A "disklike n-category" $\mathcal{C}$ consists of (§6 of 1009.5025)

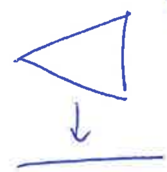
- a collection of functors $\mathcal{C}^k : \underset{\circlearrowleft \ \{\text{"isomorphisms"}\}}{\{k\text{-balls}\}} \longrightarrow \mathrm{Set}$

- for each (k-1)-ball $Y^k \hookrightarrow \partial X^{k+1}$, a restriction map☆ $\mathcal{C}^{k+1}(X) \longrightarrow \mathcal{C}^k(Y)$

  ☆: not necessarily defined everywhere!

- gluing maps $\mathcal{C}^{k+1}(X_1) \underset{\mathcal{C}^k(Y)}{\times} \mathcal{C}^{k+1}(X_2) \longrightarrow \mathcal{C}^{k+1}(X_1 \underset{Y}{\cup} X_2)$

  which compose <u>strictly</u> associatively (injectivity?)

- identities, for balls modelled on "pinched products" (e.g. )

  $\mathcal{C}^k(X) \longrightarrow \mathcal{C}^{k+1}(X \times I)$

  (no axiom yet that these behave like identities)

- <u>at level n</u>, two isotopic isomorphisms $f, g : X^n \xrightarrow{\cong} Y^n$

  induce the same map $\mathcal{C}^n(X) \xrightarrow{\cong} \mathcal{C}^n(Y)$

  and collar maps

  act by the identity.

isotopy actually we want something stronger:

If $c \in \mathcal{C}^n(X^n)$, $f \overset{\text{not fixed } \partial!}{\simeq} g : X \xrightarrow{\cong} Y$, via h_t

and $h_t(\partial c)$ is independent of t,

then $f(c) = g(c)$.

We now define $\vec{\mathcal{C}}_b : \{k\text{-manifolds}\} \circlearrowleft \{\text{isomorphisms}\} \longrightarrow \mathrm{Set}$

via the decomposition poset $\mathcal{D}(W^k)$ for any k-manifold W.

A decomposition of W is a sequence of manifolds

$$M_0 \to M_1 \to \cdots \to M_m = W$$

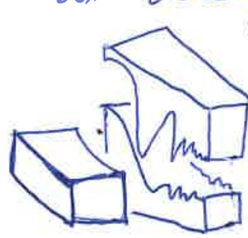
so ~~M_0 is a disjoint union of balls, and~~ each map is the gluing map coming from gluing all two copies of a closed codim-0 submanifold of the bdy.

~~A permiss~~ It is a ball decomposition if M_0 is a disjoint union of balls.

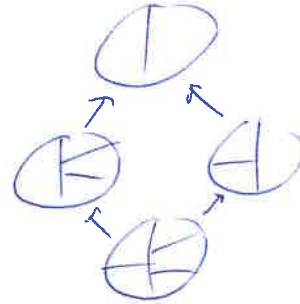
A permissible decomposition of W is a map

$$\coprod_\alpha X_\alpha \longrightarrow W$$

which can be completed (in an unspecified way) to a ball decomposition, which restricts to ∂W as a decomposition.



There's an poset structure on permissible decompositions, given by ~~[illegible]~~ realisation as a prefix.



For any n-category $\mathcal{C}$, we have a functor

$$\Psi_{W;\mathcal{C}} : \mathcal{D}(W) \to \mathrm{Set}$$

defined by $\Psi_{W;\mathcal{C}}(x) = \coprod_{\beta} \prod_{\alpha} \mathcal{C}(X_\alpha ; \beta)$

where β runs over all possible "boundary data" for $\coprod X_\alpha$ which is compatible (for all ball decompositions completing x) with the gluing maps,

and α runs over the balls.

Alternatively, $\Psi_{W;\mathcal{C}}(x)$ is the subset of $f \in \prod_\alpha \mathcal{C}(X_\alpha)$ so that for every ball decomposition completing x, the restrictions of f to the two gluing loci agree at each step.

~~With these defn~~

Then $\underset{\rightarrow}{\mathcal{C}}^k(W) = \operatorname{colim}_{\mathcal{D}(W)} \Psi_{W;\mathcal{C}}$

With this definition, we get restriction maps

$$\partial : \underset{\rightarrow}{\mathcal{C}}^k(W) \to \underset{\rightarrow}{\mathcal{C}}^{k-1}(\partial W)$$

as well as gluing maps

$$\underset{\to}{\mathcal{C}}(W_1) \times_{\underset{\to}{\mathcal{C}}(Y)} \underset{\to}{\mathcal{C}}(W_2) \to \underset{\to}{\mathcal{C}}(W_1 \cup_Y W_2)$$

(Say something about enriching and fixing boundary data)

Examples

- Maps to T, $\pi_{\le n}(T)$, (more generally, $\mathcal{C}(X) = \mathrm{Maps}(X \times F \to T)$)
- "String diagrams" for a "traditional" n-category with strong duality

$$\mathcal{C}(X^{k<n}) = \{C\text{-labeled "string-diagram" stratifications of } X\}$$

$$\mathcal{C}(X^n) = \text{formal linear combos / kernel of evaluation.}$$

~~(more generally, C an $n+m$ cat and F an m-mfld,~~

~~$\mathcal{C}(X) = C$-string diagrams in $X \times F$)~~

- Dimensional reduction:

$$\left(\int_F \mathcal{C}\right)(X) = \underset{\to}{\mathcal{C}}(X \times F)$$

- Bordism n-category of d-manifolds

$$\mathrm{Bord}^{n,d}(X^k) = \{(d-n+k)\text{-dim PL submanifolds } W \text{ of } X \times \mathbb{R}^\infty \text{ so } \partial W = W \cap (\partial X \times \mathbb{R}^\infty)\}$$

e.g. take $n=1$, $n=d$, or $n>2d$ + capture the symmetric monoidal structure.

How do we handle 'homotopical' higher categories?

We call them A_∞-disklike n-categories (but perhaps disklike (∞,n)-categories would have been better)

Everything is the same, except:

- we enrich in Chain (or Top, or ....)
- instead of asking for homeomorphisms to act on n-disks, for X, X', X'' n-disks,
- ask for a chain map

$$C_*(\mathrm{Homeo}(X;c \to X';c')) \otimes \mathcal{C}(X;c) \to \mathcal{C}(X';c')$$

which is associative up to coherent homotopy, i.e.

$$\begin{tikzcd}[column sep=tiny]
 & C_*(\mathrm{Homeo}(X;c \to X';c')) \otimes C_*(\mathrm{Homeo}(X';c' \to X'';c'')) \otimes \mathcal{C}(X;c) \arrow[dl] \arrow[dr] & \\
C_*(\mathrm{Homeo}(X;c \to X'';c'')) \otimes \mathcal{C}(X;c) \arrow[dr] & & C_*(\mathrm{Homeo}(X;c \to X';c')) \otimes \mathcal{C}(X';c') \arrow[dl] \\
 & \mathcal{C}(X'';c'') &
\end{tikzcd}$$

are homotopic, and higher arity actions give higher homotopies.

(Awkward: do we insist this is compatible with the existing 'strictly associative' action on the 0-cells? or replace it?)

Modules

We don't attempt to define functors between disklike n-categories, but instead define (higher) bimodules, which we call <u>sphere modules</u>

Lets begin with just a module.

A <u>marked k-ball</u> is a pair (B,N) diffeomorphic to

$$\Big(\underbrace{B^k}_{\text{the standard } k\text{-ball}},\ \underbrace{\{\textstyle\sum x_i^2=1,\ x_1 \geq 0\}}_{\text{the northern hemisphere}}\Big)$$

A $\mathcal{C}$-module $\mathcal{M}$ is a collection of functors, for $1 \leq k \leq n$

$$\mathcal{M}^k : \underset{\circlearrowleft\ \{\text{isos}\}}{\{\text{marked } k\text{-balls}\}} \longrightarrow \text{Set}$$

with restriction maps for

$\underbrace{Y^k}_{\text{unmarked}} \hookrightarrow$  $X^{k\text{-ball}}$ giving $\mathcal{M}(X) \longrightarrow \mathcal{C}(Y)$

and $\underbrace{Y^k}_{\text{marked}} \hookrightarrow$  giving $\mathcal{M}(X) \to \mathcal{M}(Y)$

and gluing maps corresponding to

 and