

where n denotes the distance from the $*$ -point to the first branch point. The pair in case (2) should be considered as an unordered pair, i.e. either $(\Gamma, \Gamma') = (\Gamma_n, \Gamma'_n)$ or $(\Gamma, \Gamma') = (\Gamma'_n, \Gamma_n)$ for some $n \equiv 3 \pmod{4}$. An elementary computation shows that $\|\Gamma_3\|^2 = \|\Gamma'_3\|^2 = \frac{5+\sqrt{13}}{2}$. Hence we have

Corollary 1.2. *If an irreducible inclusion $N \subset M$ has index in the interval*

$$4 < [M : N] < \frac{5 + \sqrt{13}}{2}$$

then $\Gamma = \Gamma' = A_\infty$. In particular, the inclusion is non-amenable in the sense of Popa [P2].

We expect that all the above graphs Γ_n and Γ'_n , $n \equiv 3 \pmod{4}$ can be realized as principal graphs of subfactors of the hyperfinite II_1 -factor R , but so far we have only been able to construct subfactors of R with principal graphs Γ_3 and Γ'_3 . (cf. Section 7).

Theorem 1.1 shows that the examples of subfactors of R constructed by Schou and the author in [S] based on the “3-stars” $T(2, 2, n)$, $n \geq 3$ and $T(3, 3, n)$, $n \geq 4$ $n \not\equiv 3 \pmod{4}$ all have infinite depth with principal graph A_∞ , and the same holds for Ocneanu’s subfactor based on the graph E_{10} .

It is easy to check that $\|\Gamma_n\|^2 = \|\Gamma'_n\|^2$ for all n . Numerically,

$$\|\Gamma_3\|^2 \approx 4.30278$$

$$\|\Gamma_7\|^2 \approx 4.37720$$

$$\|\Gamma_{11}\|^2 \approx 4.38271$$

and

$$\lim_{n \rightarrow \infty} \|\Gamma_n\|^2 \approx 4.38298.$$

The upper bound for the index range considered in Theorem 1.1 is $3 + \sqrt{2} \approx 4.41421$.

The rest of this paper is divided in 6 sections:

2. Principal graphs and Hilbert bimodules.
3. Ocneanu’s biunitary condition and the triple point obstruction.
4. Limiting the possibilities for one of the principal graphs Γ and Γ' .
5. Reconstructing Γ' from Γ .
6. Excluding $n \equiv 1 \pmod{4}$.
7. Announcement of further results.

§2. Principal graphs and Hilbert bimodules

Let $N \subset M$ be an inclusion of II_1 -factors of finite index, and let

$$N \subset M \subset M_1 \subset M_2 \subset \dots$$

be the associated Jones tower. Following [GHJ] the principal graph Γ of the inclusion $N \subset M$ is the principal part of the Bratteli diagram for the tower

$$C \subseteq N' \cap M \subset N' \cap M_1 \subset \dots$$

and similarly the principal graph Γ' of the inclusion $M \subset M_1$ is the principal part of the Bratteli diagram for the tower

$$C \subseteq M' \cap M_1 \subset M' \cap M_2 \subset \dots$$

We let $*$ (resp. $'$) denote the vertex in Γ (resp. Γ') corresponding to the C -component. Γ and Γ' are bipartite graphs

$$\Gamma = \Gamma_{\text{even}} \cup \Gamma_{\text{odd}}, \quad \Gamma' = \Gamma'_{\text{even}} \cup \Gamma'_{\text{odd}}$$

where Γ_{even} (resp. Γ_{odd}) is the set of vertices in Γ of even (resp. odd) distance to the $*$ -vertex, and $\Gamma'_{\text{even}}, \Gamma'_{\text{odd}}$ are defined in the same way.

Following Ocneanu's picture [O1], the four sets $\Gamma_{\text{even}}, \Gamma_{\text{odd}}, \Gamma'_{\text{even}}$ and Γ'_{odd} can be identified with sets of Hilbert bimodules: To the inclusion $N \subset M$ is associated the N - M Hilbert bimodule

$$\alpha = {}_N L^2(M)_M$$

given by

$$\alpha(a, b)x = axb, \quad a \in N, \quad b \in M, \quad x \in L^2(M).$$

We let $\bar{\alpha}$ be the contragredient M - N Hilbert bimodule

$$\bar{\alpha}(a, b)x = axb, \quad a \in M, \quad b \in N, \quad x \in L^2(M).$$

Moreover $\mathcal{E}_N$ (resp. $\mathcal{E}_M$) denote the trivial N - N bimodule $L^2(N)$ (resp. the trivial M - M bimodule $L^2(M)$). For two Hilbert bimodules α and β :

$$\alpha = {}_P H_Q, \quad \beta = {}_Q K_R, \quad P, Q, R \in \{M, N\},$$

we let $\alpha\beta$ denote the P - R -Hilbert bimodule $\alpha \otimes_Q \beta$. With this notation we may write

Γ_{even} = set of irreducible N - N bimodules occuring in the decompositions of $\mathcal{E}_N, \alpha\bar{\alpha}, (\alpha\bar{\alpha})^2, \dots$

Γ_{odd} = Set of irreducible N - M bimodules occuring in the decompositions of $\alpha, \alpha\bar{\alpha}\alpha, \alpha(\bar{\alpha}\alpha)^2, \dots$

Γ'_{even} = set of irreducible M - M bimodules occuring in the decompositions of $\mathcal{E}_M, \bar{\alpha}\alpha, (\bar{\alpha}\alpha)^2, \dots$

Γ'_{odd} = set of irreducible M - N bimodules occuring in the decompositions of $\bar{\alpha}$, $\bar{\alpha}\alpha\bar{\alpha}$, $\bar{\alpha}(\alpha\bar{\alpha})^2, \dots$

with these identifications, $*$ = $\mathcal{E}_N$ and $*'$ = $\mathcal{E}_M$. For two P - Q Hilbert bimodules γ, δ ($P, Q \in \{M, N\}$) we let $\langle \gamma, \delta \rangle$ denote the dimension of the space of intertwiners for γ and δ . With this notation, the adjacency matrix Δ_Γ for Γ can be written in the following way

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G' & 0 \end{pmatrix}$$

where G is the " $\Gamma_{\text{even}} \times \Gamma_{\text{odd}}$ -matrix" given by

$$G_{\gamma\delta} = \langle \gamma\alpha, \delta \rangle = \langle \gamma, \delta\bar{\alpha} \rangle, \quad \gamma \in \Gamma_{\text{even}}, \quad \delta \in \Gamma_{\text{odd}}.$$

(The last equality follows from Frobenius reciprocity, cf. [Su, p.245, Proposition 6]). The adjacency matrix

$$\Delta_{\Gamma'} = \begin{pmatrix} 0 & G' \\ (G')^t & 0 \end{pmatrix}$$

is given similarly:

$$G'_{\gamma'\delta'} = \langle \gamma'\bar{\alpha}, \delta' \rangle = \langle \gamma', \delta'\alpha \rangle, \quad \gamma' \in \Gamma'_{\text{even}}, \quad \delta' \in \Gamma'_{\text{odd}}.$$

The following three observations are probably well known:

Proposition 2.1. For every $n \in \mathbb{N}$,

$$\dim(N' \cap M_{n-1}) = \dim(M' \cap M_n).$$

Proof. For n odd, $n = 2k + 1$, the statement is equivalent to that the dimension of the space of selfintertwiners for $\alpha(\bar{\alpha}\alpha)^k$ is the same as the dimension of selfintertwiners of $\bar{\alpha}(\alpha\bar{\alpha})^k$, but this is clear, since $\alpha(\bar{\alpha}\alpha)^k$ and $\bar{\alpha}(\alpha\bar{\alpha})^k$ are contragredient Hilbert bimodules. (Actually in this case $N' \cap M_{n-1}$ and $M' \cap M_n$ are isomorphic).

For n : even, $n = 2k$; we shall prove that the dimension of the space of selfintertwiners of $(\alpha\bar{\alpha})^k$, is the same as the dimension of the space of selfintertwiners for $(\bar{\alpha}\alpha)^k$, i.e.

$$\langle (\alpha\bar{\alpha})^k, (\alpha\bar{\alpha})^k \rangle = \langle (\bar{\alpha}\alpha)^k, (\bar{\alpha}\alpha)^k \rangle.$$

However, this follows from Frobenius' reciprocity, because

$$\begin{aligned} \langle (\alpha\bar{\alpha})^k, (\alpha\bar{\alpha})^k \rangle &= \langle (\alpha\bar{\alpha})^{k-1}\alpha, (\alpha\bar{\alpha})^k\alpha \rangle \\ &= \langle \alpha(\bar{\alpha}\alpha)^{k-1}, \alpha(\bar{\alpha}\alpha)^k \rangle \\ &= \langle (\bar{\alpha}\alpha)^k, (\bar{\alpha}\alpha)^k \rangle. \end{aligned}$$

Proposition 2.2.

1) The contragredience map $\gamma \rightarrow \bar{\gamma}$ on Γ_{even} is a permutation π of order 2, and

$$\text{dist}_{\Gamma}(*, \gamma) = \text{dist}_{\Gamma}(*, \bar{\gamma}), \quad \forall \gamma \in \Gamma_{\text{even}}.$$

2) The contragredience map $\gamma' \rightarrow \bar{\gamma}'$ on Γ'_{even} is a permutation π' of order 2, and

$$\text{dist}_{\Gamma'}(*, \gamma') = \text{dist}_{\Gamma'}(*, \bar{\gamma}'), \quad \forall \gamma' \in \Gamma'_{\text{even}}.$$

3) The contragredience map $\gamma \rightarrow \bar{\gamma}$ is a bijection of Γ_{odd} onto Γ'_{odd} and

$$\text{dist}_{\Gamma}(*, \gamma) = \text{dist}_{\Gamma'}(*, \bar{\gamma}), \quad \forall \gamma \in \Gamma_{\text{odd}}$$

Proof.

1) Clearly $\pi^2 = 1$. Moreover for $\gamma \in \Gamma_{\text{even}}$

$$\begin{aligned} \text{dist}_{\Gamma}(*, \gamma) &= \min\{2k \mid \langle (\alpha\bar{\alpha})^k, \gamma \rangle \neq 0\} \\ &= \min\{2k \mid \langle (\alpha\bar{\alpha})^k, \bar{\gamma} \rangle \neq 0\} \\ &= \text{dist}_{\Gamma}(*, \bar{\gamma}). \end{aligned}$$

2) Same as 1).

3) It is clear that $\gamma \rightarrow \bar{\gamma}$ is a bijection of Γ_{odd} onto Γ'_{odd} . For $\gamma \in \Gamma_{\text{odd}}$,

$$\begin{aligned} \text{dist}_{\Gamma}(*, \gamma) &= \min\{2k+1 \mid \langle (\alpha\bar{\alpha})^k, \gamma \rangle \neq 0\} \\ &= \min\{2k+1 \mid \langle \bar{\alpha}(\alpha\bar{\alpha})^k, \bar{\gamma} \rangle \neq 0\} \\ &= \text{dist}_{\Gamma'}(*, \bar{\gamma}). \quad \blacksquare \end{aligned}$$

Proposition 2.3. If we identify Γ_{odd} and Γ'_{odd} according to the contragredient map $\gamma \rightarrow \bar{\gamma}$, then

- 1) $\pi G(G')^t = G(G')^t \pi'$,
- 2) $G^t \pi G = (G')^t \pi' G'$,

where π and π' are now considered as permutation matrices.

Proof. Let $Z(\Gamma_{\text{even}})$, $Z(\Gamma_{\text{odd}})$, $Z(\Gamma'_{\text{even}})$, $Z(\Gamma'_{\text{odd}})$ be the set of formal linear combinations over Z of the given sets. Let L_{α} and $L_{\bar{\alpha}}$ (resp. R_{α} and $R_{\bar{\alpha}}$) be left multiplication with α and $\bar{\alpha}$ (resp. right multiplication with α and $\bar{\alpha}$), whenever it is defined. Then we have the following commuting diagram of Z -linear maps:

$$\begin{array}{ccccc} Z(\Gamma'_{\text{odd}}) & \xrightarrow{R_{\alpha}} & Z(\Gamma'_{\text{even}}) & \xrightarrow{R_{\bar{\alpha}}} & Z(\Gamma'_{\text{odd}}) \\ L_{\alpha} \uparrow & & L_{\alpha} \uparrow & & L_{\alpha} \uparrow \\ Z(\Gamma_{\text{even}}) & \xrightarrow{R_{\alpha}} & Z(\Gamma_{\text{odd}}) & \xrightarrow{R_{\bar{\alpha}}} & Z(\Gamma_{\text{even}}). \end{array}$$

If we consider the given Z modules as spaces of row vectors, then the 7 maps can be considered as multiplication from right by Z -matrices. The corresponding 7 matrices are

$$\begin{array}{ccccc} Z(\Gamma'_{\text{odd}}) & \xrightarrow{(G')^t} & Z(\Gamma'_{\text{even}}) & \xrightarrow{G'} & Z(\Gamma'_{\text{odd}}) \\ \pi G \uparrow & & \uparrow (G')^t \pi' & & \uparrow \pi G \\ Z(\Gamma_{\text{even}}) & \xrightarrow{G} & Z(\Gamma_{\text{odd}}) & \xrightarrow{G^t} & Z(\Gamma'_{\text{even}}). \end{array}$$

under the identification $Z(\Gamma'_{\text{odd}}) \cong Z(\Gamma_{\text{odd}})$ given by the contragredience map. Hence

$$\pi G(G')^t = G(G')^t \pi' \quad \text{and} \quad (G')^t \pi' G' = G^t \pi G. \quad \blacksquare$$

§3. Ocneanu's biunitary condition and the triple point obstruction

Throughout this section $N \subset M$ is an irreducible inclusion of $\mathbb{I}_1$ -factors of finite index, and Γ and Γ' denote the principal graphs of the inclusions $N \subset M$ and $M \subset M_1$ as in section 2. For any P - Q -Hilbert bimodule $\gamma = {}_P H_Q$, $P, Q \in \{M, N\}$ we let $\xi(\gamma)$ denote the square root of the index of the corresponding inclusion $L_\gamma(P) \subset R_\gamma(Q)'$. Here L_γ (resp. R_γ) denotes the left action of P on H (resp. right action of Q on H). Since $N \subset M$ is irreducible, it is extremal in the sense of Popa [P1]. Therefore the restrictions ξ_Γ (resp. $\xi_{\Gamma'}$) of ξ to Γ (resp. Γ') is an eigenvector for the adjacency matrix Δ_Γ (resp. $\Delta_{\Gamma'}$) with eigenvalue $[M : N]^{1/2}$ i.e.

$$\Delta_\Gamma \xi_\Gamma = [M : N]^{1/2} \xi_\Gamma, \quad \Delta_{\Gamma'} \xi_{\Gamma'} = [M : N]^{1/2} \xi_{\Gamma'}.$$

From the definition of ξ we have immediately

Proposition 3.1.

- 1) $\xi_\Gamma(*) = \xi_{\Gamma'}(*)' = 1$
- 2) For all $\gamma \in \Gamma_{\text{even}}$, $\xi_\Gamma(\bar{\gamma}) = \xi_\Gamma(\gamma)$
- 3) For all $\gamma' \in \Gamma'_{\text{even}}$, $\xi_{\Gamma'}(\bar{\gamma}') = \xi_{\Gamma'}(\gamma')$
- 4) For all $\gamma \in \Gamma_{\text{odd}}$, $\xi_{\Gamma'}(\bar{\gamma}) = \xi_\Gamma(\gamma)$.

The following observation is well known and very useful:

Proposition 3.2. (The $2 \cos(\pi/n)$ -rule)

- 1) For all $\gamma \in \Gamma$, $\xi_\Gamma(\gamma) \in \{2 \cos \frac{\pi}{n}, n \geq 3\} \cup [2, \infty[$.
- 2) Let Γ_1 be a finite graph. If Γ_1 is the principal graph for some irreducible subfactor, and if ξ_1 is the unique Perron-Frobenius eigenvector for Δ_{Γ_1} on Γ_1 , for which

$$\min_{\gamma \in \Gamma_1} \xi_1(\gamma) = 1,$$

then for all $\gamma \in \Gamma_1$, $\xi_1(\gamma) \in \{2 \cos \frac{\pi}{n}, n \geq 3\} \cup [2, \infty[$

Proof.

1) It follows immediately from Jones' fundamental result [J], that

$$\xi(\gamma)^2 = [R_\gamma(Q) : L_\gamma(P)]^2 \in \{4 \cos^2 \frac{\pi}{n}, n \geq 3\} \cup [4, \infty[$$

for any P - Q -Hilbert bimodule γ of finite index.

2) From 1) we get

$$\min\{\xi_{\Gamma_1}(\gamma), \gamma \in \Gamma_1\} = \xi_{\Gamma_1}(*) = 1.$$

Hence 2) follows by uniqueness of the Perron-Frobenius eigenvector of Δ_{Γ_1} up to multiplication with positive real numbers, and the fact that $\|\Gamma_1\|^2 = [M : N]$ in the finite depth case. ■

We state next Ocneanu's biunitary condition. It suffices for our applications to consider only graphs with simple edges:

Proposition 3.3. (Ocneanu's biunitary condition) Assume that both Γ and Γ' have only simple edges. Let

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}, \quad \Delta_{\Gamma'} = \begin{pmatrix} 0 & G' \\ (G')^t & 0 \end{pmatrix}$$

and let π and π' be the contragredience maps on Γ_{even} and Γ'_{odd} respectively as in section 2. Set

$$H = (G')^t, \quad K = \pi G, \quad L = (\pi' G')^t$$

Then G, H, K, L are $\{0, 1\}$ -matrices, and

$$GL = KH \quad \text{and} \quad G^t K = LH^t$$

Moreover there exist (possibly infinite) unitary matrices u, v of the form

$$u = \oplus u^{(i,k)}, \quad (i,k) \in \Gamma_{\text{even}} \times \Gamma'_{\text{even}}, \quad (GL)_{ik} > 0, \\ v = \oplus v^{(j,l)}, \quad (j,l) \in \Gamma_{\text{odd}} \times \Gamma'_{\text{odd}}, \quad (G^t K)_{jl} > 0,$$

such that

- Each direct summand $u^{(i,k)}$ is a finite unitary matrix with $(GL)_{ik} = (KH)_{ik}$ rows and columns
- Each direct summand $v^{(j,l)}$ is a finite unitary matrix with $(G^t K)_{jl} = (LH^t)_{jl}$ rows and columns
-

$$v_{i,k}^{(j,l)} = \sqrt{\frac{\xi(i)\xi(k)}{\xi(j)\xi(l)}} u_{j,l}^{(i,k)}$$

whenever $(i, j, k, l) \in \Gamma_{\text{even}} \times \Gamma_{\text{odd}} \times \Gamma'_{\text{even}} \times \Gamma'_{\text{odd}}$ and

$$G_{ij} > 0, \quad H_{lk} > 0, \quad K_{il} > 0, \quad L_{jk} > 0.$$

Proof. This follows from Ocneanu's (mostly unpublished) work on commuting squares and connections (cf. [O1], [O2]). For convenience of the reader, we sketch a proof based on [S, pp. 2-11]. Note that the equalities $GL = KH$ and $G^t K = LH^t$ were proved in proposition 2.3.

Assume first that $N \subset M$ has finite depth. Then the double tower of commuting squares

$$\begin{array}{ccccccc} \mathbb{C} & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & M' \cap M_3 & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} & \subset & M'_1 \cap M_2 & \subset & M'_1 \cap M_3 & \subset & \dots \end{array}$$

eventually becomes periodic with period 2 (in the sense of Wenzl [W]), and for large n the commuting square

$$\begin{array}{ccc} & (G')^t & \\ M' \cap M_{2n+1} & \subset & M' \cap M_{2n+2} \\ \pi G \cup & & \cup (\pi' G')^t \\ M'_1 \cap M_{2n+1} & \subset & M'_1 \cap M_{2n+2} \\ & G & \end{array}$$

have the inclusion matrices indicated on the figure. It is clear that the horizontal inclusion matrices are G and $(G')^t$, and to prove that the vertical inclusion matrices are as indicated on the figure, one uses that the contragredience map π of Γ_{even} (resp. π' of Γ'_{even}) corresponds to the permutations of the full matrix blocks of $M'_1 \cap M_{2n+1}$ (resp. $M' \cap M_{2n+2}$) induced by the conjugate linear involution $J : a \rightarrow a^*$ in $L^2(M_{n+1})$. Hence in this case prop. 3.3 is exactly the condition given in [S, Thm. 1.1.10] for the existence of a commuting square of multimatrix algebras with given inclusion matrices.

Consider next the case, where $N \subset M$ has infinite depth. Then for every $n \in \mathbb{N}$, the commuting square

$$\begin{array}{ccc} M' \cap M_{2n-1} & \cap & M' \cap M_{2n} \\ \cup & & \cup \\ M'_1 \cap M_{2n-1} & \subset & M'_1 \cap M_{2n} \end{array}$$

is canonically isomorphic to the commuting square obtained by reducing

$$\begin{array}{ccc} M' \cap M_{2n+1} & \subset & M' \cap M_{2n+2} \\ \cup & & \cup \\ M'_1 \cap M_{2n+1} & \subset & M'_1 \cap M_{2n+2} \end{array}$$

with the Jones' projection e_{2n+1} . Thus by forming the inductive limit with respect to the imbeddings

$$x \rightarrow e_{2n+1} x e_{2n+1}, \quad n = 1, 2, \dots,$$

we obtain a commuting square of "generalized multimatrix algebras" (infinite direct sums of type I_∞ -factors)

$$\begin{array}{ccc} & (G')^t & \\ B_0 & \subset & B_1 \\ \pi G \cup & & \cup (\pi' G')^t \\ A_0 & \subset & A_1 \\ & G & \end{array}$$

with inclusion matrices as indicated on the figure, and the proof of [S, Thm. 1.1.10] can easily be modified to generalized multimatrix algebras. ■

It was discovered first by Ocneanu, and later (independently) by a number of other people, that in many concrete examples, the existence of a triple point in a given bipartite graph $\Gamma = \Gamma_{\text{even}} \cup \Gamma_{\text{odd}}$ is an obstruction to build a commuting square of multi-matrix algebras of the form

$$\begin{array}{ccc} & G^t & \\ B_0 & \subset & B_1 \\ G \cup & & \cup G^t \\ A_0 & \subset & A_1 \\ & G & \end{array}$$

where G is the $\Gamma_{\text{even}} \times \Gamma_{\text{odd}}$ part of the adjacency matrix for Γ . In prop. 3.5 we state and prove the "triple point obstruction" in the form needed in the following sections. The presentation here is based on [S]. First we need a lemma:

Lemma 3.4. *If u is a unitary 3×3 matrix and a_1, a_2, a_3 are strictly positive real numbers such that*

$$|u_{ij}|^2 = a_i a_j, \quad i \neq j,$$

then $a_1 + a_2 + a_3 \leq 2$.

Proof. (cf. [S, p.30]). Set $\delta_i = |u_{ii}| \geq 0$, $i = 1, 2, 3$. Since the rows in u have square norm equal to 1,

$$\delta_1^2 = 1 - a_1 a_2 - a_1 a_3$$

$$\delta_2^2 = 1 - a_1 a_2 - a_2 a_3$$

$$\delta_3^2 = 1 - a_1 a_3 - a_2 a_3$$

Since $u_{11}\bar{u}_{21} + u_{12}\bar{u}_{22} + u_{13}\bar{u}_{23} = 0$, the absolute values of these three terms:

$$\delta_1 \sqrt{a_1 a_2} \quad \delta_2 \sqrt{a_1 a_2} \quad a_3 \sqrt{a_1 a_2}$$

are the lengths of the 3 edges in a (possible degenerate) triangle in the complex plane. Hence

$$|\delta_1 - \delta_2| \leq a_3 \leq \delta_1 + \delta_2$$

Squaring this equation yields

$$|a_3^2 - \delta_1^2 - \delta_2^2| \leq 2\delta_1\delta_2,$$

and by squaring once more, we have

$$a_3^4 - 2a_3^2(\delta_1^2 + \delta_2^2) + (\delta_1^2 - \delta_2^2)^2 \leq 0$$

Inserting now the expressions for δ_1^2 and δ_2^2 in terms of a_1 , a_2 and a_3 and reducing, one gets

$$a_3^2((a_1 + a_2 + a_3)^2 - 4) \leq 0.$$

Hence $a_1 + a_2 + a_3 \leq 2$. ■

Recall, that an n -cycle in a graph Γ_1 is a set of n distinct vertices $\gamma_1, \dots, \gamma_n \in \Gamma_1$ connected by n edges in the cyclic order:

$$\gamma_1 - \gamma_2 - \gamma_3 - \dots - \gamma_n - \gamma_1.$$

Note that in a bipartite graph, all cycles have even length. The matrices G , G' , π and π' in the next proposition are defined as in prop. 2.2 and 2.3.

Proposition 3.5. (The triple point obstruction) *Let $N \subset M$ be an irreducible inclusion of $\mathbb{I}_1$ -factors with $4 < [M : N] < \infty$. Assume that the principal graphs Γ and Γ' for $N \subset M$ and $M \subset M_1$ have only simple edges and no 4-cycles. If $|\Gamma_{\text{even}}| = |\Gamma'_{\text{even}}|$, and if for some identification of Γ_{even} and Γ'_{even} ,*

$$G = \pi G = G' = \pi' G'$$

then Γ and Γ' have no triple points (i.e. have no vertices of valency 3)

Proof. As in section 2 we identify Γ_{odd} and Γ'_{odd} by the contragredience map. Hence the condition $G = G'$ means that we altogether have identified Γ and Γ' as graphs. By prop. 3.1 the ξ -function respects the identification Γ_{odd} and Γ'_{odd} . The fact that ξ_Γ and $\xi_{\Gamma'}$ are eigenvectors for Δ_Γ and $\Delta_{\Gamma'}$ respectively with eigenvalue $[M : N]^{1/2}$, then implies that the ξ -function also respects the identification of Γ_{even} and Γ'_{even} . Note that $\pi G = G$ and $\pi' G' = G'$ implies, that we can neglect π and π' , when we apply the biunitary condition (prop. 3.3). Assume now that Γ has a triple point $\gamma \in \Gamma_{\text{even}}$, and let u, v be as in prop. 3.3. The direct summand $u^{(\gamma, \gamma)}$ of u is a 3×3 matrices with entries

$$u_{\delta_i, \delta_j}^{(\gamma, \gamma)} \quad i, j = 1, 2, 3,$$

where $\delta_1, \delta_2, \delta_3 \in \Gamma_{\text{odd}} = \Gamma'_{\text{odd}}$ are the three neighbour points. Since Γ has no 4-cycles, the only way to pass from δ_i to δ_j in two steps, when $i \neq j$, is through γ . Therefore the 6 v -summands

$$v^{(\delta_i, \delta_j)}, \quad i, j = 1, 2, 3, \quad i \neq j$$

are just 1×1 unitary matrices so by c) in prop. 3.3

$$|u_{(\delta_i, \delta_j)}^{(\gamma, \gamma)}| = \sqrt{a_i a_j}, \quad i \neq j,$$

where $a_i = \xi(\delta_i)/\xi(\gamma)$. Since ξ is an eigenvector for Δ_Γ with eigenvalue $[M : N]^{1/2}$,

$$\sum_{i=1}^3 \xi(\delta_i) = [M : N]^{1/2} \xi(\gamma)$$

and so $a_1 + a_2 + a_3 = [M : N]^{1/2} > 2$, which contradicts lemma 3.4. If a triple point γ is in Γ_{odd} , we can apply the same proof with u and v interchanged. ■

The following special case of the triple point obstruction appears to be well known:

Corollary 3.6.

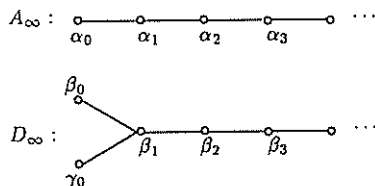
- 1) If $\Gamma = A_\infty$, then $\Gamma' = A_\infty$.
- 2) If $\Gamma = D_\infty$, then $\Gamma' = D_\infty$ and $[M : N] = 4$.

Proof.

1) Recall that $\|\Gamma\|^2 = \|\Gamma'\|^2 \leq [M : N]$, (cf. [P2]). Clearly Γ' is infinite if and only if Γ is infinite, and the only infinite graphs of norm 2 are A_∞ , D_∞ and $A_{\infty, \infty}$, but the last one is excluded since $N \subset M$ (and hence also $M \subset M_1$) is assumed to be irreducible. Therefore

$$\Gamma = A_\infty \quad \text{or} \quad D_\infty \quad \text{implies that} \quad \Gamma' = A_\infty \quad \text{or} \quad D_\infty.$$

By irreducibility the $*$ -point is located at a vertex of valence 1, i.e. with the labeling of vertices



we have $*$ = α_0 in A_∞ , and we can assume that $*$ = β_0 in D_∞ . Setting $\lambda = [M : N]^{1/2}$, there is in both cases a unique solution to $\Delta_\Gamma \xi = \lambda \xi$ satisfying $\xi(*) = 1$. For A_∞ , we get

$$\xi(\alpha_0) = 1, \quad \xi(\alpha_1) = \lambda, \quad \xi(\alpha_2) = \lambda^2 - 1, \quad \xi(\alpha_3) = \lambda^3 - 2\lambda,$$

and for D_∞ , we get

$$\xi(\beta_0) = \xi(\gamma_0) = 1, \quad \xi(\beta_1) = \lambda, \quad \xi(\beta_2) = \lambda^2 - 2, \quad \xi(\beta_3) = \lambda^3 - 3\lambda.$$

Assume $(\Gamma, \Gamma') = (A_\infty, D_\infty)$ or (D_∞, A_∞) . Then $\bar{\alpha}_3 = \beta_3$, because the contragredience map $\Gamma_{\text{odd}} \rightarrow \Gamma'_{\text{odd}}$ preserves the distance to the $*$ -points in Γ and Γ' (cf. prop. 2.3). However β_3 cannot be contragradient to α_3 , because $\xi(\beta_3) \neq \xi(\alpha_3)$. Hence

$$\Gamma = \Gamma' = A_\infty \quad \text{or} \quad \Gamma = \Gamma' = D_\infty.$$

In both cases $[M : N] \geq \|\Gamma\|^2 = 4$.

2) Assume now that $\Gamma = \Gamma' = D_\infty$. Since the vertices in $(D_\infty)_{\text{odd}}$ are uniquely determined by their distance to $*$, the contragredience map $\Gamma_{\text{odd}} \rightarrow \Gamma'_{\text{odd}}$ is the identity on $(D_\infty)_{\text{odd}}$. Moreover by prop. 2.2 (1) the contragredience map π on $(D_\infty)_{\text{even}}$ is trivial, except for a possible permutation of the two vertices β_2 and γ_0 of distance 2 to the $*$ -point. But since $\xi(\beta_2) \neq \xi(\gamma_0)$ it follows that π is trivial. The same argument applies to π' , so by the triple point obstruction (prop. 3.5) we get $[M : N] = 4$. ■

§4. Limiting the possibilities for one of the principal graphs Γ and Γ'

Let $(\Gamma, *)$ be the principal graph of an irreducible inclusion of Π_1 -factors of finite index. We set

$$n(\Gamma, *) = \min\{\text{dist}(*, \gamma) \mid \gamma \in \Gamma, \text{valency}(\gamma) \geq \xi\}.$$

In particular, if $\Gamma = A_n$, $n \geq 2$ or $\Gamma = A_\infty$, then $n(\Gamma, *) = +\infty$.

Lemma 4.1. *Let $N \subset M$ be an irreducible inclusion of Π_1 -factors of finite index, and let Γ and Γ' be the principal graphs of $N \subset M$ and $M \subset M_1$ respectively.*

1) *Then $n(\Gamma, *) = n(\Gamma', *')$.*

2) *In case Γ and Γ' have only simple edges, and $n = n(\Gamma, *) = n(\Gamma', *') < \infty$, then the branch points γ and γ' of distance n to $*$ and $*$ ' have the same valency.*

Proof. Set $n = \min\{n(\Gamma, *), n(\Gamma', *')\}$. If $n = +\infty$, there is nothing to prove, so we can assume that $n < \infty$. By symmetry, we may further assume that

$$n = n(\Gamma, *) \leq n(\Gamma', *').$$

In particular Γ is not of type A. Let $N^\circ \subset M^\circ$ be an inclusion of Π_1 -factors with principal graph A_∞ (for instance, the Jones' subfactor of index 4). Then the first n steps of the Bratteli diagrams

$$C \subset N' \cap M \subset N' \cap M_1 \subset \dots$$

and

$$C \subset (N^\circ)' \cap M^\circ \subset (N^\circ)' \cap M_1^\circ \subset \dots$$

are isomorphic, while the $(n+1)$ 'th step has more edges in the first row than in the second row. Hence

$$\dim(N' \cap M_{k-1}) = \dim((N^\circ)' \cap M_{k-1}^\circ), \quad k = 1, \dots, n$$

where

$$\dim(N' \cap M_n) > \dim((N^\circ)' \cap M_n^\circ).$$

Hence by proposition 2.1,

$$\dim(M' \cap M_k) = \dim((N^\circ)' \cap M_{k-1}^\circ), \quad k = 1, \dots, n$$

$$\dim(M' \cap M_{n+1}) > \dim((N^\circ)' \cap M_n^\circ), \quad k = 1, \dots, n.$$

Thus the n 'th point in the principal graph of $M \subset M_1$ connected from the $*$ -point is also a branch point.

2) Let $v(\gamma)$ and $v(\gamma')$ denote the valencies of γ and γ' . Then

$$\begin{aligned} v(\gamma) - 2 &= \dim(N' \cap M_n) - \dim((N^\circ)' \cap M_n^\circ) \\ &= \dim(M' \cap M_{n+1}) - \dim((N^\circ)' \cap M_n^\circ) \\ &= v(\gamma') - 2 \end{aligned}$$

■

The following Lemma is our main key to reduce the number of possibilities for the pair of principal graphs (Γ, Γ') .

Lemma 4.2. *Let $N \subset M$ be an irreducible inclusion of $\mathbb{I}_1$ -factors of finite index, and let Γ and Γ' be the principal graphs of $N \subset M$ and $N \subset M_1$ respectively. Assume*

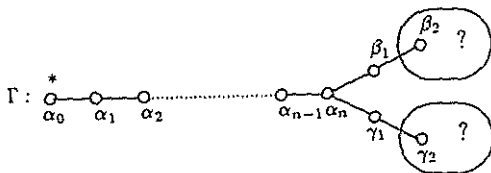
- a) $[M : N] > 4$.
- b) Γ and Γ' are not isomorphic to A_∞ .
- c) Γ and Γ' have only simple edges.
- d) Γ and Γ' have no 4-cycles.
- e) The branch point $\gamma \in \Gamma$ (resp. $\gamma' \in \Gamma'$) of smallest distance to $*$ (resp. $*$ ') is a triple point.

Then either γ or γ' have a neighbouring branch point, i.e. γ or γ' is connected to a vertex of valency greater or equal to 3.

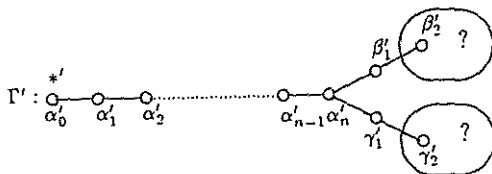
Proof. Assume a), b), c), d) and e), but that neither γ or γ' have a neighbouring branch point. With the notation of lemma 4.1, set

$$n = n(\Gamma, *) = n(\Gamma', *').$$

Then b) implies that $n < \infty$, and by e) Γ and Γ' contain exactly 2 vertices of distance $n + 1$ from their respective $*$ -points. By e) these altogether 4 vertices have valency less or equal to 2. We first treat the case, where all 4 vertices have valency 2. In the proof below we also assume $n \geq 2$, but with trivial modifications, the proof also covers the case $n = 1$. Thus we will first assume that Γ and Γ' have the forms



and



for same $n \geq 2$. Note that d) implies that $\beta_2 \neq \gamma_2$ and $\beta'_2 \neq \gamma'_2$.

Case n even:

By the identification $\Gamma_{\text{odd}} = \Gamma'_{\text{odd}}$ we have

$$\alpha_{n-1} = \alpha'_{n-1}$$

because the contragredience map preserves distance to the respective $*$ -points. Also after possibly relabeling the two "rays" from α'_n , we can assume

$$\beta_1 = \beta'_1 \quad \text{and} \quad \gamma_1 = \gamma'_1.$$

Moreover, since the contragredience maps π and π' on Γ_{even} and Γ'_{even} preserve distance to the respective $*$ -points $\pi(\alpha_{n-2}) = \alpha_{n-2}$, $\pi'(\alpha'_{n-2}) = \alpha'_{n-2}$. Moreover π leaves the set $\{\beta_2, \gamma_2\}$ invariant and π' leaves the set $\{\beta'_2, \gamma'_2\}$ invariant. Since the matrix element $(G^t \pi G)_{\beta_1 \beta_1}$ is equal to the number of elements $\delta \in \Gamma_{\text{even}}$, for which both δ and $\pi(\delta)$ is connected to β_1 , we get:

$$(G^t \pi G)_{\beta_1 \beta_1} = \begin{cases} 2 & \text{if } \pi(\beta_2) = \beta_2 \text{ and } \pi(\gamma_2) = \gamma_2 \\ 1 & \text{if } \pi(\beta_2) = \gamma_2 \text{ and } \pi(\gamma_2) = \beta_2 \end{cases}$$

The corresponding formula for $((G')^t \pi' G')$, together with the equality $G^t \pi G = (G')^t \pi' G'$ (cf. prop. 2.3) now shows that only the following two cases can occur:

- (i) β_2 and γ_2 are fixed by π , and β'_2 and γ'_2 are fixed by π' .
- (ii) π interchanges β_2 and γ_2 , and π' interchanges β'_2 and γ'_2 .

Since ξ_Γ and $\xi_{\Gamma'}$ are eigenvectors for Δ_Γ and $\Delta_{\Gamma'}$ respectively with eigenvalue $\lambda = [M : N]^{1/2}$, we have

$$\xi(\alpha_n) = \frac{1}{\lambda}(\xi(\alpha_{n-1}) + \xi(\beta_1) + \xi(\gamma_1)) = \xi(\alpha'_n).$$

In case (i), we can repeat the proof of the triple point obstruction (prop. 3.5) almost word by word, and obtain a contradiction to the condition $[M : N] > 4$. In case (ii) we get

$$\xi(\beta_2) = \xi(\gamma_2), \quad \xi(\beta'_2) = \xi(\gamma'_2)$$

from prop. 3.1. Hence by the eigenvector equations

$$\xi(\beta_1) = \frac{1}{\lambda}(\xi(\alpha_n) + \xi(\beta_2)) = \frac{1}{\lambda}(\xi(\alpha_n) + \xi(\gamma_2)) = \xi(\gamma_1)$$

and similarly $\xi(\beta'_1) = \xi(\gamma'_1)$. Now let u, v be the two unitaries in the biunitary condition (prop. 3.3). The 3×3 block of the matrix $G^t \pi G = (G')^t \pi' G'$ corresponding to rows and columns labelled by $\{\alpha_{n-1}, \beta_1, \gamma_1\}$ is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Let (u, v) be the pair of unitaries in prop. 3.3. Then $v^{(i,j)}$ is a 1×1 unitary matrix, whenever $(i, j) \in \{\alpha_{n-1}, \beta_1, \gamma_1\}^2$ corresponds to one of the 6 positions with 1's in the above 3×3 matrix. Hence by condition c) in prop. 3.3, the absolute values of the corresponding 6 entries of the 3×3 unitary matrix $u^{(\alpha_n, \alpha_n)}$ are:

$$\left(|u_{ij}^{(\alpha_n, \alpha_n)}| \right)_{i,j} = \begin{pmatrix} \cdot & \sqrt{a_1 a_2} & \sqrt{a_1 a_2} \\ \sqrt{a_1 a_2} & a_2 & \cdot \\ \sqrt{a_1 a_2} & \cdot & a_2 \end{pmatrix},$$

where $a_1 = \frac{\xi(\alpha_{n-1})}{\xi(\alpha_n)}$ and $a_2 = \frac{\xi(\beta_1)}{\xi(\alpha_n)} = \frac{\xi(\gamma_1)}{\xi(\alpha_n)}$. After permuting the last two rows, we get a 3×3 unitary matrix with absolute value of the off diagonal elements equal to $\sqrt{a_i a_j}$, where $a_3 = a_2$. Hence by lemma 3.4, $a_1 + 2a_2 \leq 2$. However

$$a_1 + 2a_2 = \frac{\xi(\alpha_{n-1}) + \xi(\beta_1) + \xi(\gamma_1)}{\xi(\alpha_n)} = [M : N]^{1/2} > 2$$

by the eigenvector equation for ξ . Therefore we can also exclude case (ii).

Case n odd:

By the identification $\Gamma_{\text{odd}} = \Gamma'_{\text{odd}}$ we have

$$\alpha_{n-2} = \alpha'_{n-2} \quad \text{and} \quad \alpha_n = \alpha'_n.$$

Moreover, we can assume that $\beta_2 = \beta'_2$ and $\gamma_2 = \gamma'_2$. Hence

$$\xi(\alpha_{n-1}) = \frac{1}{\lambda}(\xi(\alpha_{n-2}) + \xi(\alpha_n)) = \xi(\alpha'_{n-1})$$

and similarly $\xi(\beta_1) = \xi(\beta'_1)$ and $\xi(\gamma_1) = \xi(\gamma'_1)$. Moreover π leaves the set $\{\beta_1, \gamma_1\}$ invariant, and π' leaves the set $\{\beta'_1, \gamma'_1\}$ invariant. Using the equality

$$(\pi G(G')^t)_{\beta_1, \beta'_1} = (G(G')^t \pi')_{\beta_1, \beta'_1}$$

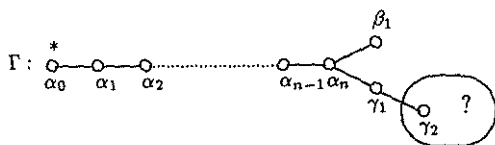
it follows that only the following two cases can occur.

(iii) β_1 and γ_1 are fixed by π and β'_1 and γ'_1 are fixed by π'

(iv) π interchanges β_1 and γ_1 and π' interchanges β'_1 and γ'_1 .

The proofs of impossibility of (iii) and (iv) are now completely parallel to the proofs of impossibility of (i) and (ii) in the case n even (with u and v interchanged).

Finally we treat the case, where at least one of the four vertices $\beta_1, \gamma_1, \beta'_1, \gamma'_1$ has valency 1. Without loss of generality we may assume $v(\beta_1) = 1$. Then $v(\gamma_1) = 2$, because $v(\gamma_1) = 1$ would imply that $\Gamma = D_{n+3}$, contradicting that $[M : N] > 4$. Thus Γ has the form



Case n even:

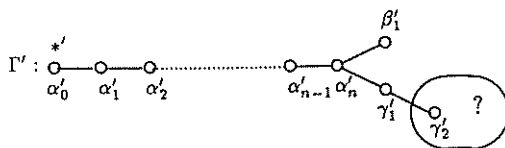
As for $v(\beta_1) = 2$, we can assume $\beta_1 = \beta'_1$, $\gamma_1 = \gamma'_1$, and moreover $\xi(\alpha'_n) = \xi(\alpha_n)$. By the eigenvector equation $\Delta_\Gamma \xi_\Gamma = \lambda \xi_\Gamma$,

$$\lambda \xi(\beta'_1) = \lambda \xi(\beta_1) = \lambda \xi(\alpha_n) = \xi(\alpha'_n).$$

This shows that $v(\beta'_1) = 1$ (valency in Γ'), because if $v(\beta'_1) = 2$, then the eigenvector equation $\Delta_{\Gamma'} \xi_{\Gamma'} = \xi_{\Gamma'}$ would imply

$$\lambda \xi(\beta'_1) = \xi(\alpha'_n) + \xi(\cdot) > \xi(\alpha'_n)$$

where “.” is the second vertex connected to β_1 . Since $\Gamma' \neq D_{n+3}$ it has the form.



Clearly $\pi(\gamma_2) = \gamma_2$ and $\pi'(\gamma'_2) = \gamma'_2$, because π and π' preserves distance to the $*$ -points. Thus the proof of the triple point obstruction (prop. 3.5) can again be used to give a contradiction.

Case n odd:

Since the contragredience map $\Gamma_{\text{odd}} \rightarrow \Gamma'_{\text{odd}}$ preserves distance to the $*$ -points, Γ' has only one vertex $\gamma'_2 = \overline{\gamma_2}$ of distance $n+2$ to $*$. Therefore we can label the two vertices of distance $n+1$ to $*$ by β'_1 and γ'_1 in such a way that Γ' takes the form show in the above figure. The contragredience map gives the identifications $\alpha_n = \alpha'_n$ and $\gamma_2 = \gamma'_2$. Therefore

$$\xi(\beta'_1) = \frac{1}{\lambda} \xi(\alpha'_n) = \frac{1}{\lambda} \xi(\alpha_n) = \xi(\beta_1)$$

and

$$\xi(\gamma_1) = \frac{1}{\lambda} (\xi(\alpha'_n) + \xi(\gamma'_2)) = \frac{1}{\lambda} (\xi(\alpha_n) + \xi(\gamma_2)) = \xi(\gamma_1).$$

It also follows that $\xi(\beta_1) < \xi(\gamma_1)$ and $\xi(\beta'_1) < \xi(\gamma'_1)$. Therefore π (resp. π') cannot intertwine β_1 and γ_1 (resp. β'_1 and γ'_1), and the proof of the triple point obstruction (prop. 3.5) can again be used to give a contradiction. ■

Lemma 4.3. *Let $N \subset M$ be an irreducible inclusion of Π_1 -factors with index in the interval $4 < [M : N] < 3 + \sqrt{2}$, and principal graph Γ . Then*

- 1) Γ has only simple edges.
- 2) Γ has no 4-cycles or 6-cycles.
- 3) Every $\gamma \in \Gamma$ has valency $v(\gamma)$ less or equal to 3.

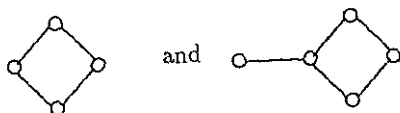
Proof. Since $\|\Gamma\|^2 \leq [M : N]$ with equality, when Γ is finite, then either $4 < \|\Gamma\|^2 < 3 + \sqrt{2}$, or Γ is an infinite graph satisfying $\|\Gamma\| = 2$, i.e. $\Gamma = A_\infty, D_\infty$ or $A_{\infty, \infty}$. By irreducibility $\Gamma \neq A_{\infty, \infty}$ and by corollary 3.6, $\Gamma \neq D_\infty$. Clearly A_∞ satisfies 1), 2) and 3) above. Hence we may assume that

$$4 < \|\Gamma\|^2 < 3 + \sqrt{2}.$$

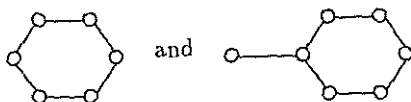
- 1) Clearly the multiplicity of an edge is at most $\|\Gamma\|$, so only multiplicity 1 and 2 can occur. Since the graph $\text{---}\text{---}\text{---}$ has $(\text{norm})^2 = 4$, and Γ is connected. Γ must contain the graph $\text{---}\text{---}\text{---}$ which has $(\text{norm})^2 = 5$, contradicting, that

$$\|\Gamma\|^2 < 3 + \sqrt{2} < 5.$$

2) Since the graphs

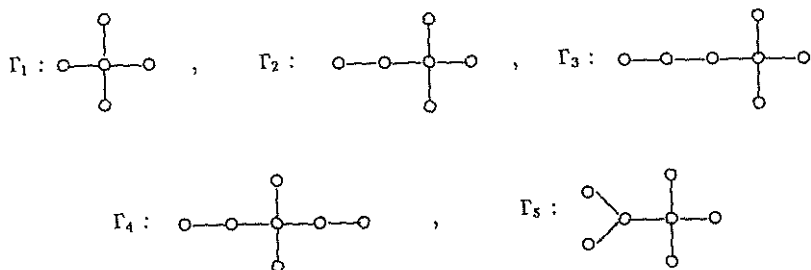


have $(\text{norm})^2$ equal to 4 and $\frac{5+\sqrt{17}}{2} (> 3 + \sqrt{2})$, 4-cycles are excluded, and since the graphs

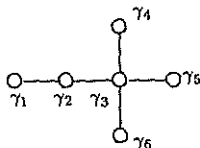


have $(\text{norm})^2$ equal to 4 and $3 + \sqrt{2}$, also 6-cycles are excluded.

3) Clearly the valency of every vertex is less or equal to $\|\Gamma\|^2$, so $v(\gamma) \in \{1, 2, 3, 4\}$ for every $\gamma \in \Gamma$. Consider the five graphs



Then $\|\Gamma_1\|^2 = 4$, $\|\Gamma_2\|^2 = \frac{5+\sqrt{13}}{2}$, $\|\Gamma_3\|^2 = 3 + \sqrt{2}$, $\|\Gamma_4\|^2 = \frac{5+\sqrt{17}}{2}$, and $\|\Gamma_5\|^2 = 3 + \sqrt{3}$. If Γ has a vertex γ of valency $v(\gamma) = 4$ it must therefore contain a connected extension of Γ_1 in which no vertex have valence ≥ 5 , i.e. it must contain Γ_2 . However Γ_2 itself cannot be a principal graph, because it violates the “ $2 \cos \frac{\pi}{n}$ -rule” (prop. 3.2): To see this, let us label the vertices in the following way:



Then the unique Perron-Frobenius eigenvector for Δ_{Γ_2} normalized to have mini-

imum value 1 is

$$\begin{aligned}\xi(\gamma_1) &= 1, \quad \xi(\gamma_2) = \lambda, \quad \xi(\gamma_5) = \lambda^2 - 1, \quad \text{and} \\ \xi(\gamma_4) &= \xi(\gamma_5) = \xi(\gamma_6) = \frac{\lambda^2 - 1}{\lambda}\end{aligned}$$

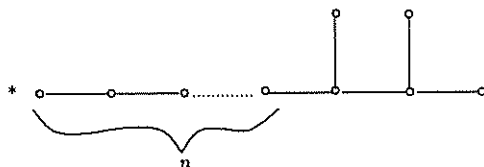
where $\lambda = \left(\frac{5+\sqrt{13}}{2}\right)^{1/2}$. However $1 < \frac{\lambda^2-1}{\lambda} < \sqrt{2}$, and since $2 \cos \frac{\pi}{3} = 1$ and $2 \cos \frac{\pi}{4} = \sqrt{2}$ the graph violates the $2 \cos \frac{\pi}{n}$ -rule. Hence Γ must contain a connected extension of Γ_2 , i.e. it contains Γ_3 , Γ_4 or Γ_5 , because no vertex have valence ≥ 5 . However

$$\|\Gamma_i\|^2 \geq 3 + \sqrt{2}, \quad i = 3, 4, 5,$$

which contradicts that $\|\Gamma\|^2 < 3 + \sqrt{2}$. ■

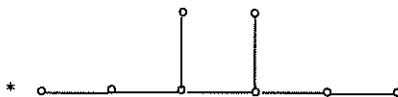
Proposition 4.4. *Let $N \subset M$ be an irreducible inclusion of $\mathbb{II}_1$ -factors of index $4 < [M : N] < 3 + \sqrt{2}$, and let Γ and Γ' be the principal graphs of the inclusions $N \subset M$ and $M \subset M_1$ respectively. Then one of the following three condition holds*

- 1) $\Gamma = \Gamma' = A_\infty$.
- 2) One of the graphs Γ, Γ' is



for some $n \geq 3$.

- 3) One of the graphs Γ, Γ' is



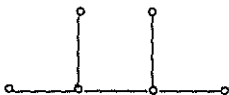
Proof. Since $\|\Gamma\| = \|\Gamma'\|$, we get from the start of the proof of lemma 4.3 that either $\Gamma = \Gamma' = A_\infty$ or

$$4 < \|\Gamma\|^2 = \|\Gamma'\|^2 < 3 + \sqrt{2},$$

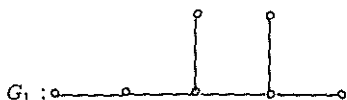
so is the rest of the proof we can assume that this double inequality holds. With the notation of lemma 4.1, let

$$n = n(\Gamma, *) = n(\Gamma', *')$$

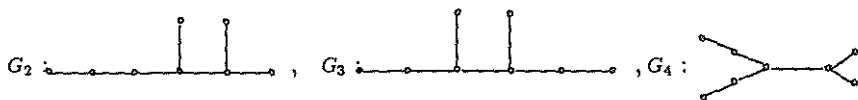
be the distance from $*$ (resp. $*$ ') to the nearest branch-point $\gamma \in \Gamma$ (resp. $\gamma' \in \Gamma'$). By lemma 4.2 and lemma 4.3 either γ or γ' is connected to another triple point, so we may in the following assume that $\gamma \in \Gamma$ has this property. Hence Γ contains the graph,



where one of the triple points is equal to γ . As this graph has $(\text{norm})^2 = 4$, Γ must contain a connected extension of this graph in which all vertices have valence ≤ 3 , i.e. it contains



This graph has $\|G_1\|^2 \approx 4.21432$, and it can be easily checked that the graph violates the $2 \cos \frac{\pi}{n}$ -rule. Hence Γ contains a further extension with no vertex of valence ≥ 4 , i.e. it contains one of the three graphs

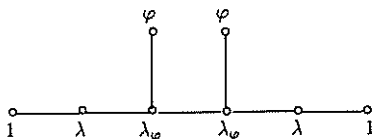


The latter case is excluded, because $\|G_4\|^2 = 3 + \sqrt{2}$. If G_3 is strictly contained in Γ , then Γ must contain



because no vertex have valence ≥ 4 , and Γ does not contain G_4 . However $\|G_5\|^2 > 3 + \sqrt{2}$ (To five decimals, $\|G_5\|^2 = 4.46050$). Thus $G_3 \subseteq \Gamma$ implies $\Gamma = G_3$. The

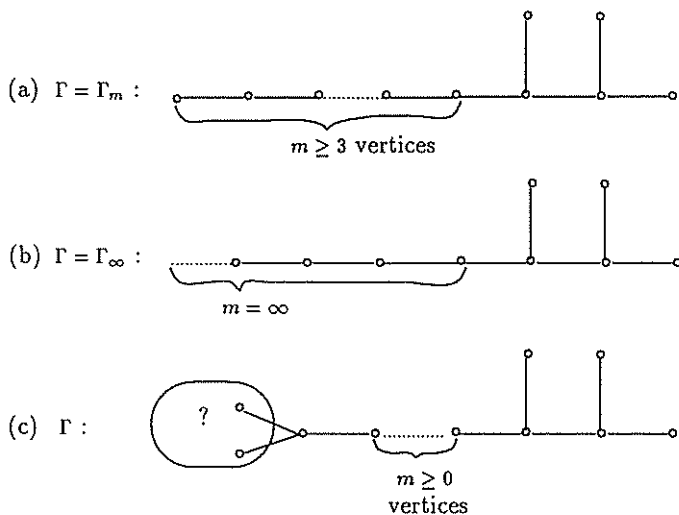
Perron-Frobenius vector of Δ_{G_3} , normalized to have minimum value 1 is



where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio, and

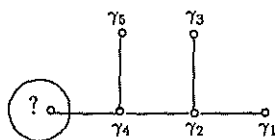
$$\lambda^2 = \|G_3\|^2 = \frac{1}{2}(\varphi + \sqrt{\varphi^2 + 4}).$$

Hence the $*$ -point is located at the endpoint of one of the “legs” with 2 vertices, i.e. we are in case 3). Finally, we have to consider the possible extensions Γ of G_2 . From the above, Γ cannot contain a strict extension of G_3 or G_4 , so it must have one of the following three forms:



Let ξ be solution to $\Delta_\Gamma \xi = \lambda \xi$, $\lambda = \|\Gamma\|$ defined as the square root of the local indices (cf. section 3). In all three cases (a), (b) and (c) we can label 5 of the

vertices by γ_i , $i = 1, \dots, 5$ as follows:



Set $a = \xi(\gamma)$. By solving the eigenvector equation, one gets

$$\xi(\gamma_2) = \lambda a, \quad \xi(\gamma_3) = a, \quad \xi(\gamma_4) = (\lambda^2 - 2)a \quad \text{and} \quad \xi(\gamma_5) = \frac{\lambda^2 - 2}{\lambda} a.$$

In case (b) and in case (c) the $*$ -point must be located at γ_1 , γ_3 or γ_5 , because the $*$ -point has valence 1, and because either γ_2 or γ_4 is the branchpoint closest to $*$. Since $\lambda^2 - 1 > \lambda$, we have

$$\xi(\gamma_5) > \xi(\gamma_1) = \xi(\gamma_3),$$

so the $*$ -point is located at either γ_1 or γ_3 . Therefore $a = \xi(\gamma_1) = \xi(\gamma_5) = 1$, which gives

$$\xi(\gamma_5) = \frac{\lambda^2 - 2}{\lambda}.$$

However

$$1 < \lambda < (3 + \sqrt{2})^{1/2} \quad \text{implies} \quad 1 < \frac{\lambda^2 - 2}{\lambda} < \sqrt{2}$$

so the $2 \cos \frac{\pi}{n}$ -rule is violated. This excludes case (b) and (c), and it also shows that in case (a) the $*$ -point must be at the end of the "leg" of length m . ■

Remark 4.5. Already from prop. 4.4, we can conclude that if an irreducible inclusion of Π_1 -factors has index in the interval

$$4 < [M : N] < \frac{5 + \sqrt{13}}{2}$$

then $\Gamma = \Gamma' = A_\infty$, because all the graphs in case 2) and 3) have $(\text{norm})^2$ greater or equal to $\frac{1}{2}(5 + \sqrt{13})$. Indeed the lowest value is case 2) is obtained for $n = 3$, and $\|\Gamma_3\|^2 = \frac{1}{2}(5 + \sqrt{13})$. The $(\text{norm})^2$ of the graph in case 3) is $\frac{1}{2}(\varphi + \sqrt{\varphi^2 + 4}) \approx 4.39026$, where φ is the golden ratio, and this number is bigger than $\frac{1}{2}(5 + \sqrt{13})$.

§5. Reconstructing Γ' from Γ

The following two combinatorial lemmas will allow us exclude some of the graphs listed in proposition 4.4 and to reconstruct the principal graph Γ' from the principal graph Γ (or Γ from Γ') in the remaining cases.

Lemma 5.1. Let $\Gamma = \Gamma_{\text{even}} \cup \Gamma_{\text{odd}}$ be a bipartite graph with adjacency matrix

$$\Delta_{\Gamma} = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}.$$

Assume that

- (a) Γ is a connected graph with only simple edges
- (b) All vertices in Γ have finite valency.
- (c) Γ contains no 4-cycles or 6-cycles.
- (d) $|\Gamma_{\text{odd}}| \geq 2$.

For $\gamma \in \Gamma_{\text{even}}$, let $E_{\gamma} \subseteq \Gamma_{\text{odd}}$ denote the set of vertices connected to γ . Then:

- 1) The map, $\gamma \rightarrow E_{\gamma}$, is a bijection from the set of vertices $\gamma \in \Gamma_{\text{even}}$ of valence $v(\gamma) \geq 2$ onto the set of subsets E of Γ_{odd} , which are maximal with respect to the condition

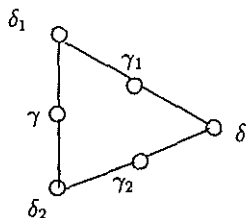
$$(*) \quad (G^t G)_{\delta_1 \delta_2} > 0 \quad \text{for all } \delta_1, \delta_2 \in E.$$

- 2) If we for $\delta \in \Gamma$, let $v_1(\delta)$ denote the number of neighbouring vertices to δ of valency 1, then for any $\delta \in \Gamma_{\text{odd}}$, the number

$$v(\delta) - v_1(\delta)$$

is equal to the number of maximal subsets E of Γ_{odd} with respect to condition (*), for which $\delta \in E$.

Proof. 1) Let $\gamma \in \Gamma_{\text{even}}$ with $v(\gamma) \geq 2$. Then $|E_{\gamma}| = v(\gamma) \geq 2$ so we may choose two distinct elements $\delta_1, \delta_2 \in E_{\gamma}$. Suppose there exists $\delta \in \Gamma_{\text{odd}} \setminus E_{\gamma}$ such that $E = E_{\gamma} \cup \{\delta\}$ satisfies (*). Since $(G^t G)_{\delta_1 \delta} > 0$ there is a path of length 2 from δ_1 to δ through an element $\gamma_1 \in \Gamma_{\text{even}}$, and similarly, there is a path of length 2 from δ_2 to δ through an element $\gamma_2 \in \Gamma_{\text{even}}$. Thus we have the following situation:



where δ_1, δ_2 and δ are distinct elements of Γ_{odd} . However, since Γ has no 4-cycles or 6-cycles, $\gamma_1 = \gamma_2 = \gamma$, which proves that $\delta \in E_{\gamma}$. Hence E_{γ} is a maximal set satisfying (*).

Assume conversely that $E \subset \Gamma_{\text{odd}}$ is a maximal set satisfying (*). Since Γ is connected and $|\Gamma_{\text{odd}}| \geq 2$, E has two distinct elements δ_1, δ_2 . Moreover δ_1 and δ_2 are connected by a path of length 2 through an element $\gamma \in \Gamma_{\text{even}}$. If $\delta \in E$ is

distinct from both δ_1 and δ_2 , then again using that Γ has no 4-cycles or 6-cycles $\delta \in E_\gamma$. Hence $E_\gamma \subseteq E$ and so by maximality of E_γ , $E = E_\gamma$. Moreover γ is unique, because $E = E_{\gamma_1} = E_{\gamma_2}$ for two distinct elements $\gamma_1, \gamma_2 \in \Gamma_{\text{even}}$, would imply that $\gamma_1 - \delta_1 - \gamma_2 - \delta_2 - \gamma_1$ is a 4-cycle in Γ . This proves 1).

2) Set

$$\Gamma_{\text{even}}^{(1)} = \{\gamma \in \Gamma_{\text{even}} | v(\gamma) = 1\}.$$

Let $\delta \in \Gamma_{\text{odd}}$. By definition of $v_1(\delta)$, the number $v(\delta) - v_1(\delta)$ is equal to the number of $\gamma \in \Gamma_{\text{even}} \setminus \Gamma_{\text{even}}^{(1)}$ connected to δ , which again is equal to the number of $\gamma \in \Gamma_{\text{even}} \setminus \Gamma_{\text{even}}^{(1)}$ for which $\delta \in E_\gamma$. Hence 2) follows from 1).

Lemma 5.2. If $\Gamma = \Gamma_{\text{even}} \cup \Gamma_{\text{odd}}$ and $\Gamma' = \Gamma'_{\text{even}} \cup \Gamma'_{\text{odd}}$ are two bipartite graphs with adjacency matrices

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix} \quad \text{and} \quad \Delta_{\Gamma'} = \begin{pmatrix} 0 & G' \\ (G')^t & 0 \end{pmatrix}.$$

Assume that

- (a) Γ and Γ' are connected graphs with only simple edges,
- (b) Γ and Γ' contain no 4-cycles or 6-cycles,
- (c) All vertices in Γ and Γ' have finite valency,
- (d) $|\Gamma_{\text{odd}}| = |\Gamma'_{\text{odd}}| \geq 2$.

Assume further that for a given identification of Γ_{odd} with Γ'_{odd} ,

$$G^t G = (G')^t G'.$$

Then $|\Gamma_{\text{even}}| = |\Gamma'_{\text{even}}|$ and for some permutation matrix ρ ,

$$G' = \rho G$$

i.e. Γ and Γ' are isomorphic as graphs with an isomorphism that respects the identification of Γ_{odd} with Γ'_{odd} .

Proof.

Set

$$\begin{aligned} \Gamma_{\text{even}}^{(1)} &= \{\gamma \in \Gamma_{\text{even}} | v(\gamma) = 1\}, \\ (\Gamma'_{\text{even}})^{(1)} &= \{\gamma' \in \Gamma'_{\text{even}} | v(\gamma') = 1\}. \end{aligned}$$

By lemma 5.1 (1) there is a unique bijection ρ of $\Gamma \setminus \Gamma_{\text{even}}^{(1)}$ onto $\Gamma' \setminus (\Gamma'_{\text{even}})^{(1)}$ such that

$$E_{\rho(\gamma)} = E_\gamma, \gamma \in \Gamma_{\text{even}} \setminus \Gamma_{\text{even}}^{(1)}.$$

For $\delta \in \Gamma_{\text{odd}}$ we let δ' denote the corresponding element in Γ'_{odd} . Then

$$v(\delta') = ((G')^t G')_{\delta' \delta'} = (G^t G)_{\delta \delta} = v(\delta).$$

Moreover, by lemma 5.1 (2)

$$v(\delta') - v_1(\delta') = v(\delta) - v_1(\delta).$$

Thus $v_1(\delta') = v_1(\delta)$ for all $\delta \in \Gamma_{\text{odd}}$, i.e the two sets

$$\{\gamma \in \Gamma_{\text{even}}^{(1)} | \gamma \text{ connected (only) to } \delta\}$$

and

$$\{\gamma' \in (\Gamma'_{\text{even}})^{(1)} | \gamma' \text{ connected (only) to } \delta'\}$$

have the same number of elements, when δ and δ' are corresponding elements of Γ_{odd} and Γ'_{odd} . Therefore we can extend ρ to a bijection (also called ρ) of Γ_{even} onto Γ'_{even} such that ρ for each $\delta \in \Gamma_{\text{odd}}$ is a bijection of the first of the above 2 sets onto the other.

It is now easy to check, that ρ together with the identification $\delta \rightarrow \delta'$ of Γ_{odd} onto Γ'_{odd} defines a graph isomorphism of Γ onto Γ' . In particular

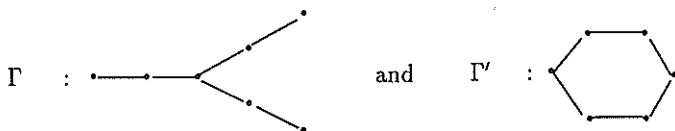
$$G' = \rho G,$$

when we consider ρ as a permutation matrix with rows labeled by Γ'_{even} and columns labeled by Γ_{even} .

Remark 5.3. Condition b) in lemma 5.2 is essential. Consider

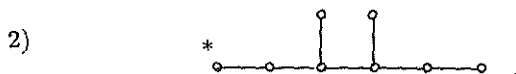
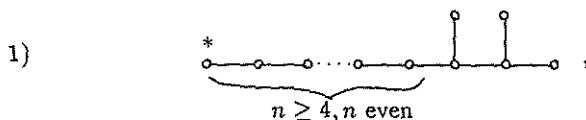
$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad G' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then $G^t G = (G')^t G'$, but the corresponding bipartite graphs,

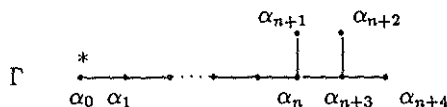


are not isomorphic.

Lemma 5.4. *Let $N \subset M$ be an irreducible inclusion of II_1 -factors with index in the interval $4 < [M : N] < 3 + \sqrt{2}$. Then the principal graph $(\Gamma, *)$ of $N \subset M$ cannot be one of the following graphs*



Proof. 1) Let $n = 2k$. Then with respect to the labeling



$\Gamma_{\text{even}} = \{\alpha_0, \alpha_2, \dots, \alpha_{n+4}\}$, $\Gamma_{\text{odd}} = \{\alpha_1, \alpha_3, \dots, \alpha_{n+3}\}$, and G is a matrix with $|\Gamma_{\text{even}}| = k + 3$ rows and $|\Gamma_{\text{odd}}| = k + 2$ columns, e.g. for $n = 4$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Assume $(\Gamma, *)$ is the principal graph of an irreducible inclusion of II_1 -factors. Let G', π and π' be defined as in section 2. Then by prop. 2.3

$$\pi G(G')^t = G(G')^t \pi' \quad \text{and} \quad G^t \pi G = (G')^t \pi' G'.$$

Since the contragredience map $\pi : \Gamma_{\text{even}} \rightarrow \Gamma_{\text{even}}$ preserves distance to $*$, then either π is trivial or π intertwines the two vertices of distance $n + 2$ to $*$, and leaves the rest of Γ_{even} fixed. In both cases $\pi G = G$. Hence

$$G^t G = G^t \pi G = (G')^t \pi' G'$$

and

$$G(G^t G) = G(G')^t \pi' G' = \pi G(G')^t G' = G(G')^t G'$$

i.e.

$$G(G^t G - (G')^t G') = 0.$$

One checks easily, that in the present case $\ker(G) = 0$. Therefore

$$G^t G = (G')^t G'.$$

Hence by lemma 4.3 and lemma 5.3.

$$G' = \rho G$$

for some 4×4 permutation matrix ρ , and by renumbering the vertices in Γ'_{even} , we may assume, that

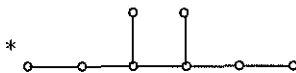
$$G' = G$$

so the principal graphs of Γ and Γ' for $N \subset M$ and $M \subset M_1$ are identified. This identification preserves $*$ -points, since the $*$ -point is necessarily at the end of the “ n -leg” by the proof of proposition 4.4. The argument for $G = \pi G$ applies now also to show $G' = \pi' G'$. Hence

$$G = \pi G = G' = \pi' G',$$

but this is not possible, because of the triple point obstruction (prop. 3.5).

2) Assume that the principal graph is



It is easily checked that the two vertices of distance 4 to the $*$ -point have distinct Perron-Frobenius weights. Hence by prop. 2.3 and prop. 3.1, π is trivial. Moreover $\ker(G) = 0$ also in this case, so as above, we may identify the sets Γ_{even} and Γ'_{even} such that $G' = G$. This identification preserves the $*$ -points, since the $*$ -point is the only vertex in Γ_{even} (resp Γ'_{even}) of lowest Perron-Frobenius weight. Hence the proof of π trivial also gives that π' is trivial, and the triple point obstruction can again be applied.

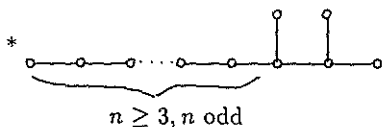
Proposition 5.5. *Let $N \subset M$ be an irreducible inclusion of II_1 -factors, with index in the interval $4 < [M : N] < 3 + \sqrt{2}$. Then the unordered pair (Γ, Γ') of principal graphs for the inclusions $N \subset M$ and $M \subset M_1$ is*

$$1) \quad \Gamma = \Gamma' = A_\infty$$

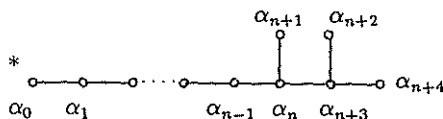
or

[illegible]

Proof. By prop. 4.4 and lemma 5.4 either $\Gamma = \Gamma' = A_\infty$ or one of the principal graphs Γ, Γ' is



We need only to consider the latter case. It is no loss of generality to assume that the above graph is the principal graph Γ of $N \subset M$. We use again the labeling



where n now is odd, $n = 2k + 1$. Then

$$\Gamma_{\text{even}} = \{\alpha_0, \alpha_2, \dots, \alpha_{n+3}\}, \Gamma_{\text{odd}} = \{\alpha_1, \alpha_3, \dots, \alpha_{n+4}\}$$

and G is a matrix with $|\Gamma_{\text{even}}| = k + 3$ rows and $|\Gamma_{\text{odd}}| = k + 3$ columns, e.g. for $n = 5$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Since the contragredience map $\pi : \Gamma_{\text{even}} \rightarrow \Gamma_{\text{even}}$ preserves distance to $*$, π leaves $\alpha_1, \alpha_3, \dots, \alpha_{n-1}$ fixed, but could possibly intertwine α_{n+1} and α_{n+3} . However the computation in the proof of lemma 4.4 (with $\gamma_5 = \alpha_{n+1}$ and $\gamma_2 = \alpha_{n+3}$), shows that

$$\xi(\alpha_{n+3}) > \xi(\alpha_{n+1}).$$

Hence π is trivial. As in the proof of lemma 5.4 we therefore have

$$(*) \quad G(G^t G - (G')^t G') = 0.$$

It is easy to compute that $\ker(G)$ is the one-dimensional subspace of $\mathbf{R}(\Gamma_{\text{odd}})$ spanned by the vector

$$\eta = \alpha_{n+2} - \alpha_{n+4}.$$

By selfadjointness of $G^t G - (G')^t G'$ it follows from $(*)$ that the matrix is proportional to the projection P on the one-dimensional subspace of $\mathbf{R}(\Gamma_{\text{even}})$ spanned by η . Set $L = 2P$. Then the entries of L are either $+1, -1$ or 0 . e.g. for $n = 5, L$ is the matrix:

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Hence $(G')^t G' - G^t G = cL$ for some $c \in \mathbf{Z}$. Moreover, by proposition 2.3

$$cL = (G')^t(1 - \pi')G',$$

which is non-negative in the C^* -ordering of matrices. Therefore $c \geq 0$. The entries of the lower-right 2×2 -block of $G^t G$ are all equal to 1. Since $(G')^t G' = G^t G + cL$ have non-negative entries, we must have either $c = 0$ or $c = 1$. If $c = 0$ we can apply lemma 5.2 together with the triple point obstruction as in the proof of lemma 5.4 to reach a contradiction. Hence $c = 1$. In the rest of the proof we take as an example $n = 5$, but the proof generalizes trivially to any other odd $n \geq 3$. From the above matrices G and L one gets easily that for $c = 1$:

$$G^t G = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad (G')^t G' = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Since $4 < [M : N] < 3 + \sqrt{2}$, it follows from lemma 4.3 that the graph Γ' satisfies the conditions a), b), c) and d) in lemma 5.1. Hence by lemma 5.1 (1) the map

$\gamma \rightarrow E_\gamma$ is a one-to-one correspondence between $\Gamma'_{\text{even}} \setminus (\Gamma'_{\text{even}})^{(1)}$ and the maximal subsets of Γ'_{odd} with respect to condition (*) in lemma 5.1. As the rows and columns in $(G')^t G'$ are labeled by

$$\bar{\alpha}_1, \bar{\alpha}_3, \bar{\alpha}_5, \bar{\alpha}_7 \quad \text{and} \quad \bar{\alpha}_9$$

(for $n = 5$), these maximal sets are

$$x = \{\bar{\alpha}_1, \bar{\alpha}_3\}, y = \{\bar{\alpha}_3, \bar{\alpha}_5\}, z = \{\bar{\alpha}_5, \bar{\alpha}_7\}, w = \{\bar{\alpha}_5, \bar{\alpha}_9\}.$$

Hence we may set $\Gamma'_{\text{even}} \setminus (\Gamma'_{\text{even}})^{(1)} = \{x, y, z, w\}$ and x (resp y, z, w) is connected to $\bar{\alpha}_i$ iff $\bar{\alpha}_i$ is in the set x (resp y, z, w). Moreover from lemma 5.1 (2), $v(\bar{\alpha}_i) - v_1(\bar{\alpha}_i)$ is the number of sets among x, y, z, w which $\bar{\alpha}_i$ is contained in. Since $v(\bar{\alpha}_i)$ is the diagonal element corresponding to $\bar{\alpha}_i$ in $(G')^t G'$, we have

$$v_1(\bar{\alpha}_1) = 2 - 1 = 1$$

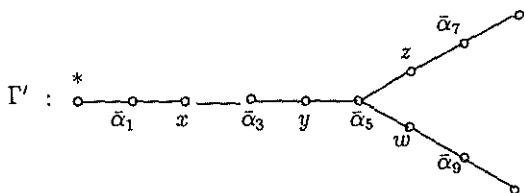
$$v_1(\bar{\alpha}_3) = 2 - 2 = 0$$

$$v_1(\bar{\alpha}_5) = 3 - 3 = 0$$

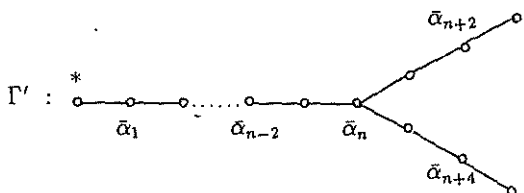
$$v_1(\bar{\alpha}_7) = 2 - 1 = 1$$

$$v_1(\bar{\alpha}_9) = 2 - 1 = 1.$$

Hence $\bar{\alpha}_1, \bar{\alpha}_7$ and $\bar{\alpha}_9$ are each connected to a vertex of $(\Gamma'_{\text{even}})^{(1)}$ i.e. a vertex of Γ'_{even} of valence 1. Therefore Γ' has the form

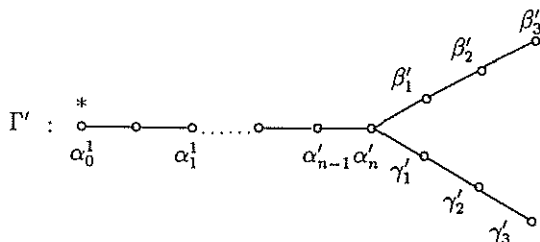


For a general odd $n \geq 3$ the same argument shows that Γ' has the form:



As the contagredience map $\alpha_i \rightarrow \bar{\alpha}_i, i$ odd preserves distance to the *-points the *-point of Γ' is located at the vertex of Γ' of valence 1, which is connected to $\bar{\alpha}_1$.

Remark 5.6. The contragredience map $\pi' : \Gamma'_{\text{even}} \rightarrow \Gamma'_{\text{even}}$ preserves the distance to the $*$ -point, Γ' . Hence with the following labeling of the vertices



we have $\pi'(\alpha'_i) = \alpha'_i, i = 0, 2, \dots, n-1$, and π' leaves the two sets $\{\beta', \gamma'_1\}$ and $\{\beta'_3, \gamma'_3\}$ invariant. It is easy checked that the equality

$$(G')^t(\pi')G' = G^t\pi G$$

is fulfilled if and only if either

a) π' interchanges β'_1, γ'_1 and fixes β'_3 and γ'_3 .

or

b) π' fixes β'_1, γ'_1 and interchanges β'_3 and γ'_3 .

Moreover, in both cases the other equality in Prop.2.3:

$$\pi G(G')^t = G(G')^t \pi'$$

is also fulfilled. It can be shown that only case b) can occur, but the proof involves a computation of the full table of "fusion rules" on Γ'_{even} , i.e. the multiplication table for the tensor products of the Hilbert bimodules coming from Γ'_{even} . For $n = 3$ the proof is elementary. In this case α'_0, β'_3 and γ'_3 are the vertices of Γ'_{even} , for which the Perron-Frobenius weights $\xi(\cdot)$ are 1. Therefore $\{\alpha'_0, \beta'_3, \gamma'_3\}$ form a group of three elements with identity α'_0 . As the two non-zero elements of $\mathbb{Z}_3$ are inverses of each other, it follows that $\beta'_3 = \gamma'_3$, so we are in case b).

§6. Excluding $n \equiv 1 \pmod{4}$

Let $N \subset M$ be an inclusion of Π_1 -factors of finite index, and let Γ and Γ' be the principal graphs of $N \subset M$ and $M \subset M_1$. As in section 2, we let α denote the N - M Hilbert bimodule

$$\alpha = {}_N L^2(M)_M,$$

and we identify Γ_{even} (resp. $\Gamma_{\text{odd}}, \Gamma'_{\text{even}}, \Gamma'_{\text{odd}}$) with a set of irreducible N - N (resp. N - M, M - M, M - N) Hilbert bimodules. Consider the $\mathbb{Z}$ -modules

$$\mathbb{Z}(\Gamma_{\text{even}}), \quad \mathbb{Z}(\Gamma_{\text{odd}}), \quad \mathbb{Z}(\Gamma'_{\text{even}}), \quad \mathbb{Z}(\Gamma'_{\text{odd}})$$

of formal $\mathbb{Z}$ -linear combinations of the sets indicated in the brackets. Then the product of Hilbert bimodules $\gamma = {}_P H_Q$, $\delta = {}_Q H_R$, $P, Q, R \in \{M, N\}$ defined by

$$\gamma\delta = \gamma \otimes_Q \delta$$

extends to a bilinear product

$$\mathbb{Z}(\cdot) \times \mathbb{Z}(\cdot) \rightarrow \mathbb{Z}(\cdot)$$

for matching choices of $\cdot$, $\cdot$ and $\cdot$ among Γ_{even} , Γ_{odd} , Γ'_{even} and Γ'_{odd} . In this way we can for every polynomial $q(t)$ with coefficients in $\mathbb{Z}$, define elements,

$$\begin{aligned} q(\alpha\bar{\alpha}) &\in \mathbb{Z}(\Gamma_{\text{even}}), \\ q(\bar{\alpha}\alpha) &\in \mathbb{Z}(\Gamma'_{\text{even}}), \\ \alpha q(\bar{\alpha}\alpha) &= q(\alpha\bar{\alpha})\alpha \in \mathbb{Z}(\Gamma_{\text{odd}}), \\ \bar{\alpha} q(\alpha\bar{\alpha}) &= q(\bar{\alpha}\alpha)\bar{\alpha} \in \mathbb{Z}(\Gamma'_{\text{odd}}). \end{aligned}$$

As in section 2, Δ_Γ and $\Delta_{\Gamma'}$ denote the adjacency matrices of Γ and Γ' , and $*$ = $\mathcal{E}_N$, $*'$ = $\mathcal{E}_M$ are the trivial N - N and M - M bimodules.

Lemma 6.1.

1) If $p(t) = q(t^2)$ is an even polynomial in t with integer coefficients. Then the following 3 conditions are equivalent

- a) $q(\alpha\bar{\alpha}) = 0$ and $q(\bar{\alpha}\alpha) = 0$
- b) $*$ $\in \ker p(\Delta_\Gamma)$ and $*'$ $\in \ker p(\Delta_{\Gamma'})$.
- c) $p(\Delta_\Gamma) = p(\Delta_{\Gamma'}) = 0$.

2) If $p(t) = tq(t^2)$ is an odd polynomial in t with integer coefficient. Then the following 6 conditions are equivalent

- a) $\alpha q(\bar{\alpha}\alpha) = 0$.
- a') $\bar{\alpha} q(\alpha\bar{\alpha}) = 0$.
- b) $*$ $\in \ker p(\Delta_\Gamma)$.
- b') $*'$ $\in \ker p(\Delta_{\Gamma'})$.
- c) $p(\Delta_\Gamma) = 0$.
- c') $p(\Delta_{\Gamma'}) = 0$.

Proof.

1) Note that Δ_Γ can be considered as a $\mathbb{Z}$ -linear map $\mathbb{Z}(\Gamma) \rightarrow \mathbb{Z}(\Gamma)$ with matrix

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$$

Thus Δ_Γ^2 leaves $\mathbb{Z}(\Gamma_{\text{even}})$ and $\mathbb{Z}(\Gamma_{\text{odd}})$ invariant and the corresponding matrices are

$$(\Delta_\Gamma^2)_{\text{even}} = GG^t \quad \text{and} \quad (\Delta_\Gamma^2)_{\text{odd}} = G^t G.$$

Hence Δ_{Γ}^2 is given by right multiplication by $\alpha\bar{\alpha}$ on $\mathbf{Z}(\Gamma_{\text{even}})$ and by right multiplication by $\bar{\alpha}\alpha$ on $\mathbf{Z}(\Gamma_{\text{odd}})$. Similarly $\Delta_{\Gamma'}^2$ is given by right multiplication by $\bar{\alpha}\alpha$ on $\mathbf{Z}(\Gamma'_{\text{even}})$ and by right multiplication by $\alpha\bar{\alpha}$ on $\mathbf{Z}(\Gamma'_{\text{odd}})$. Therefore a) $\Rightarrow$ c). Obviously c) $\Rightarrow$ b), and b) $\Rightarrow$ a) is clear, because

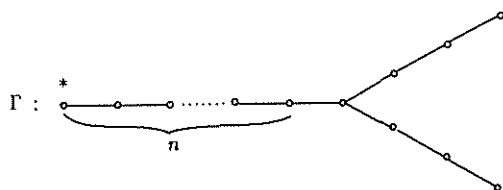
$$q(\alpha\bar{\alpha}) = \mathcal{E}_N q(\alpha\bar{\alpha}) \quad \text{and} \quad q(\bar{\alpha}\alpha) = \mathcal{E}_M q(\bar{\alpha}\alpha)$$

2) The contragradience map extends to a $\mathbf{Z}$ -linear bijection of $\mathbf{Z}(\Gamma_{\text{odd}})$ onto $\mathbf{Z}(\Gamma'_{\text{odd}})$. Therefore a) $\Leftrightarrow$ a'). Hence by symmetry, we only need to check, that a), b) and c) are equivalent. For every $k \in \mathbb{N}_0$, Δ_{Γ}^{2k+1} maps $\mathbf{Z}(\Gamma_{\text{even}})$ into $\mathbf{Z}(\Gamma_{\text{odd}})$ and $\mathbf{Z}(\Gamma_{\text{odd}})$ into $\mathbf{Z}(\Gamma_{\text{even}})$. Moreover on $\mathbf{Z}(\Gamma_{\text{even}})$, the map is given by right multiplication by $\alpha(\bar{\alpha}\alpha)^k$ and on $\mathbf{Z}(\Gamma_{\text{odd}})$, it is given by right multiplication by $\bar{\alpha}(\alpha\bar{\alpha})^k$. From this one gets easily that a) together with the equivalent condition a') implies c). c) $\Rightarrow$ b) is trivial and b) $\Rightarrow$ a) follows from

$$\alpha q(\bar{\alpha}\alpha) = \mathcal{E}_N \alpha q(\bar{\alpha}\alpha). \quad \blacksquare$$

end of proof of Theorem 1.1.

Let $N \subset M$ be an irreducible inclusion of Π_1 -factors of index in the interval $4 < [M : N] < 3 + \sqrt{2}$. By prop. 5.5 the principal graph of either $N \subset M$ or of $M \subset M_1$ is



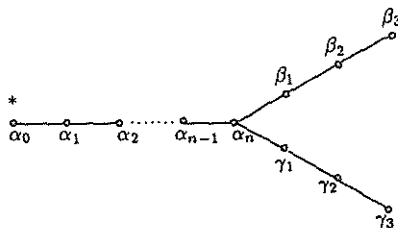
for some odd $n \geq 3$. To complete the proof of theorem 1.1 we have to show that $n \equiv 3 \pmod{4}$. As in [S] we let $(R_n(t))_{n=0}^{\infty}$ be the sequence of polynomials defined recursively by

$$R_0(t) = 1, \quad R_1(t) = t,$$

and

$$R_{n+1}(t) + R_{n-1}(t) = tR_n(t)$$

For $t = 2 \cos \theta$, $R_n(2 \cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$, $n = 0, 1, 2, \dots$. Let us label the vertices of Γ in the following way.



Using the recursion formula for $(R_n(t))_{n=1}^\infty$ we get

$$R_k(\Delta_\Gamma)(*) = \alpha_k, \quad k = 0, \dots, n$$

and

$$\begin{aligned} R_{n+1}(\Delta_\Gamma)(*) &= \beta_1 + \gamma_1 \\ R_{n+2}(\Delta_\Gamma)(*) &= \beta_2 + \gamma_2 + \alpha_n \\ R_{n+3}(\Delta_\Gamma)(*) &= \beta_3 + \gamma_3 + \beta_1 + \gamma_1 + \alpha_{n-1} \\ R_{n+4}(\Delta_\Gamma)(*) &= \beta_2 + \gamma_2 + 2\alpha_n + \alpha_{n-2} \end{aligned}$$

Hence, if we set

$$P_n(t) = R_{n+4}(t) - R_{n+2}(t) - R_n(t) - R_{n-2}(t)$$

then

$$P_n(\Delta_\Gamma)(*) = 0.$$

Note that since n is odd, $P_n(t)$ is an odd polynomial so by lemma 5.1, $P_n(\Delta_\Gamma) = 0$. Consider now the vector $\eta \in \mathbb{R}(\Gamma)$ given by

$$\eta = \beta_1 + \sqrt{2}\beta_2 + \beta_3 - \gamma_1 - \sqrt{2}\gamma_2 - \gamma_3.$$

Then it is easily seen that η is an eigenvector for Δ_Γ with eigenvalue $\sqrt{2}$. Moreover

$$R_k(\sqrt{2}) = R_k(2 \cos \frac{\pi}{4}) = \frac{\sin((k+1)\frac{\pi}{4})}{\sin \frac{\pi}{4}}$$

Therefore

$$R_k(\sqrt{2}) = \begin{cases} \sqrt{2}, & k \equiv 1 \pmod{8} \\ -\sqrt{2}, & k \equiv 5 \pmod{8} \\ 0, & k \equiv 3, 7 \pmod{8} \end{cases}$$

and hence,

$$R_n(\sqrt{2}) = \begin{cases} -2\sqrt{2}, & n \equiv 1 \pmod{8} \\ +2\sqrt{2}, & n \equiv 5 \pmod{8} \\ 0, & n \equiv 3, 7 \pmod{8} \end{cases}$$

which shows that

$$\begin{aligned} P_n(\Delta_\Gamma)\eta &= -2\sqrt{2}\eta, & n &\equiv 1 \pmod{8} \\ P_n(\Delta_\Gamma)\eta &= +2\sqrt{2}\eta, & n &\equiv 5 \pmod{8} \end{aligned}$$

Therefore $P(\Delta_\Gamma) \neq 0$, when $n \equiv 1 \pmod{4}$, and we have reached a contradiction in this case. Hence $n \equiv 3 \pmod{4}$.

Remark 6.2. The proof given above to exclude $n \equiv 1 \pmod{4}$ can be considered as a refinement of the algebraic proofs given independently by [I] and [SuV] for excluding D_n , n odd, as a principal graph. Indeed this can easily be proved by use of lemma 6.1:

First observe that if $\Gamma = D_n$, then $\Gamma' = D_n$, because $\|\Gamma'\|^2 = \|\Gamma\|^2$, so the A - D - E -classification of graphs of norm less than two, together with the equality $|\Gamma_{\text{odd}}| = |\Gamma'_{\text{odd}}|$ excludes all other possibilities. Set $n = m + 3$, and assume that n is odd (equivalently m is even). As the $*$ -point must be located at the vertex of lowest weight for the Perron-Frobenius vector, the $*$ -point is at the end of the "leg" of length m in both Γ and Γ' . It is now easy to check that

$$(R_{m+2}(\Delta_\Gamma) - R_m(\Delta_\Gamma))(*) = 0$$

As the same equality holds for $\Delta_{\Gamma'}$, we get from lemma 6.1 (1) that

$$R_{m+2}(\Delta_\Gamma) - R_m(\Delta_\Gamma) = 0.$$

However, the vector η , that takes the values $+1$ and -1 at the end of the two legs of length 1 in D_{n+3} , and 0 otherwise, is an eigenvector for Δ_Γ with eigenvalue 0. Hence, by the recursion formula for $(R_n(t))_{n=0}^\infty$, $R_{2k}(\Delta_\Gamma)\eta = (-1)^k\eta$, which shows that

$$R_{m+2}(\Delta_\Gamma)\eta - R_m(\Delta_\Gamma)\eta \neq 0,$$

when m is even, and we have reached a contradiction.

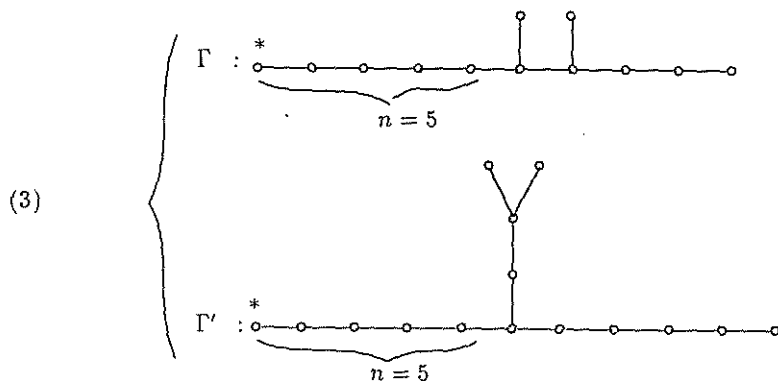
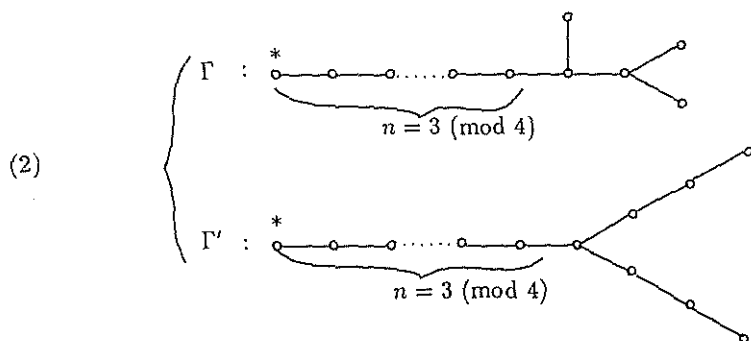
Lemma 6.1 cannot be used to exclude the graph E_7 as a principal graph; but is an easy (and well known) observation that E_7 can be excluded by the $2 \cos \frac{\pi}{n}$ -rule (see e.g. [I, p.967]).

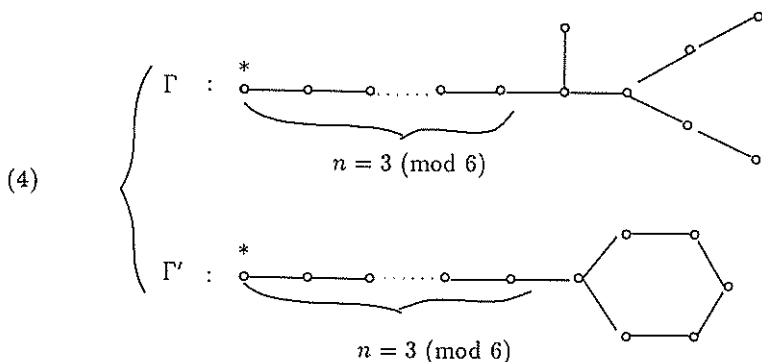
§7. Announcement of further results.

With minor improvements of the techniques used to prove Theorem 1.1, we have been able to extend the "at most list" of principal groups up to index $3 + \sqrt{3}$. The result is:

Let $N \subset M$ be an irreducible inclusion of II_1 -factors of index in the interval $4 < [M : N] < 3 + \sqrt{3}$. Then the unordered pair (Γ, Γ') of principal graphs for $N \subset M$ and $M \subset M_1 = \langle M, e_N \rangle$ is one of the following:

(1) $\Gamma = \Gamma' = A_\infty$





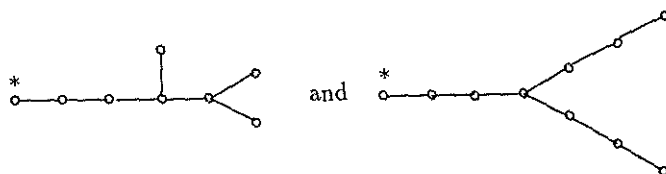
Numerically, $[M : N] = \frac{1}{2}(5 + \sqrt{17}) \approx 4.56155$, in case (3), and in case (4), the corresponding sequence of indices starts at $\frac{1}{2}(7 + \sqrt{5}) = 4.61803$ for $n = 3$ and it converges rapidly to ≈ 4.65897 for $n \rightarrow \infty$.

It can be shown, that in the cases (2), (3) and (4) the contragredience maps π and π' are uniquely determined from the pair of graphs. Moreover, there is up to isomorphism only one commuting square of multi-matrix algebras A_0, A_1, B_0, B_1 of the form.

$$\begin{array}{ccccc} & & (G')^t & & \\ & B_0 & \subset & B_1 & \\ \pi G & \cup & & \cup & (\pi' G')^t \\ & A_0 & \subset & A_1 & \\ & & G & & \end{array}$$

for any given dimension vector on A_0 . Hence by the generating property for finite depth subfactors ([O1], [P1]), there is up to conjugacy at most one subfactor of the hyperfinite Π_1 -factor for each of the graphs listed in case (2), (3) and (4). The main problem now is to prove the existence of subfactors with the graphs listed in (2), (3) and (4) as principal graphs. So far we have only managed to do this for $n = 3$ in case (2), i.e. we have shown:

There are precisely two conjugacy classes of finite depth subfactors of the hyperfinite Π_1 -factor with index $\frac{1}{2}(5 + \sqrt{13})$, and the corresponding principal graphs are



The proofs of the results listed in this section will be given elsewhere.

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