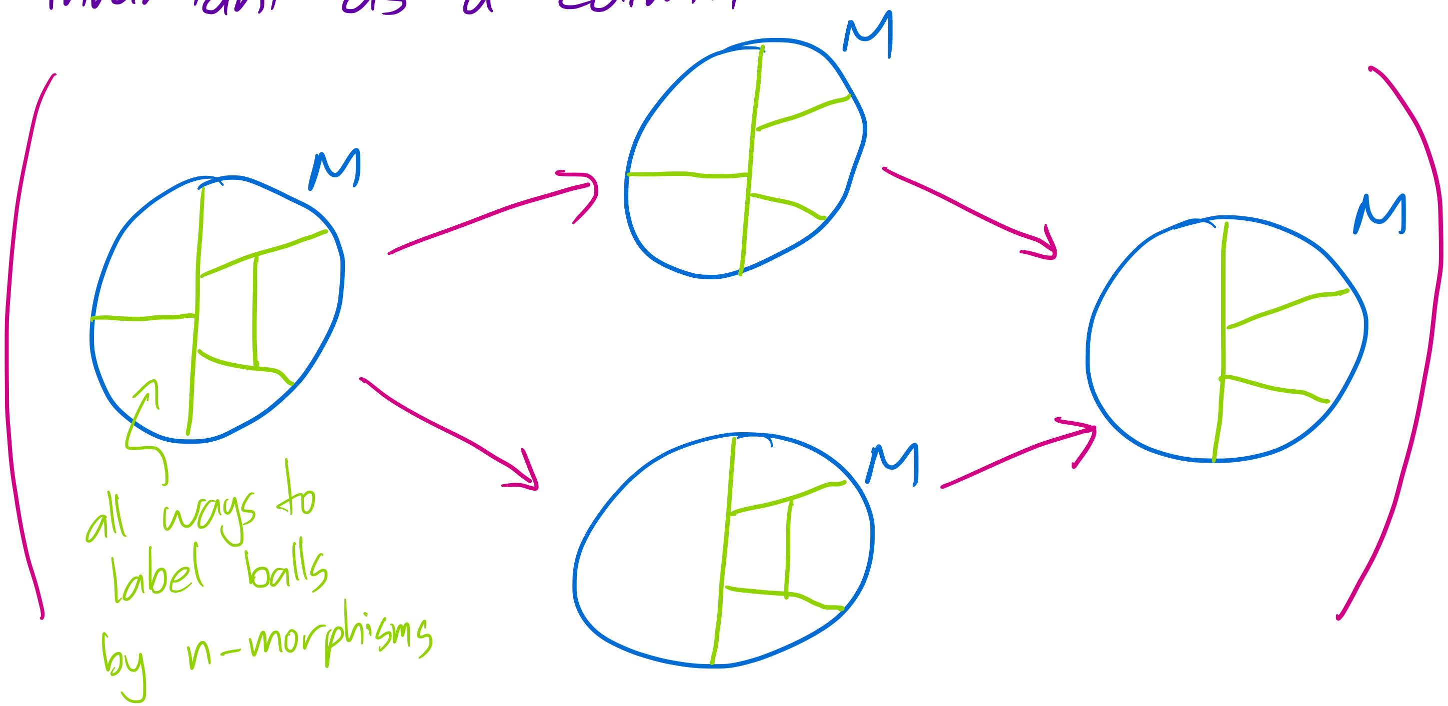
<u>Example</u> $\mathcal{C} = TL$ at any value of q.

$\int_{\mathbb{T}^2} TL = \Bbbk\Big\{$  $\Big\}\Big/$ mod TL relations, $\bigcirc = q+q^{-1}$

(The "Kauffman skein module")

We can express this invariant as a colimit:

$\int_M \mathcal{C} = \operatorname{colim}$ (over ball decompositions)



This construction is diffeomorphism invariant, and satisfies good gluing rules.

If the category satisfies some nondegeneracy and finiteness conditions, then

- these vector spaces are finite dimensional
- we also have linear maps associated to $(n+1)$-dimensional cobordisms.

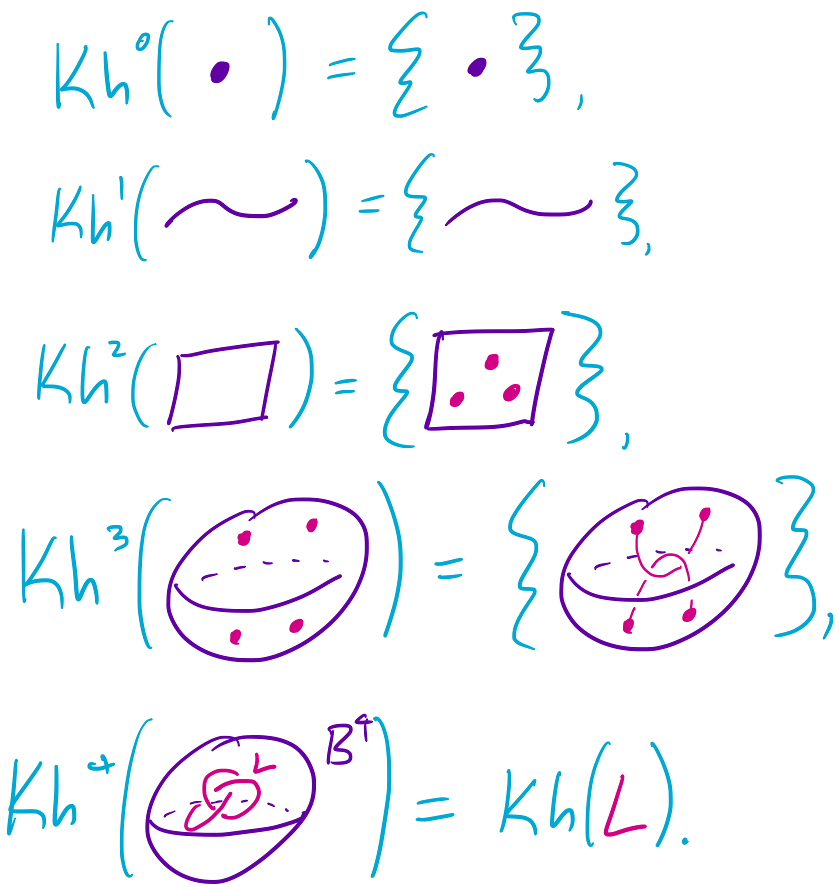
Example the right sort of 2-category is a pivotal (multi) fusion category, and then this construction is describing the Turaev-Viro TQFTs.

If the category further satisfies a "trivial centre" property, then these invariants become trivial (in arbitrary dimension, this is speculation!) (i.e. 1-d vector spaces for closed n-manifolds) but there is still information in their relative versions.

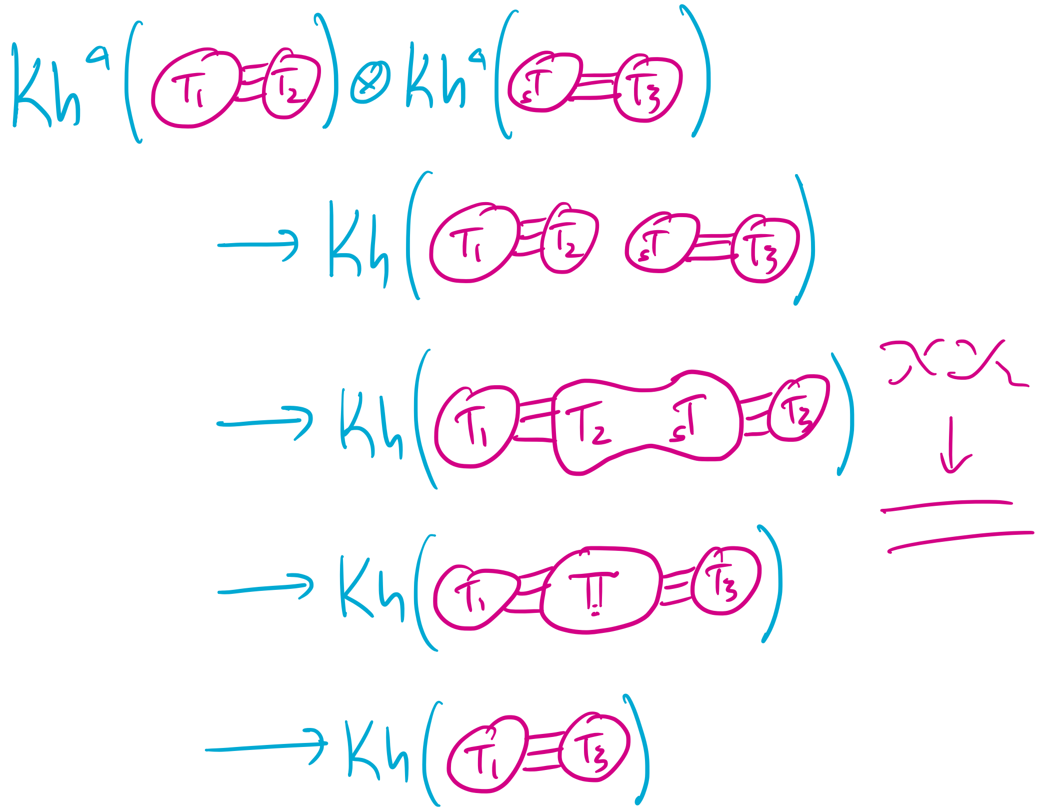
Example
- We can think of a braided tensor category (e.g. $\mathrm{Rep}\, U_q\, \mathfrak{g}$) as a "2-boring 3-category".
- We automatically get vector spaces for 3-manifolds.
- If the category is finitely semisimple, we get numerical invariants of 4-manifolds (the Crane-Yetter TQFTs).
- If the category is modular, the invariant of a 3-manifold only depends on its 2-manifold boundary, and similarly for 4-manifolds. These are the Reshetikhin-Turaev invariants.

- All of this discussion is to say what we are <u>not</u> doing.
- Categorifying RT 3-manifold invariants for $\mathrm{Rep}\, U_q \mathfrak{g}$, $q^l = 1$, is a fine goal.
- We're content to merely categorify the ~~Crane-Yetter~~ 3-manifold invariants, to produce vector space valued invariants of 4-manifolds.

Khovanov homology gives a 4-category in an almost tautological way:



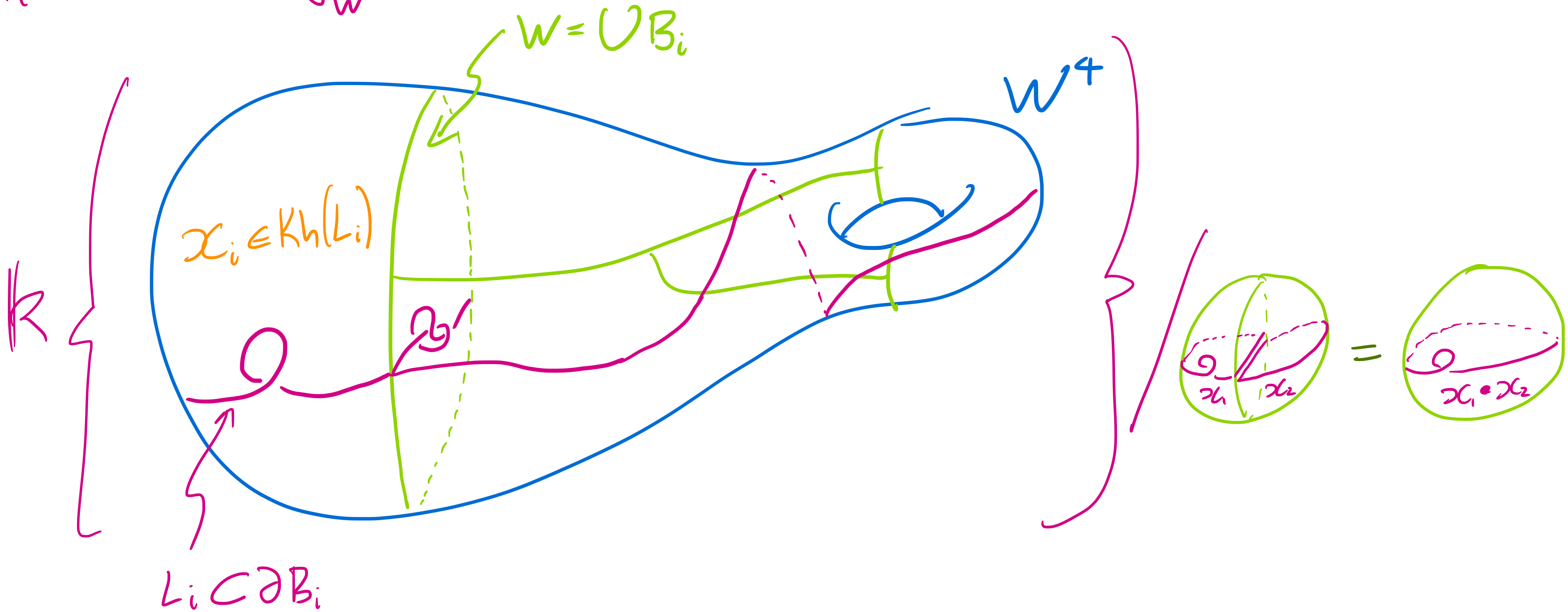
To say this constitutes a 4-category we need to specify how to compose 4-morphisms — this is provided by functoriality:



There's one wrinkle: we need Kh(L) to be a functorial invariant of links in the 3-sphere.

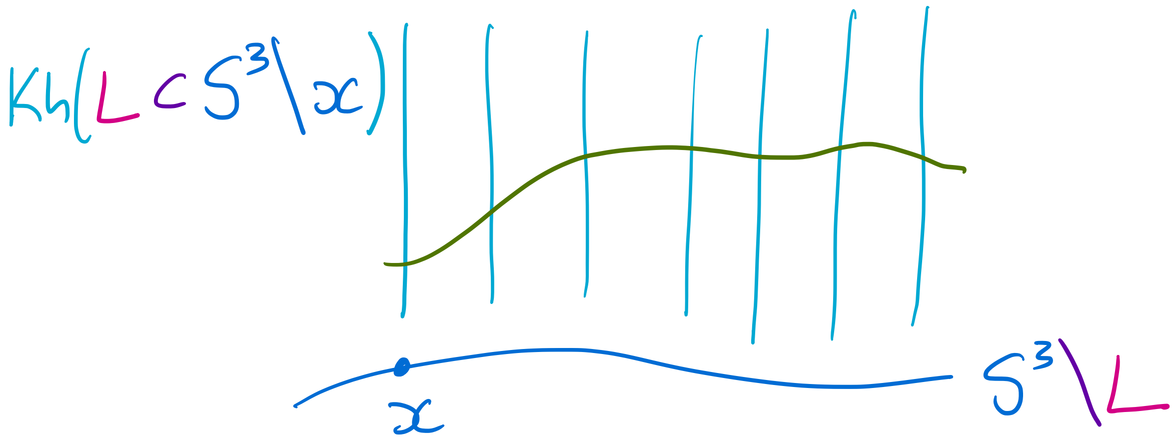
Putting all this together, our standard recipe gives a 4-manifold invariant:

$$Kh(L \subset \partial W^4) = \int_W Kh =$$

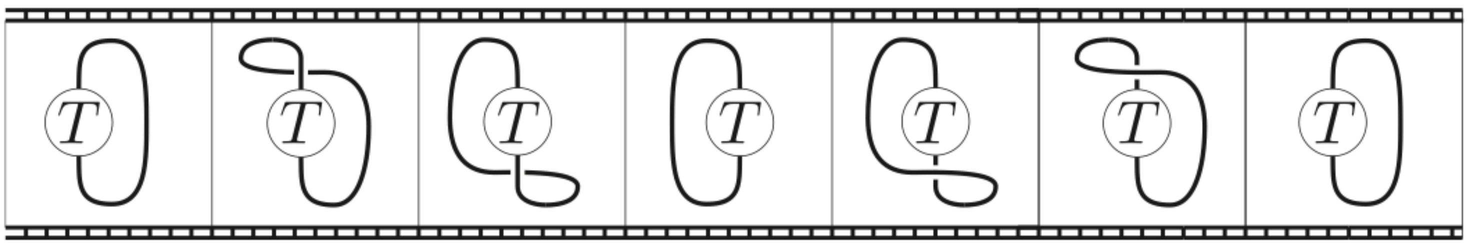


Generically, links in S^3 avoid the north pole, and indeed cobordisms do too.
However isotopies of cobordisms do not.

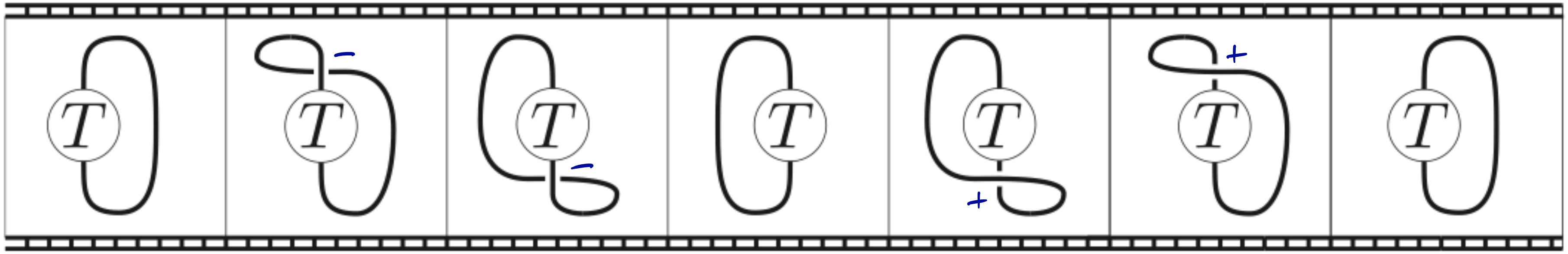
We define $Kh(L)$ as the flat sections of a bundle:

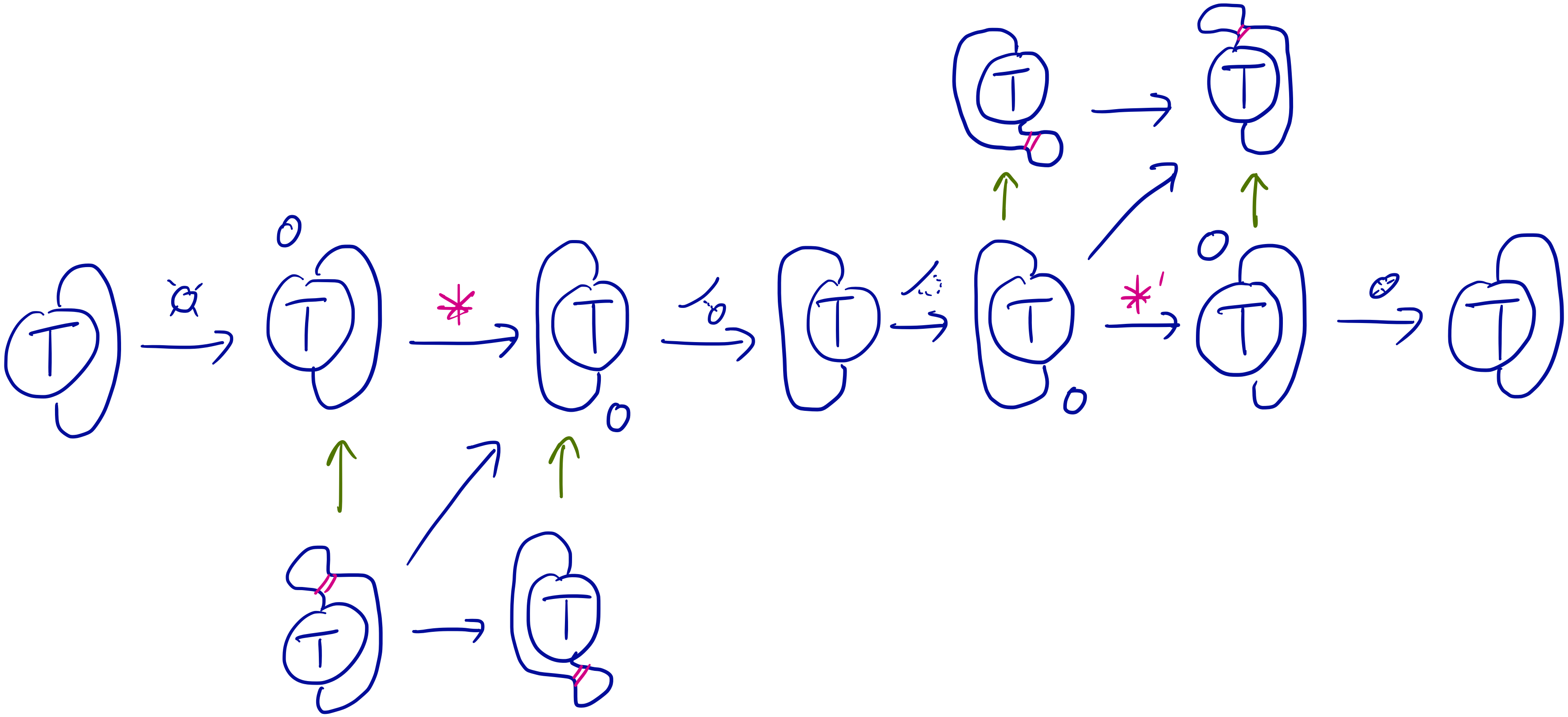


For this to give a sensible answer, we need to know the bundle has trivial monodromy, or equivalently that:



induces the identity map for every tangle T.

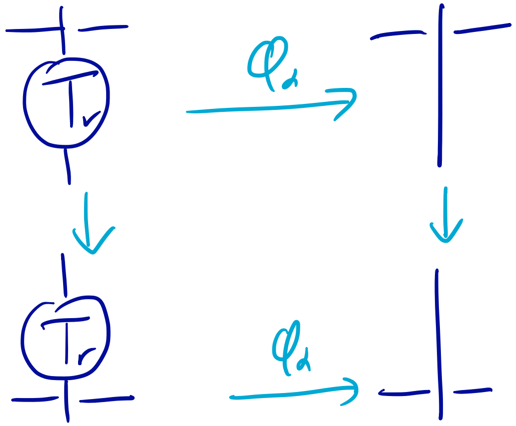




If, by an isotopy, we assume T is a braid closure, $*$ is diagonal w.r.t. resolutions of T, with components induced by cobordisms.

For each resolution T_r, we construct an isomorphism $T_r \overset{\varphi}{\cong} \bigoplus_{\alpha}$ |

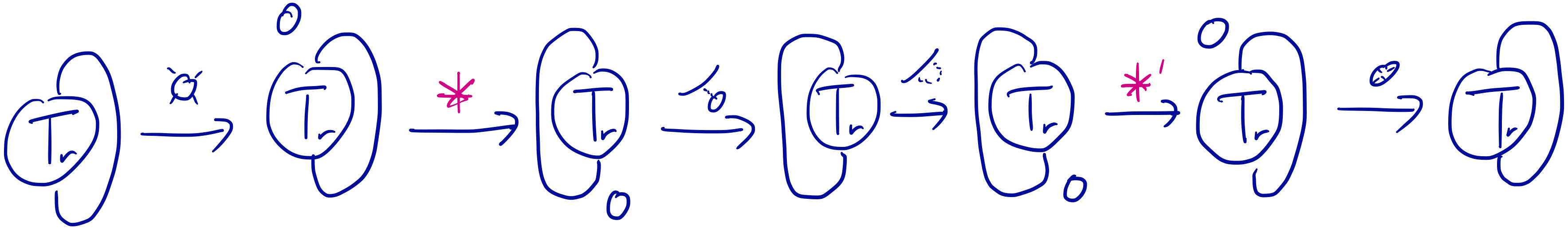
and show



commutes on the nose.

(φ_α is defined in steps; only certain movie moves occur)

Then



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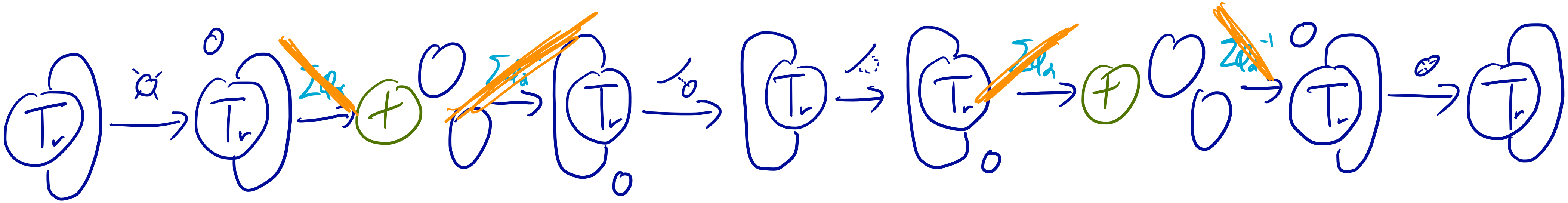

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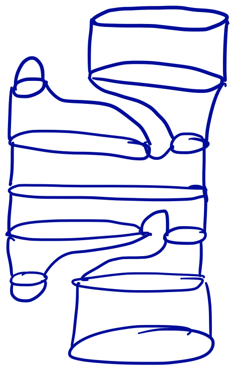

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Kevin will say more on Thursday about gradings and genus bounds!

Notes

- we work in the "gl_2" version of Khovanov homology, which is functorial over $\mathbb{Z}$.
- we partially define an extension to an invariant of tangled gl_2 webs, but only prove enough about functoriality for our purposes.