Chirality for operator algebras

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Abstract

For a finite system of bimodules over II_1 factors, we define a notion of "nondegenerate braided system of bimodules". If a system of bimodules is braided and nondegenerate, then the resulting 3-dimensional topological quantum field theory (TQFT) based on triangulation splits as a tensor product of another TQFT based on vertical surgery and its complex conjugate. This is a generalization of a theorem of Turaev that the Turaev-Viro TQFT is a tensor product of the Reshetikhin-Turaev TQFT and its complex conjugate.

1 Introduction

We first explain a general relation between operator algebras and topological quantum field theory. See [4] for more details.

We start with a subfactor $N \subset M$ with finite Jones index and finite depth. We use the *standard bimodule* $h = {}_N L^2(M)_M$ to make irreducible decompositions of finite tensor products $\cdots \otimes_N h \otimes_M \bar{h} \otimes_N h \otimes_M \bar{h} \otimes_N \cdots$, where $\bar{h}$ is the contragredient bimodule ${}_M L^2(M)_N$. The finite depth condition means that we have only finitely many equivalence classes of (four kinds of) bimodules in the decompositions. We denote this system of bimodules by $\mathcal{M}$. Each bimodule $x \in \mathcal{M}$ has the corresponding Jones index $[x]$. The global index $[\mathcal{M}]$ is defined by $\sum_{{}_N x_N}[x]$, which is also equal to $\sum_{{}_M x_M}[x]$. The tensor product operation makes the system $\mathcal{M}$ a finite tensor category. With intertwiners, we can define quantum $6j$-symbols satisfying unitarity, tetrahedral symmetry, and the pentagon relation as in [4]. This gives another (equivalent) formulation of *paragroups* introduced in [2]. Then the general method of [8] gives a 3-dimensional TQFT based on triangulation. Roughly speaking, each oriented tetrahedron with bimodules on edges and intertwiners on faces has a value given by the composition of the four intertwiners on the faces (with a suitable normalization). Then we make a state sum, or a partition function, over all the configurations of bimodules and intertwiners for a fixed triangulation. We can prove that this number does not depend on the triangulation. We further study this TQFT with an operator algebraic technique.

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First we look at the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. (Note that for the subfactor of type A_n, the asymptotic inclusion is given by $\langle e_j \rangle_{j\neq 0} \subset \langle e_j \rangle_{j\in \mathbf{Z}}$, where the projections $\{e_j\}_{j\in\mathbf{Z}}$ satisfy the Jones relation for $4\cos^2 \pi/(n+1)$.) This construction of the asymptotic inclusion can be regarded as an analogue of the "quantum double" construction of Drinfeld for paragroups. The index $[M_\infty : M \vee (M' \cap M_\infty)]$ is given by the global index $[\mathcal{M}]$ of the original inclusion $N \subset M$. The principal graph of the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ is given by the fusion graph of the original subfactor $N \subset M$ as follows. First we take the (finite) system of M-M bimodules arising from the initial inclusion $N \subset M$. Then we make the graph $\mathcal{G}_0$ so that its even vertices are labeled by pairs (x, y) of M-M bimodules, its odd vertices are labeled by M-M bimodules z, and the number of edges from (x, y) to z is the multiplicity of z in $x \otimes_M y$. We call this graph the *fusion graph*. The principal graph of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ is given by the connected component $\mathcal{G}$ of the fusion graph containing the even vertex $(*, *)$. (See [3, §III. 1].)

Next we study the system $\mathcal{M}_\infty$ of the M_∞-M_∞ bimodules arising from the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$. We call this system the *asymptotic system*. In the TQFT, the Hilbert space associated to the torus $S^1 \times S^1$ has a special meaning. The system $\mathcal{M}_\infty$ of the M_∞-M_∞ bimodules gives a natural basis of this Hilbert space $H_{S^1\times S^1}$, if the fusion graph is connected. To see this, we introduce the *tube algebra* as follows [5].

Now all the bimodules x, x', y, y', a, a', b are N-N bimodules in the system $\mathcal{M}$. We define a finite dimensional C^*-algebra Tube $\mathcal{M}$

$$\text{Tube } \mathcal{M} = \oplus_{x,y,a} \text{Hom}(x \otimes a, a \otimes y),$$

as in Figure 5 (a), with the product defined by

$$\xi \cdot \xi' = \delta_{y,x'} \sum_{b,\alpha} (\alpha \otimes 1_{y'}) \cdot (1_a \otimes \xi') \cdot (\xi \otimes 1_{a'}) \cdot (1_x \otimes \alpha^*)$$

where $\xi \in \text{Hom}(x \otimes a, a \otimes y), \xi' \in \text{Hom}(x' \otimes a', a' \otimes y')$ and α is in a basis of $\text{Hom}(a \otimes a', b)$. (See Figure 5 (b).)

The Hilbert space $H_{S^1\times S^1}$ in the TQFT has a commutative multiplicative structure, and the minimal projections in the center of the tube algebra naturally give a basis of this Hilbert space. Denote the label set for the center of the tube algebra by $\mathcal{T}$ and call it the *tube system*. Furthermore this set $\mathcal{T}$ is naturally identified with the set of the irreducible M_∞-M_∞ bimodules arising from the asymptotic inclusion, if the fusion graph is connected.

The group $SL(2, \mathbf{Z})$ has a natural unitary representation ρ on the Hilbert space $H_{S^1\times S^1}$ and we define

$$S = \rho\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \qquad T = \rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

Then S diagonalizes the fusion algebra of the M_∞-M_∞ bimodules and T is diagonal. This is an analogue of the Verlinde identity [9]. In particular, the fusion algebra of M_∞-M_∞ bimodules is commutative, which is not trivial at all. We can actually construct all the data satisfying the combinatorial axioms for RCFT in the sense of Moore and Seiberg [1].

We further have a formula for the dimension of the Hilbert space H_S associated to a surface S in the case where the fusion graph is connected. We denote the global index of this system $\sum_{X\in\mathcal{M}_\infty}[X]$ by $[\mathcal{M}_\infty]$, and define $[[X]]_\infty = [X]^{-1/4}[\mathcal{M}_\infty]^{1/4}$ for each M_∞-M_∞ bimodule X. Then the dimension of the Hilbert space H_S is given by $\sum_X [[X]]_\infty^{-2\chi(S)}$, where $\chi(S)$ is the Euler characteristic of the surface S. This number is also the topological invariant associated to the manifold $S \times S^1$. This is also an analogue of the Verlinde formula. Note that if S is the 2-dimensional sphere S^2, we get 1 as the dimension, and that if S is the 2-dimensional torus $S^1 \times S^1$, we get the number of the M_∞-M_∞ bimodules as the dimension.

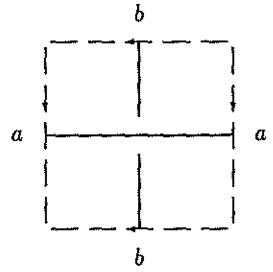
Example 1.1 Start with a subfactor $N \subset M$ of type A_4. In this case, the principal graph of the asymptotic inclusion is D_6 as in [3, §III. 1], so the dual principal graph is also D_6 and its even vertices give the M_∞-M_∞ bimodules. The Perron-Frobenius weight of the even vertices of D_6 is given by $1, \varphi^2, \varphi, \varphi$, where we have $\varphi^2 = \varphi + 1$. Then the global index $[\mathcal{M}_\infty]$ is $1 + \varphi^4 + \varphi^2 + \varphi^2 = 5\varphi^2$, and the above formula give the dimension $5\varphi^2(1 + \varphi^{-4} + \varphi^{-2} + \varphi^{-2}) = 25$, if $\chi = -2$, for example.

If we start from a subfactor of type A_n, $n \geq 4$, then the labeling of the M_∞-M_∞ bimodules is given by the following. We take all the pairs (a, b) of integers $a, b \in \{1, 2, \dots, n\}$ with $a \equiv b \bmod 2$ and define an equivalence relation $(a, b) \sim (\bar{a}, \bar{b})$, where $\bar{a} = n+1-a$. If n is even, the M_∞-M_∞ bimodules are labeled by the equivalence classes of these pairs, and in particular, the number of the M_∞-M_∞ bimodules is $n^2/4$. If n is odd, we take all the equivalence classes of these pairs and split the class $((n+1)/2, (n+1)/2)$ into two, $((n+1)/2, (n+1)/2)_+$ and $((n+1)/2, (n+1)/2)_-$. Then the M_∞-M_∞ bimodules are labeled by these, and in particular, the number of the M_∞-M_∞ bimodules is $(n^2+7)/4$. These are the dimensions of the Hilbert space $H_{S^1\times S^1}$ associated to the two dimensional torus $S^1 \times S^1$.

2 Braided systems of bimodules

We start with a finite system $\mathcal{M}$ of N-N bimodules, e.g., arising from a subfactor with finite depth. (Here we consider only one kind of bimodule.) The tensor product operation of the N-N bimodules is not commutative in general, but we *assume* that it is commutative. Even under such an assumption, we have no distinguished homomorphism (intertwiner) $\xi \in \mathrm{Hom}(a \otimes b, b \otimes a)$ for two N-N bimodules a, b in the system in general. We define a notion of "braiding" as a possibility of systematic choices of intertwiners from $a \otimes b$ to $b \otimes a$.

Definition 2.1 We say that the system $\mathcal{M}$ is *braided* if for any pair a, b of the N-N bimodules in $\mathcal{M}$, there is a system ε of non-zero intertwiners $\varepsilon_{ab} = \xi \in \mathrm{Hom}(a \otimes b, b \otimes a)$ denoted by the following picture

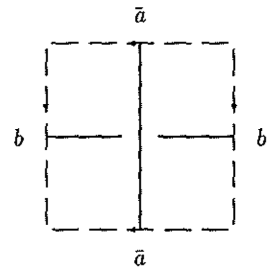


with the following three axioms.

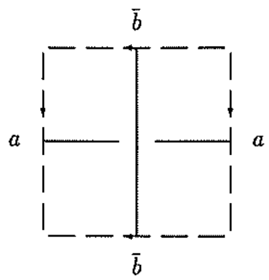
First, we define actions of the two matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as follows. For an intertwiner $\xi : a \otimes b \to b \otimes a$ denoted by the above picture, we apply the Frobenius reciprocity map [4] twice to get the intertwiner ${}^a\xi^a : b \otimes \bar{a} \to \bar{a} \otimes b$. This is the image of ξ with the action of S, and denoted by the following picture.

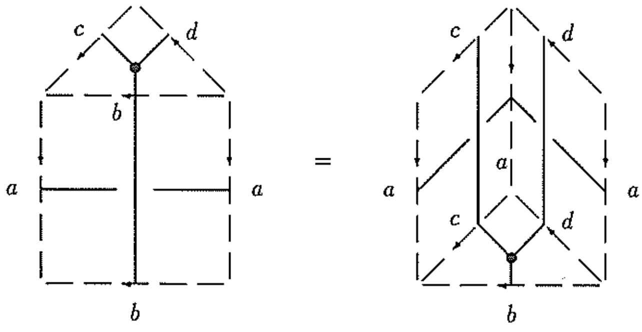


Graphically, this action is regarded as a counterclockwise 90 degree rotation. Note that the action of S is a linear unitary. Next we define an action of Γ on ξ by $\overline{{}^a\xi^a} : a \otimes \bar{b} \to \bar{b} \otimes a$. Graphically, this is regarded as a reflection with respect to a vertical axis, and denoted by the following picture.

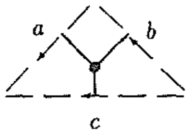


Note that this action is a conjugate linear unitary. The first axiom for ε states that the system ε is invariant under the action S^2. The second axiom states that the system ε is invariant under the action ST.

The third axiom, the *crossing-fusion axiom*, is stated graphically as follows.



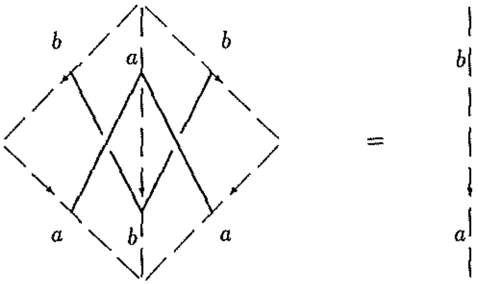
Here the following picture means an intertwiner (labeled by $\bullet$) from $a \otimes b$ to c.



The choice ε is called a *crossing*.

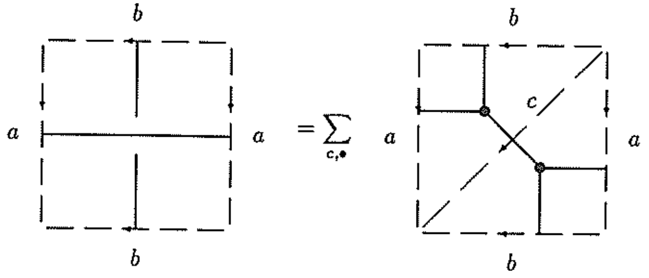
In the picture of the crossing-fusion axiom, the intertwiners denoted by $\bullet$ on the both hand sides are equal. The both pictures denote intertwiners from $a \otimes c \otimes d$ to $b \otimes a$, and we require these two intertwiners are equal.

In the picture of the crossing-fusion axiom, we can take the "tail" of the "Y" shape to be the identity bimodule, we get the following picture.

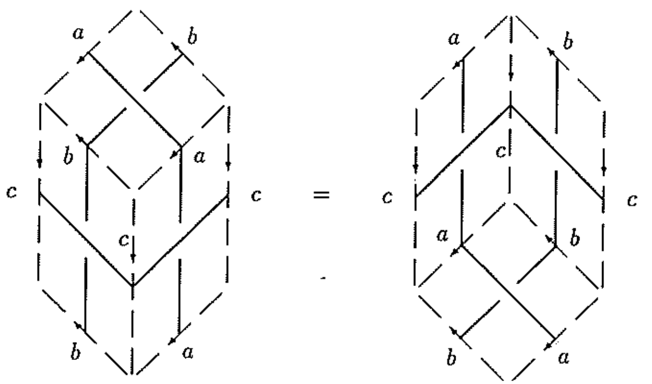


In the picture, the left picture denotes an intertwiner from $a \otimes b$ to $a \otimes b$. The right picture means the identity intertwiner on $a \otimes b$. This is regarded as invariance under Reidemeister move II.

The fact that $a \otimes b$ is decomposed into a direct sum of N-N bimodules c is represented by the following picture,

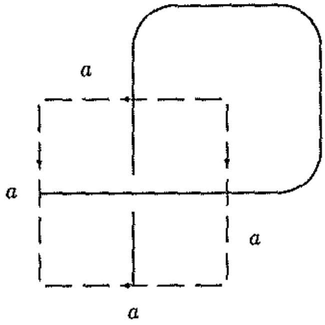


and this implies the Yang-Baxter equation together with the crossing-fusion axiom as in the following picture.



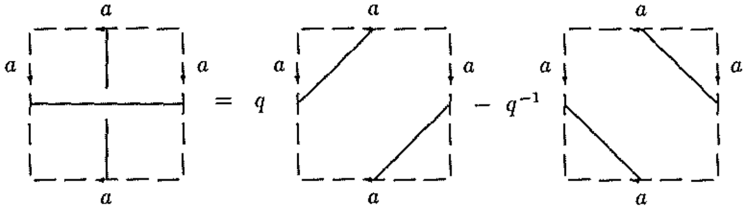
This is invariance under Reidemeister move III.

Reidemeister move I simplifies, for a given label a, to a scalar multiple of the identity $r(a)1_a$.



By using Reidemeister II move on the sphere S^2, one shows that $\tau(a)\overline{\tau(a)} = 1$.

In the crossing-fusion axiom, it is enough to give the crossing coefficients for generators of the fusion algebra, because of the fusion axiom. This is a general form of relations like the one satisfied by the Jones polynomial.

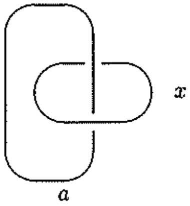


where the coefficient q satisfies $q^n = 1$.

Definition 2.2 For a crossing ε, we define the *opposite*, or *conjugate* crossing by $\bar{\varepsilon} = S(\varepsilon)$.

On the commutative fusion algebra given by $\mathcal{M}$, we define a character as follows.

Definition 2.3 For a braided system of bimodules, we define a character $\chi_x(a) \in \mathbf{C}$ for N-N bimodules a, x by the following picture, up to a normalization constant. This is the Hopf link.



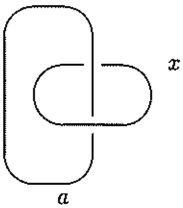
Note that this picture is regarded as an assignment of bimodules/intertwiners to the boundary of a 3-ball, then the partition function scheme with $6j$-symbols produce the number, which is the meaning of the above picture. (We assign a dotted square to each of the two crossing of the above knot. Then two squares glued together make a 2-sphere and intertwiners are assigned on it.) This is also equal to a weighted trace value of an intertwiner from $a \otimes x$ to itself.

This character is an invariant of the colored Hopf link, and gives an analogue of the S-matrix. We can prove the following.

Proposition 2.4 *For three N-N bimodules x, a, b, we have*

$$\chi_x(a)\chi_x(b) = \sum_c N_{ab}^c \chi_x(c).$$

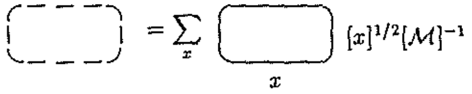
Definition 2.5 If a braided system of bimodules satisfies the following identity, then the system is called *nondegenerate*.

$$\sum_x$$


$$[x]^{1/2}[\mathcal{M}]^{-1} = \delta_{a,1}$$

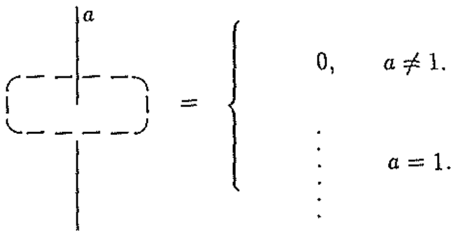
With the above, we make the following definition of a killing ring.

Definition 2.6 We define an element in the fusion algebra by the following formula, and call it a *killing ring*.



The killing ring satisfies the slide move, as in Figure 5 (c)–(f). The wire a in (c) expands as in (d) to the symmetric expression in (e) which further yields (f). The normalization coefficients have been omitted. This property was introduced (for $SU(2)_q$) by L. Kauffman.

With the above definition, the condition "nondegenerate" can also be stated as follows.



In the right hand side of the formula, the dotted vertical line means the identity bimodule ${}_N N_N$.

Here is an example of the character.

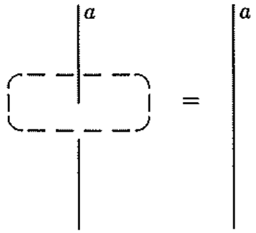
Example 2.7 Take a subfactor of type A_n. We label even vertices of the graph A_n by $1, 3, 5 \ldots$. Then the character pairing (j, k) is given by $\sin(jk\pi/(n+1))$ up to normalization constant. We list the tables for A_4 and A_5 (up to normalization constants).

$j \setminus k$	1	3
1	1	$-\varphi$
3	φ	-1

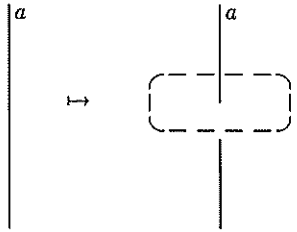
$j \setminus k$	1	3	5
1	1	2	1
3	2	-2	2
5	1	2	1

These show that the system A_4 is nondegenerate, but A_5 is degenerate.

The element a satisfying the following identity is called *degenerate*. The degenerate elements give a sub-fusion algebra.



The operator



is the projection onto the degenerates.

We have the following theorem for nondegenerate braided system of bimodules.

Theorem 2.8 *Suppose that a braided system $\mathcal{M}$ of bimodules is nondegenerate. Then the center of the corresponding tube algebra is labeled by pairs (a, b) of N-N bimodules a and b.*

Graphically, each label is denoted by a link of three components drawn on the surface of the two-dimensional torus; one is a killing ring set as a meridian, and the other two are set longitudinally and labeled by a, b respectively. The point is that the crossing of a and the killing ring is the opposite of the crossing of b and the killing ring. These label the Hilbert space of the two-dimensional torus of the TQFT.

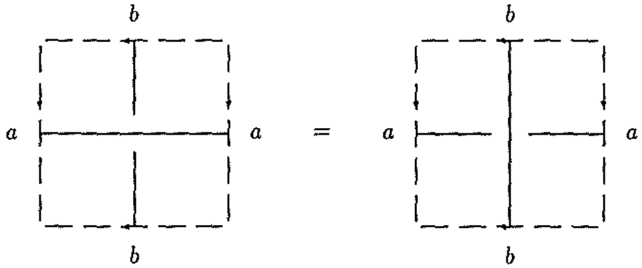
The crossing $\varepsilon_{a,b}$ between wires $a, b \in \mathcal{M}$ lives in the Hilbert space of the square with matching opposite edges, i.e., the space of the torus with one point. In general, $\varepsilon_{a,b}$ is not in the center of the tube algebra. However the sum $\varepsilon_*(a) = [\mathcal{M}]^{-1} \sum_b \varepsilon_{a,b}$ is central, and thus lives in the intrinsic space of the torus with no points. We represent $\varepsilon_*(a)$ by a longitudinal circle labeled a lying on top of the killing ring arranged as a a meridian. For $a \in \mathcal{M}$, denote by p_a the degenerate tube with two opposite edges labeled a and the other edges labeled 1 given by $1 \in \mathrm{Hom}(a \otimes 1, 1 \otimes a)$. Then p_a is a projection in the tube algebra.

Theorem 2.9 *Given a crossing ε on $\mathcal{M}$, for every $a \in \mathcal{M}$, the element $\varepsilon_*(a)$ is a minimal projection in the tube algebra Tube $\mathcal{M}$ of $\mathcal{M}$, which is central and satisfies $\varepsilon_*(a) \leq p_a$.*

The map $\varepsilon_* : \mathcal{M} \to \mathcal{T}$ gives an embedding of $(\mathcal{M}, \varepsilon)$ (with fusion and braiding) into the system $\mathcal{T}$ with the natural braiding. The vector $\varepsilon = [\mathcal{M}] \sum_{a \in \mathcal{M}} \varepsilon_*(a) \in$ Tube $\mathcal{M}$ satisfies $S^2(\varepsilon) = S\Gamma(\varepsilon) = \varepsilon$. The image $\varepsilon_*(\mathcal{M})$ is closed under the fusion and satisfies $\varepsilon_*(\mathcal{M})' = \overline{\varepsilon_*(\mathcal{M})} = \Gamma(\varepsilon_*(\mathcal{M}))$.

Conversely, suppose given a map $\rho : \mathcal{M} \to \mathcal{M}$ with $\rho(a) \leq p_a$ for any $a \in \mathcal{M}$ such that the vector $\xi_\rho = \sum_{a \in \mathcal{M}} \rho(a)$ satisfies the properties $S^2(\xi_\rho) = S\Gamma(\xi_\rho) = \xi_\rho$. Then ρ is an embedding of the system $\mathcal{M}$ into $\mathcal{T}$ and yields a braiding ε^ρ on $\mathcal{M}$, the preimage of the canonical braiding on $\mathcal{T}$. We have $\rho = (\varepsilon^\rho)_*$.

Definition 2.10 We say that the labels a and b *permutes* if the following graphically represented equality holds.



We define the *relative permutant* $\mathcal{N}' \cap \mathcal{M}$ of a subset $\mathcal{N}$ of $\mathcal{M}$ as the elements of $\mathcal{M}$ which permute with all the elements of $\mathcal{N}$.

Note that a is degenerate if it permutes with all b and that the degenerates are $\mathcal{M}' \cap \mathcal{M}$.

For the tube system $\mathcal{T}$, we have the following bipermutant property.

Theorem 2.11 *For any subsystem $\mathcal{N}$ of $\mathcal{T}$, we have $(\mathcal{N}' \cap \mathcal{T})' \cap \mathcal{T} = \mathcal{N}$.*

We say that the tube system is reflexive.

Next we study extensions of systems of bimodules.

Definition 2.12 A nondegenerate extension $\mathcal{M} \subset \mathcal{P}$ is called *minimal nondegenerate* if $\mathcal{M}' \cap \mathcal{P} = \mathcal{M}' \cap \mathcal{M}$.

We then have the following theorem.

Theorem 2.13 *For any braided bimodule system, there is an essentially unique minimal nondegenerate extension $\mathcal{M} \subset \mathcal{P}$, also called the ghost extension. The elements in the complement of $\mathcal{M}$ are called the ghosts.*

For example, for the case of A_{2n+1}, the system of N-N bimodules corresponds to the integer spin system. Its minimal nondegenerate extension is the embedding into the system of all the half-integer spins.

Besides the minimal nondegenerate extension,which is quite difficult to construct, there is another nondegenerate extension, the standard one. The intuitive idea of the standard extension is to take the picture of a crossing and drill tubes under each wire, thus obtaining a picture with wires but no crossings on a nontrivial surface. In the latter picture everything is determined by fusion alone. Here is an example of a standard nondegenerate extension.

Example 2.14 Let $\mathcal{M}$ denote the system of the M-M bimodules of the Jones subfactor $N \subset M$ of index 3 with principal graph A_5. Let G be the symmetric group S_3 with generators a, b of order 3, 2 respectively and irreducible representations $1, \sigma, \varepsilon$ of dimensions 1, 2, 1 respectively. The conjugacy orbits of G have representatives $1, a, b$. The centralizers of $1, a, b$ are the groups $S_3, \mathbf{Z}_3, \mathbf{Z}_2$ and have irreducible representations $\{1, \sigma, \varepsilon\}$, $\{1, \alpha, \bar{\alpha}\}$,$\{1, \varepsilon\}$ respectively. (Here ε denotes the sign representation and α is a primitive third root of 1). Identify the bimodules in $\mathcal{M}$ with Irr $G = \{1, \sigma, \varepsilon\}$ and the elements of $\mathcal{T}$ with the pairs (orbit representative, irreducible of centralizer), i.e., $\{(1,1),(1,\sigma),(1,\varepsilon),(a,1),(a,\alpha),(a,\overline{\alpha}),(b,1),(b,\varepsilon)\}$. Each element of $\mathcal{T}$ is selfconjugate, and Γ permutes $(a,\alpha),(a,\overline{\alpha})$ and leaves the other labels fixed. There are three different braidings of $\mathcal{M}$, one selfconjugate and the other two conjugate to each other. For any braiding, the associated embedding of $1, \varepsilon \in \mathcal{M}$ can be only $(1,1),(1,\varepsilon)$ respectively. The element σ can be mapped into any of $(1,\sigma),(a,\alpha),(a,\overline{\alpha})$, which gives the three crossings. The first map is $\rho_0 : \mathrm{Irr}\, G \ni \pi \mapsto (1,\pi)$. This corresponds to the canonical braiding on Irr G which is *completely degenerate*, i.e., the canonical homomorphism $\alpha \otimes \beta \to \beta \otimes \alpha$ for $\alpha, \beta \in \mathrm{Irr}\, G$. The other two maps ρ_+, ρ_- with images $\{(1,1),(a,\alpha),(1,\varepsilon)\}$, $\{(1,1),(a,\overline{\alpha}),(1,\varepsilon)\}$ respectively correspond to the identification of $\mathcal{M}$ with the integer spin representations of $SU(2)_q$, $q = \exp(\pi i/6)$. Since $\rho_+(\mathcal{M})$ and $\rho_-(\mathcal{M})$ are the permutant of each other, the degenerate part of each is their intersection $\{(1,1),(1,\varepsilon)\}$.

A braided bimodule system in which any two wires permute is called a *completely degenerate* system. An example is given by the natural permutation of a pair of representations of a finite (or compact) group. Using quantum deformations, this degenerate braiding is deformed to nondegenerate braidings. For completely degenerate systems, the minimal extension is easy to construct.

Theorem 2.15 *If $\mathcal{M}$ is completely degenerate, then the minimal extension coincides with the standard extension $\mathcal{T}$.*

Let $\mathcal{M}$ be a braided bimodule system with degenerate subsystem $\mathcal{D} = \mathcal{M}' \cap \mathcal{M}$ and minimal extension $\mathcal{P}$.

Consider the finite dimensional Hilbert space K consisting of vectors $\xi \in \bigoplus_{x,y\in\mathcal{P},m\in\mathcal{M}} \mathrm{Hom}(x \otimes y, m)$. The space A consisting of vectors

$$\begin{aligned}\xi &\in \bigoplus_{x_1,y_1,x_2,y_2\in\mathcal{P}} \mathrm{Hom}(x_1 \otimes x_2, y_1 \otimes y_2) \\ &= \bigoplus_{x_1,y_1,x_2,y_2\in\mathcal{P}} \bigoplus_{p\in\mathcal{P}} \mathrm{Hom}(x_1 \otimes x_2, p) \otimes_{\mathbb{C}} \mathrm{Hom}(p, y_1 \otimes y_2),\end{aligned}$$

for which

$$\xi \in \bigoplus_{x_1,y_1,x_2,y_2\in\mathcal{P}} \bigoplus_{m\in\mathcal{M}} \mathrm{Hom}(x_1 \otimes x_2, m) \otimes_{\mathbb{C}} \mathrm{Hom}(m, y_1 \otimes y_2) = K \otimes \overline{K},$$

as in Figure 5 (g), and

$$S\xi = {}^{y_1}\xi^{x_2} \in \bigoplus_{x_1,y_1,x_2,y_2\in\mathcal{P}} \bigoplus_{d\in\mathcal{D}} \mathrm{Hom}(\overline{y_1} \otimes x_1, d) \otimes_{\mathbb{C}} \mathrm{Hom}(d, y_2 \otimes \overline{x_2}),$$

as in Figure 5 (h). Then $A \subset K \otimes \overline{K} = \mathrm{End}(K)$ is closed under composition of homomorphisms, and has a finite dimensional C^*-algebra structure. The subalgebra $\mathrm{End}(K)$ is obtained from the algebra $\bigoplus_{x_1,y_1,x_2,y_2\in\mathcal{P}} \mathrm{Hom}(x_1 \otimes x_2, y_1 \otimes y_2)$ by reduction with the central projection π in Figure 5 (i), where the dotted killing ring is labeled by the degenerate elements $\mathcal{D}$; it is here that we use the minimality property $\mathcal{D}' \cap \mathcal{P} = \mathcal{M}$. The conditional expectation $E_A : \mathrm{End}(K) \to A$ is shown in Figure 5 (j), where the dashed killing ring is labeled by the observable elements $\mathcal{M}$, and we use the fact that $\mathcal{M}' \cap \mathcal{P} = \mathcal{D}$.

Theorem 2.16 *The tube algebra* Tube $\mathcal{M}$ *is isomorphic to the relative commutant* $A' \cap$ *End*(K). *The tube system* $\mathcal{T}$ *is labeled by the minimal projections in the center of the algebra* A.

The algebra $A' \cap \mathrm{End}(K) = \overline{K} \otimes_A K$ is contained into the subset

$$\bigoplus_{m_1,m_2\in\mathcal{M}} \bigoplus_{x,y\in\mathcal{P}} \mathrm{Hom}(m_1, x \otimes y) \otimes_{\mathbb{C}} \mathrm{Hom}(x \otimes y, m_2) \subset \overline{K} \otimes K$$

and map into Tube $\mathcal{M}$ as in the Figure 5 (l)–(m). The killing ring marked by the dashed line is the killing ring of the system $\mathcal{M}$. The Figure 5 (l)–(m) shows that this map is well defined on $\overline{K} \otimes_A K$. Note that in spite of the presence of elements of the minimal extension $\mathcal{P}$, by composing the elements in the tube one obtains an element of Tube $\mathcal{M}$, i.e., the "ghosts" in $\mathcal{P}$ are not observable. This motivates our terminology by analogy with the Faddeev-Popov ghosts in which non-observable half integer spin elements are introduced to remove the degeneracy of integer spin elements.

A tube (Figure 6 (a), with $x, y, a, b \in \mathcal{M}$) is written in terms of the above basis with coefficients as in the Figure 6 (b), where $t_+, t_- \in \mathcal{P}, \xi, \eta \in K$ and the dashed killing ring is labeled by $\mathcal{M}$. The proof of this crucial observation transforms (a) into the intermediate step (c), in which the dashed killing ring (labeled by $\mathcal{M}$) is killed by the dotted killing ring labeled by $\mathcal{P}$. The latter expands as in (d), and a slide move brings it to the form (e), from which (b) follows. Note that the sum in (b) is always in the image of $\overline{K} \otimes_A K$.

The Figure 6 (f) shows that the product of tubes corresponds to the product in $\overline{K} \otimes_A K$,

$$(\overline{\xi_1} \otimes_A \eta_1) \cdot (\overline{\xi_2} \otimes_A \eta_2) = \overline{\xi_1} \otimes_A \langle \eta_1, \xi_2 \rangle_A \cdot \eta_2 = \overline{\xi_1} \otimes_A E_A(\eta_1 \otimes \overline{\xi_2}) \cdot \eta_2.$$

The original system $\mathcal{M}$ embeds via the map ε_* into the tube algebra as in the Figure 6 (g). The product in (h) between a tube and $\varepsilon_*(u)$ is equal, as in (i), (j) to a scalar multiple of $\varepsilon_*(u)$. Thus $\varepsilon_*(u)$ is a minimal projection which is central. The general intertwiner in $\mathrm{Hom}(\varepsilon_*(a) \otimes \varepsilon_*(b), \varepsilon_*(c))$, shown in (k) transforms, using the slide moves in (l), into an element of $\varepsilon_*(\mathrm{Hom}(a \otimes b, c))$ in (m), thus showing that the homomorphism spaces of $\mathcal{M}$ and $\varepsilon_*(\mathcal{M})$ are naturally isomorphic.

Consider the asymptotic system $\mathcal{M}_\infty$ arising from a subfactor with finite depth. The bimodules (labels of edges) of the asymptotic system are naturally associated to labels of circles and to minimal central projections in the tube algebra in the TQFT of the initial system. The triangles in the asymptotic systems correspond to trinion (trouser) surfaces in the initial theory. The natural braiding is then obtained as follows.

Take a ball in $\mathbf{R}^3$ and remove two disjoint cylindrically holes through it, one above the other (i.e., make a solid picture of the crossing into a ball). The surface of of this body is made of two tubes together with a four-holed sphere (which decomposes into two trinions). On the tubes, put vectors corresponding to two asymptotic labels. The TQFT gives then a vector in the space of the four-holed sphere. This is a 3-dimensional thickening of the right hand side of the equality decomposing a crossing into fusions, after Definition 2.1. Take the boundary of a small tubular neighborhood of each member of this equality. Remark that the crossing tubes in the left member can be put inside inflated letter "H" in the right member, and then the boundaries of the two members can be joined on four circles, to give a 3-dimensional manifold with boundary. The invariant of this manifold gives the coefficient of the crossing. In this way, we get the following theorem.

Theorem 2.17 *The asymptotic system has a natural braiding. This braiding is non-degenerate if and only if the fusion graph is connected. The labels of the minimal nondegenerate extension of this system are the minimal projections in the center of the tube algebra.*

3 Chirality and TQFT

Given a link L in S^3, choose a projection L_0 on S^2 for which the writhe is 0 (i.e., the canonical framing of L coincides with the vertical framing — also called blackboard framing — of L_0; this can be done from any projection of L by adding twists on each component). Choose arbitrary labels for each link component. Then we have an assignment of a dotted square at each crossing, which means we have intertwiners at squares that make a 2-sphere. (Note that the first two axioms in the definition of the braiding system imply that this assignment of intertwiners is well-defined.) This is regarded as a labeling of the boundary of a 3-ball, thus the state sum gives a complex number associated to L_0. This is an invariant of L. For $SU(2)_k$ and the spin 1/2 label, this coincides with the value of the Jones polynomial at roots of unity.

Take now an arbitrary link projection and sum the previous invariants, by replacing each link component with the killing ring. Then we obtain an invariant of the 3-manifold given by Dehn surgery with vertical framing on L. The crossing-fusion axiom implies invariance under handle slides, thus our invariant of framed links is invariant under Kirby moves and we get an invariant of compact oriented 3-manifolds. This is a generalization of the Reshetikhin-Turaev TQFT [6], which is essentially a weighted sum of invariants of a framed link over all the colorings for $U_q(sl_2)$. Thus we have two methods to construct a TQFT from a system $\mathcal{M}$. One is based on triangulation and the quantum $6j$-symbols for $\mathcal{M}$ [4] and the other is based on vertical surgery as above. The relation between the two is given as follows.

Theorem 3.1 *Suppose that the system $\mathcal{M}$ of bimodules is braided and nondegenerate. Then the TQFT based on triangulation splits as a tensor product of another TQFT based on vertical surgery and its complex conjugate.*

This generalizes a result of Turaev [7], which is based on his theory of shadows and 4-dimensional manifolds.

We give a sketch of our proof of the above theorem. We use Figures 1 – 3 at the end of this paper. We start with a 3-manifold given by a surgery with a link with vertical framing, and compute the invariant based on triangulation first.

In Figure 1, the first picture represents the fact that the manifold obtained by vertical surgery on a link is obtained by joining wedges (called thin wedges, in which the two tubes are contracted to circles) on the adjacent faces; this is a purely topological observation.

In Figure 1, the dashed circles (killing rings) and the 4 wires crossing the edges of the wedge (all pictured thick black) are in the plane of the surface of the wedge. The thick darker wires (one in each hole) are slightly above the killing ring and the thick lighter wires are slightly below the killing ring; both go through the hole of the wedge. The same convention applies in the other pictures. The black wire is in the plane, dark slightly above and light slightly below.

The first one describes a basis of the Hilbert space of the surface of the wedge; the drawings are links with crossings on the surface of the wedge. Compare this situation with that in the remark after Theorem 2.8. (Only the thick wires matter on the wedge; the others merely show its shape.)

The fact used next is that the invariant for a picture on the surface of a ball B^3 is obtained by composing the respective homomorphisms. For a sphere with handles, if there is a single wire labeled a on a handle, then the result is 0 if $a \neq 1$, and if $a = 1$ then one removes the wire a and cuts the handle. (This expresses the fact that the handle is full).

The slide move is applied to the bottom killing ring, after which there are only killing rings left on the two handles. These are cut as above and the picture remaining on the sphere is simplified to obtain the final picture of Figure 1.

That is, we find a basis in the Hilbert space of an annulus with two entries and exits. The surface of the wedge is made of two such annuli. Take arbitrary labels for the thick wires and intertwiners for the eight triple points. The partition function of the wedge applied to these will be the evaluation of the last picture in Figure 1.

The we have two links with opposite crossings which slide away, and we are done.

The pictures in Figure 3 describes a proof of the equivalence between a spin model, in which there are numbers at each crossing having matching faces (the blobs are arbitrary) and a wire model. There are "bridge" moves and "close" moves, done until all faces of the knot projection are exhausted (a leftover ring cancels a normalization coefficient). The regions of the complement of the link projection must be ordered so that the union of the first k be simply connected for each k. In particular, this gives an easy proof of the equivalence between two models of Reshetikhin for $SU(2)_q$, first proved in a quite complicated way by Viro. If blobs are replaced by our computation of the wedge, one can finish the proof as in Figure 1. The point is that this proof extends easily to the degenerate case.

In summary, the pair of triple points on each face of the wedge must be matched to a pair of triple points on a neighboring wedge, and then summed. This is done in Figure 1 using the theorem proved in Figure 3. The result is two separate copies of the link. This all works in the nondegenerate case. In the degenerate case, the "over" and "under" worlds are connected by degenerates.

Let the closed 3-manifold V be obtained by vertical surgery on a link L with components L_i, $i \in I$. Let $\mathcal{M}$ be a braided system of bimodules with degenerate subsystem $\mathcal{D}$ and minimal nondegenerate extension $\mathcal{P}$. Let λ denote a labeling of the components of L by elements of $\mathcal{P}$, with L_i labeled by $p_i \in \mathcal{P}$, and denote by $\tau_\lambda(L)$ the corresponding knot invariant of L. Let $\mathcal{P}^I$ denote the set of labelings λ above. For $p, q \in \mathcal{P}$ let

$$[p,q]_{\mathcal{M}} = [\mathcal{M}]^{-1}[\mathcal{P}]^{-1} \sum_{m \in \mathcal{M}} N^m_{p,\overline{q}}[m]^{1/2}.$$

Here the fusion number $N^m_{p,\overline{q}}$ denotes the multiplicity of m in $p \otimes \overline{q}$. Since $m \in M$ induces the trivial character on the degenerate algebra $\mathcal{D}$, the only pairs $p, q \in \mathcal{P}$ with $[p,q]_{\mathcal{M}} \neq 0$ are the pairs for which p and q induce the same degenerate character.

A modification of the argument described in the Figures 1–3 proves the following result.

Theorem 3.2 *The triangulation invariant $Z^{(\mathcal{M})}(V)$ satisfies*

$$Z^{(\mathcal{M})}(V) = \sum_{\lambda,\mu\in\mathcal{P}^I} \prod_{i\in I} [\lambda_i, \mu_i]_{\mathcal{M}} \tau_\lambda(L)\overline{\tau_\mu(L)}.$$

A sketch of the proof of this result is the following. In the Figure 2 (e), one obtains a copy of the projection π in Figure 5 (i) for every link segment between two crossings of the original link. Up to this point the computations were done in $\mathcal{M}$; we switch now to the system $\mathcal{P}$. Since we are computing the invariant of the ball B^3 with marked surface, the invariant is simply multiplied in this process by a normalization factor. We now go backwards to (c), to find that we are computing the invariant of the link (c) with each tube component marked by the projection π. The inner product of π with a minimal central projection of Tube $(\mathcal{P})$ labeled with projections $p, q \in \mathcal{P}$ (recall that $\mathcal{P}$ is nondegenerate) is, up to a factor, precisely $[p, q]_{\mathcal{M}}$ described above.

In particular, this result shows that although the subfactor of type E_6 generates an abelian bimodule system, there exists no braiding on it, since Niţică and Török have shown that the invariant of the lens space $L(3,1)$ is not real.

At the end, we explain two examples of nondegenerate braided systems of bimodules.

From a finite depth subfactor, we pass to the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. Then the system of M_∞-M_∞ bimodules is braided and nondegenerate as mentioned above, if the fusion graph is connected. The above theorem implies splitting of the corresponding TQFT, but it is easy to see the splitting directly. Instead of the M_∞-M_∞ bimodules, we can use the $M \vee (M' \cap M_\infty)$-$M \vee (M' \cap M_\infty)$ bimodules. Then from the description of the corresponding quantum $6j$-symbols, the TQFT splits as a tensor product of a TQFT based on the triangulation for the original inclusion $N \subset M$ with the system of the M-M bimodules and its complex conjugate.

Consider the subfactors of type A_n. The even vertices of the principal graph correspond to N-N bimodules and odd vertices to N-M bimodules. So it is impossible to define multiplication among the odd vertices in general, but in the case of A_n subfactors, we can make a fusion algebra in which all the multiplications are possible regardless parity of the vertices by using endomorphisms. In this way, we can construct a TQFT based on triangulation from the A_n subfactors, and then this is exactly same as the one considered by Turaev-Viro [8] with $q = \exp(\pi i/(n+1))$. This system is braided and nondegenerate, and the above theorem gives another proof of a theorem of Turaev [7] that the Turaev-Viro TQFT [8] is a tensor product of the Reshetikhin-Turaev TQFT [6] and its complex conjugate.

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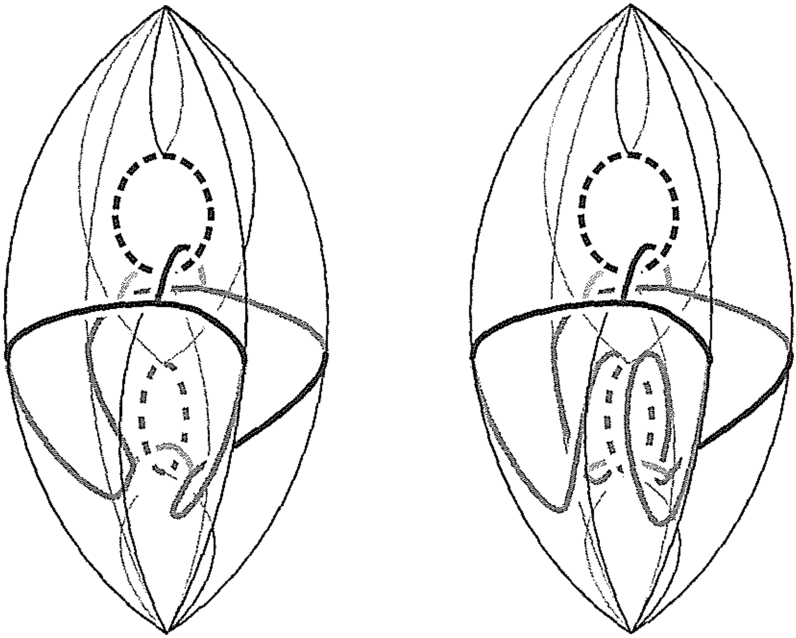
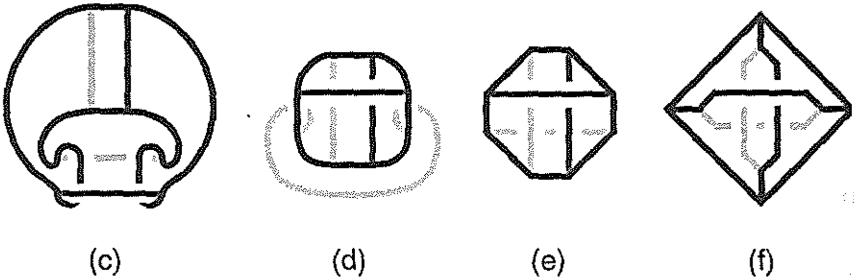
Figure 1

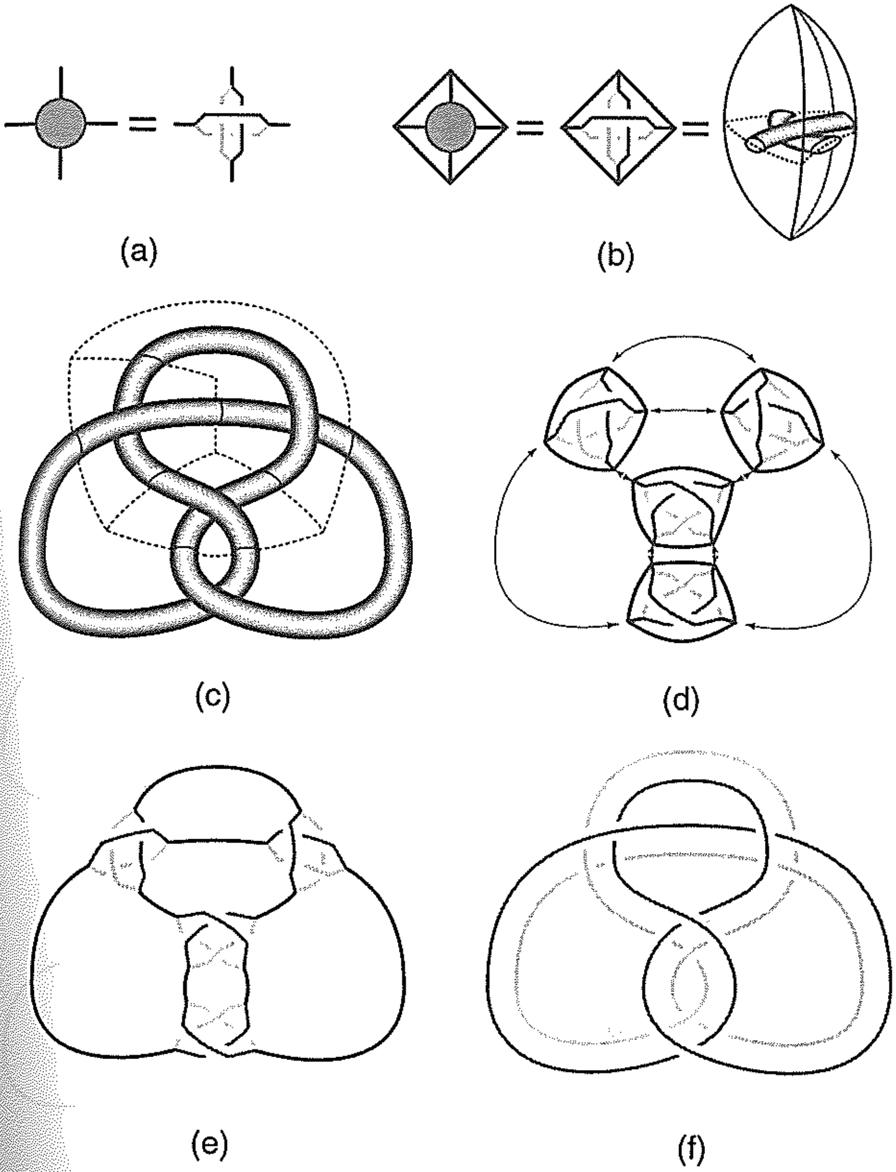


(a) (b)



(c) (d) (e) (f)

Figure 2



(a) (b) (c) (d) (e) (f)

Figure 3

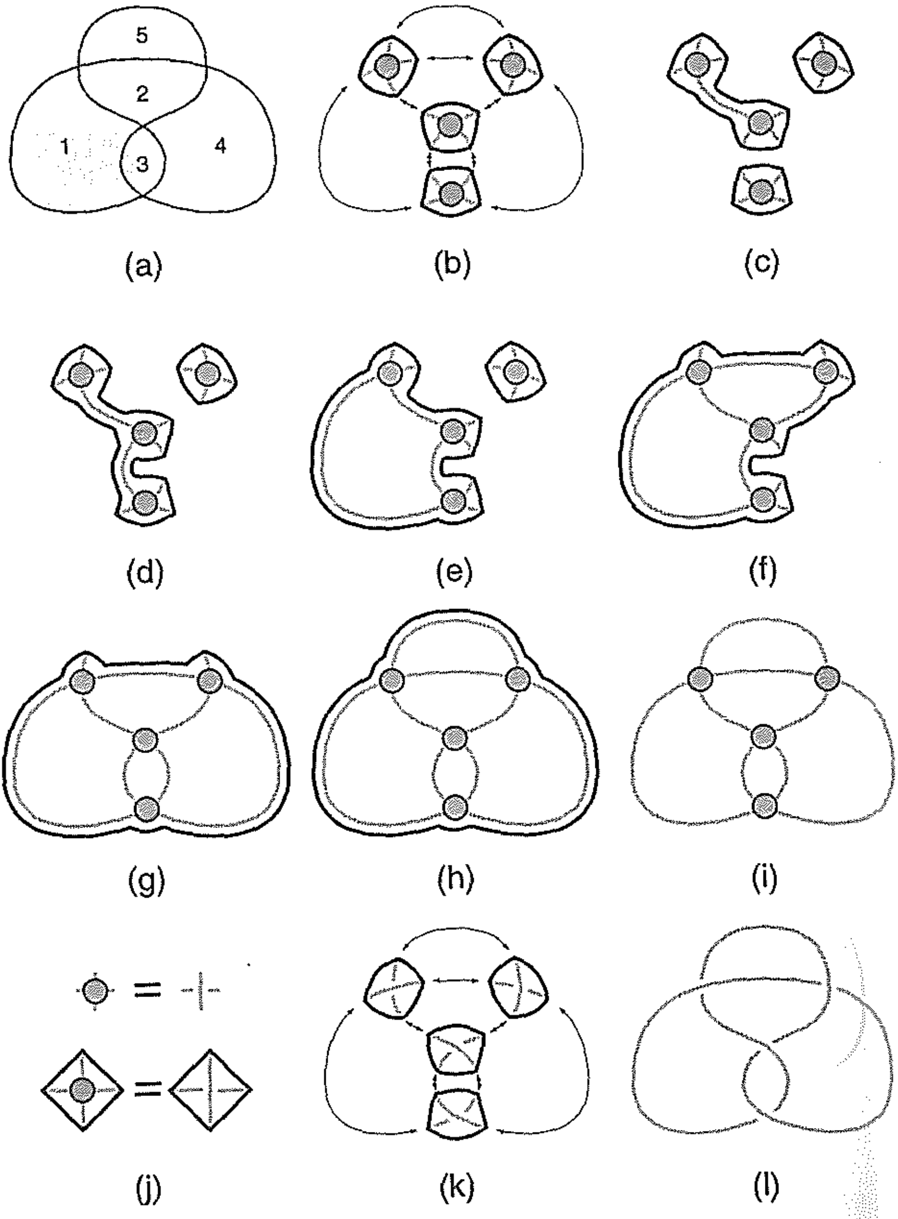


Figure 4

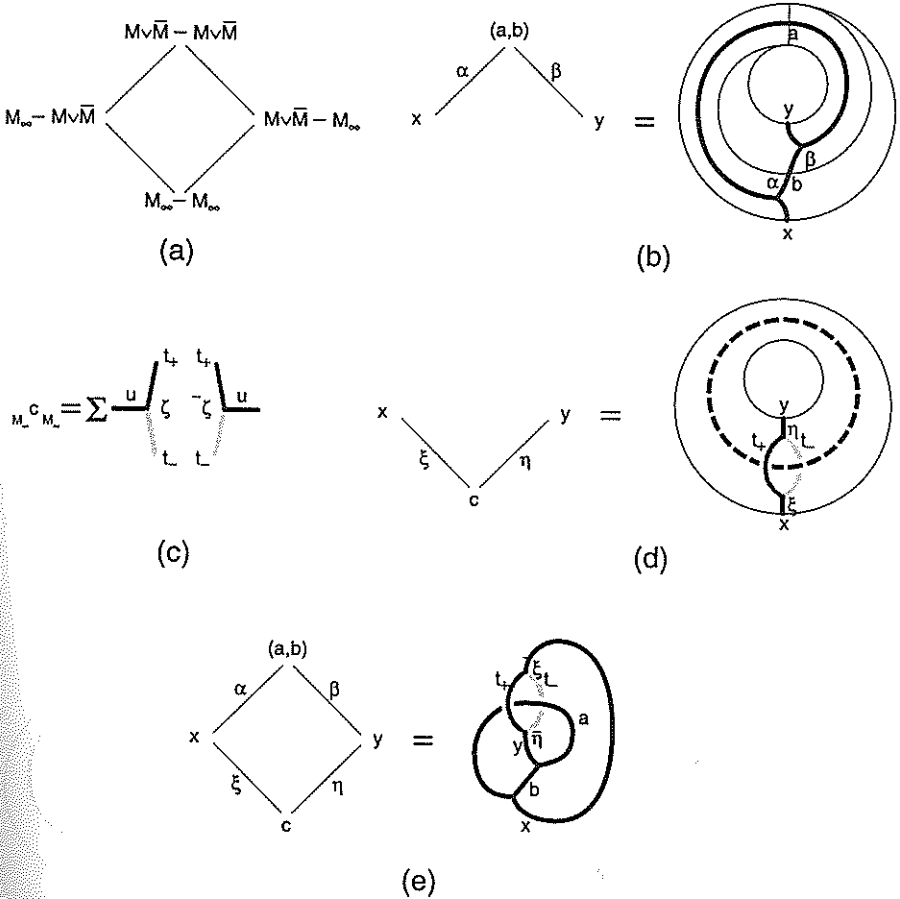

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Figure 5

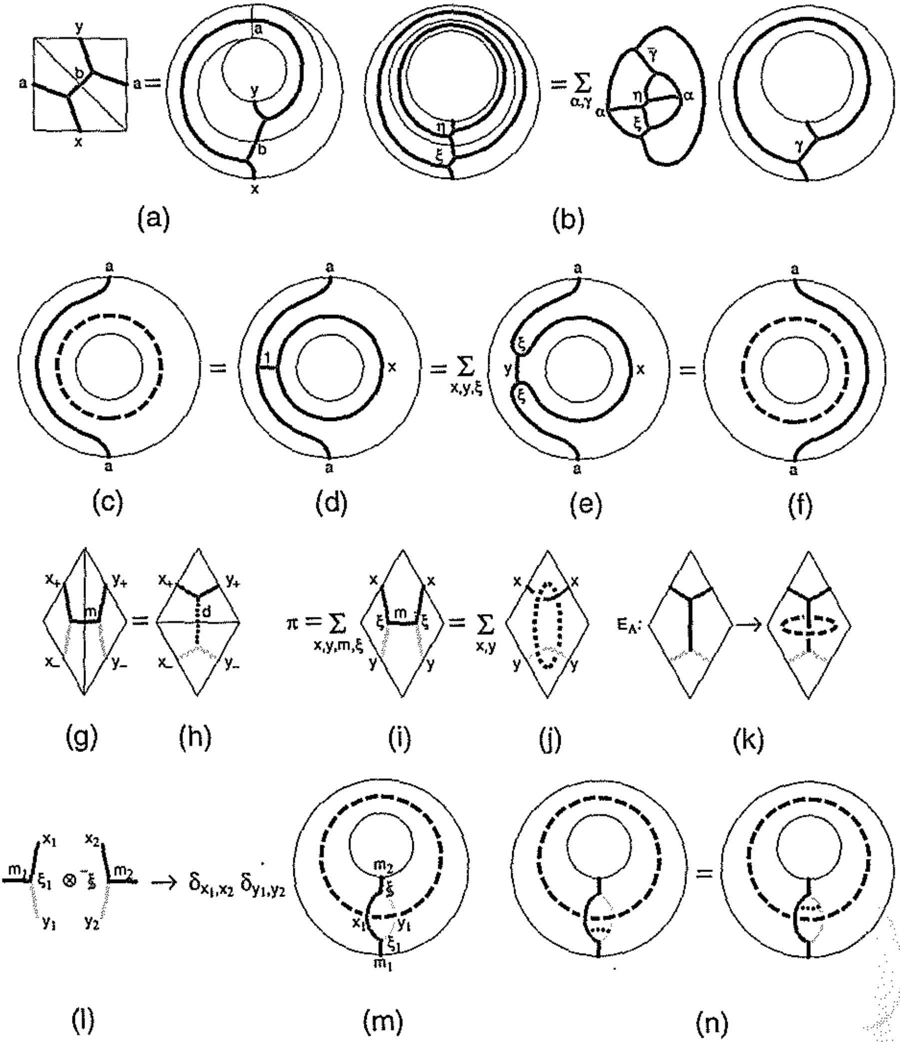
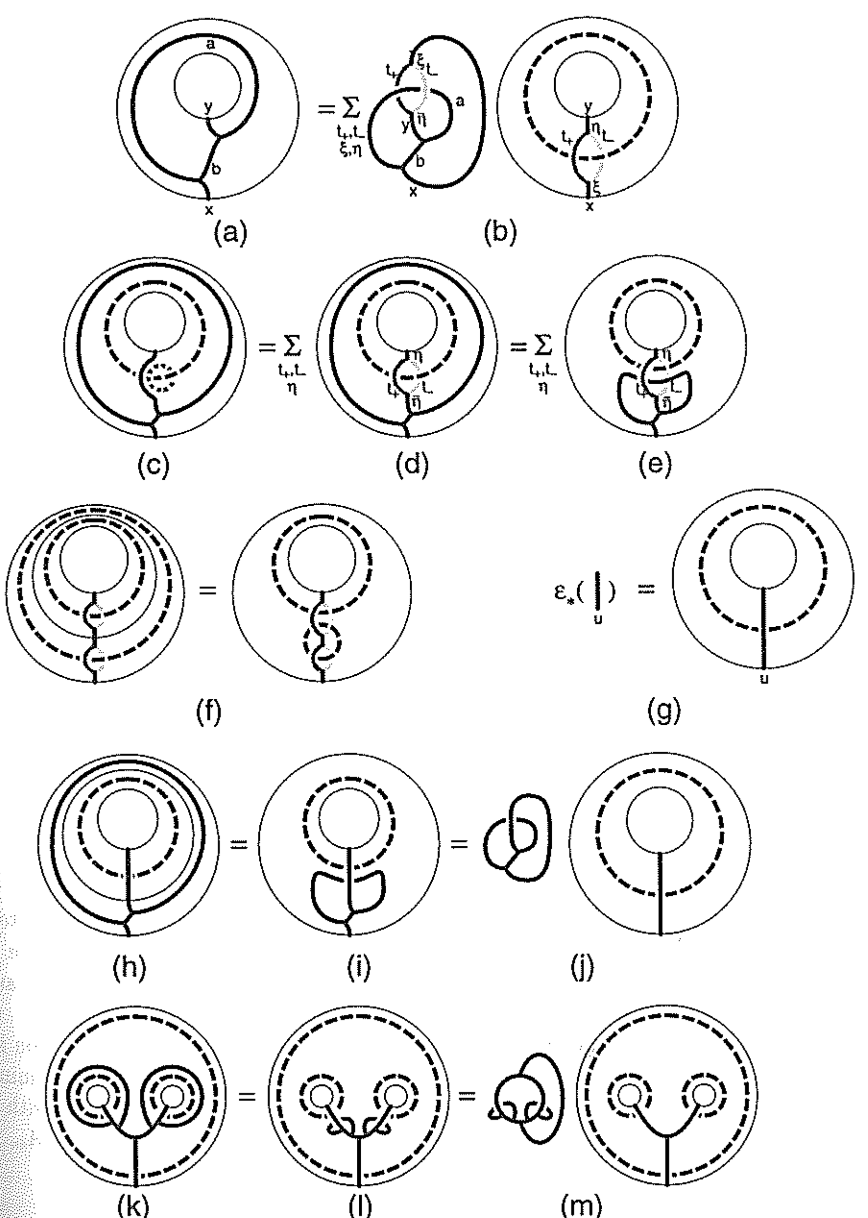


Figure 6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