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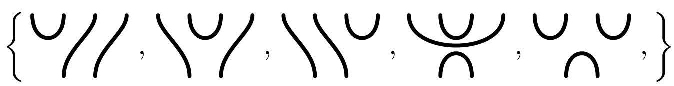
# Scott Morrison - Research statement

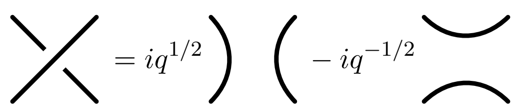
In this research statement, I'll describe three projects I'm currently working on at the interface between topology, representation theory, and algebra. I like to think about 'topological objects with algebraic structure' which can often equally well be seen as 'algebraic objects with topological structure'. Usually I think of these as formalised by 2−, 3− or 4-categories with duals. Another way of saying this is that I like to do research that lets me draw pictures!

A typical example is given by 'Temperley-Lieb diagrams'. These form a 2-category, in which the 2-morphisms are planar crossingless matchings, so for example

$$\mathrm{Hom}(2 \to 4) =$$



The two different compositions in this 2-category are vertical and horizontal juxtaposition. We also allow linear combinations of diagrams, and replace all closed loops with $q + q^{-1}$. At first sight this looks very much like a topological object being given an algebraic structure, but an old result lets us turn this on its head: the Temperley-Lieb 2-category is isomorphic to the category of representations of $U_q(\mathfrak{sl}_2)$, with morphisms $\mathrm{Hom}_{SU(2)}(\mathbb{C}^{2^{\otimes n}} \to \mathbb{C}^{2^{\otimes m}})$. Moreover, we can understand important extra structure in either setting by realising that both 2-categories are secretly 3-categories (or, equivalently in this case, braided tensor categories)! The 'extra direction of composition' for Temperley-Lieb diagrams is given by overlaying one diagram on top of another, resolving crossings using the Kauffman skein formula:




On the representation theory side, the braiding comes from the universal $\mathcal{R}$-matrix.

I've divided up my research interests into the following three sections; §1 covers Khovanov homology and blob homology, §2 covers fusion categories and planar algebras, and §3 covers diagrammatic methods in the representation theory of quantum groups (for example generalising the description above of Temperley-Lieb). The divisions aren't exactly canonical, as there are several natural overlaps.

The three sections could be characterised by the dimension of the topological objects they describe, namely 4, 2 and 3 respectively. Alternatively, they can be distinguished by the techniques they use and the connections they make with other fields. Khovanov homology and blob homology make essential use of homological algebra and triangulated categories. Fusion categories and planar algebras are related via subfactors to analysis and via representation theory to quantum groups.

## 1 Khovanov homology and homological TQFTs

During the 1980s and 1990s, surprising and deep connections were found between topology and algebra, starting from the Jones polynomial for knots, leading on through the representation theory of a quantum group (a braided tensor category) and culminating in topological quantum field theory (TQFT) invariants of 3-manifolds.

In 1999, Khovanov [2] came up with something entirely new. He associated to each knot diagram a chain complex whose homology was a knot invariant. Some aspects of this construction were familiar; the Euler characteristic of his complex is the Jones polynomial, and just as we had learnt to understand the Jones polynomial in terms of a braided tensor category, Khovanov homology can now be understood in terms of a certain braided tensor 2-category. But other aspects are more mysterious. The categories associated to quantum knot invariants are essentially *semisimple*, while those arising in Khovanov homology are far from it. Instead they are *triangulated*, and *exact triangles* play a central role in both definitions and calculations.

While Khovanov homology and its variations provide a 'categorification' of quantum knot invariants, to date there has been no corresponding categorification of TQFT 3-manifold invariants. The failure of standard TQFT methods in Khovanov homology has led me and my coauthor Kevin Walker to define 'blob homology', a certain generalisation of a TQFT which promises to be particularly useful for extracting topological information from non-semisimple categories. (See §1.1.)

My other research on Khovanov homology attempts to understand the 4-dimensional geometry of Khovanov homology (§1.2) and understand in detail some small cases of the categories associated to Khovanov homology (§1.3). Finally, I outline a conjecture using Khovanov homology to explain the relationship between the dual canonical basis for $SU(3)$ and a certain related diagrammatic basis, in §1.4.

## 1.1 Blob homology

With Kevin Walker, I've defined the "blob complex" $\mathcal{B}_*(M, \mathcal{C})$ associated to an $n$-manifold $M$ and a (suitable) $n$-category $\mathcal{C}$. This is a simultaneous generalisation of two interesting gadgets. When $n = 1$, $M = S^1$ and $\mathcal{C}$ is an algebra, the homology of the blob complex is the Hochschild homology of the algebra. On the other hand, the 0-th homology of the blob complex is the usual TQFT skein module of "pictures from $\mathcal{C}$ drawn on $M$". In this sense the blob complex is a "derived" version of a TQFT.

We can prove several interesting properties of the blob complex. It's a functorial construction, and diffeomorphisms of the manifold act on the complex. Moreover, there's an action of chains of diffeomorphisms as well. Thus for example on the torus we get not only an action of the mapping class group, but also a compatible action of rotations along rational slopes. There appears to be a good gluing formula, expressed in terms of $A_\infty$ bimodules. (The details here are still being worked out.)

We hope to apply blob homology to tight contact structures (for $n = 3$) and Khovanov homology (for $n = 4$). In both theories exact triangles play an important role. These exact triangles don't interact well with the gluing structure of the usual TQFTs, however. One of our motivations for considering blob homology is to work around these difficulties.

## 1.2 Khovanov homology for 4-manifolds

Khovanov homology as originally defined was in invariant of unoriented knots in $B^3$, associating (doubly-graded) vector spaces to knots, and linear maps (only well-defined up to sign) to cobordisms between knots.

Motivated by ideas from the representation theory of $U_q(\mathfrak{sl}_2)$, I gave a new variation of Khovanov homology in my paper

**Fixing the functoriality of Khovanov homology**
Joint with David Clark and Kevin Walker, available at arXiv:math.GT/0701339, accepted by *Geometry and Topology*.

This associates a vector space to an *oriented* knot in $B^3$, and the linear maps associated to cobordisms are now well-defined. The construction uses 'disorientations'.

However, I'd like to do more.

**Conjecture 1.1.** *Khovanov homology can be extended to* disoriented *knots. Moreover, this extension gives linear maps for cobordisms in $S^3$, not just $B^3$.*

Along with a description (partially appearing in [1]) of Khovanov homology as a 4-category with duals, this conjecture allows us to define an invariant of a pair $(W, L \subset \partial W)$, with $W$ a 4-manifold, and $L$ a link in its boundary. For $L \subset \partial B^4$, this should recover the usual Khovanov homology invariant. It also applies to 4-manifolds without boundary, potentially giving non-trivial invariants. I anticipate that this construction will give useful bounds for the slice genus of a link, generalising those of [9].

## 1.3 The 2-point and 4-point Khovanov categories

For tangles with any number of boundary points, there is an associated category of Khovanov chain complexes. Understanding these categories (in particular, finding simpler Morita-equivalent categories) is essential to computations for the 4-manifold invariant proposed in the previous paragraph. The 2-point category can completely described, improving on my earlier paper

**The Karoubi Envelope and Lee's Degeneration of Khovanov Homology**
Joint with Dror Bar-Natan, *Algebraic & Geometric Topology* 6 (2006) 1459-1469. Available at DOI:10.2140/agt.2006.6.1459 or arXiv:math.GT/0606542.

and giving easy proofs of results of Lee and Rasmussen. Moreover, it gives a more efficient method for computing the $s$-invariant.

While a complete description of the 4-point category seems to be difficult, it seems likely the Khovanov homology of a rational tangle can be computed explicitly. This should be thought of as a categorification of the modular group $PSL(2, \mathbb{Z})$ acting by linear fractional transformations.

## 1.4 Dual canonical bases from $SU(3)$ Khovanov homology.

Kuperberg's spider for $SU(3)$ gives a model for the representation theory of $U_q(\mathfrak{sl}_3)$ based on planar trivalent graphs, modulo some relations [4]. In particular, we get the 'web basis' for the invariant spaces $\mathrm{Inv}_{SU(3)}(\mathbb{C}^{3^{\otimes n}})$. Khovanov and Kuperberg showed that while the web basis had many of the nice properties of the dual canonical basis, and agreed with it in many small cases, they were not the same.

While working on a paper about a 'foam' model for $SU(3)$ Khovanov homology,

**On Khovanov's cobordism theory for $\mathfrak{su}_3$ knot homology**
Joint with Ari Nieh, *Journal of Knot Theory and its Ramifications* Vol. 17, No. 9 (2008). Available at DOI:10.1142/S0218216508006555 or arXiv:math.GT/0612754

a counterexample to a lemma we were trying to prove suggested a connection between the web basis and the dual canonical basis, via a 3-category of 'trivalent foams', described in the paper above. This prompted the following conjecture.

**Conjecture 1.2.** *The indecomposable objects in this category are Kuperberg's web basis. If we pass to the idempotent completion, these objects are no longer indecomposable, and the new indecomposable objects are the dual canonical basis.*

This parallels Lusztig's construction [5] of the canonical basis as the simple objects in a category based on quiver varieties, but starts from a completely different topological/combinatorial category.

# 2 Fusion categories and planar algebras

## 2.1 Knot polynomial identities and quantum group coincidences

Making use of the skein theory of the $D_{2n}$ planar algebras, described in

**Skein theory for the $D_{2n}$ planar algebras**
Joint with Emily Peters and Noah Snyder, available at arXiv:0808.0764, submitted to *Journal of Pure and Applied Algebra.*

I can prove a series of new identities relating quantum knot invariants (often relating different groups, and different roots of unity).

Further, in joint work with Emily Peters and Noah Snyder, we make use of the $D_{2n}$ planar algebras to explain $SO(3)$-$SO(n)$ level-rank duality. Armed with this (along with generalised Kirby-Melvin symmetry and a few coincidences of small quantum groups), we will give separate proofs of the above knot invariant identities, by lifting them to equivalences of braided tensor categories. These equivalences should be thought of as 'root of unity' analogues of coincidences between small Lie groups, such as $Spin(4) \cong SU(2) \times SU(2)$ and $PSL(4, \mathbb{C}) \cong SO(6)$.

## 2.2 Atlas of subfactors

I'm in the midst of implementing algorithms to enumerate and construct subfactors (and also singly generated fusion categories) of small index and small rank, working with Emily Peters and Noah Snyder. Preliminary progress on the project appears at http://tqft.net/svn/Atlas_of_subfactors.

A typical paper from this project might be "Simple fusion categories of global dimension at most 30", and we hope to prepare a partner for the Knot Atlas http://katlas.org/, cataloguing small subfactors and fusion categories. We hope that this will be of interest both to mathematicians and physicists. There are many opportunities for students to participate in this project!

# 3 Diagrammatic representation theory

## 3.1 Kashaev-Reshetikhin knot invariants

With Noah Snyder I'm working on computations and a paper about the Kashaev-Reshetikhin knot invariants.

We'll build on the papers of Kashaev and Reshetikhin to give a fully rigorous construction of their new knot invariants. This invariant is a function on the space $Hom(G_m(K), SL_2)/SL_2$ where $G_m(K)$ is the generalized knot group, and the action of $SL_2$ is by conjugation. We prove several basic results, for example, that the value of the function on the trivial point is $|J_m(\zeta_m)|^2$.

Further, we'll give the first computations of this invariant for nontrivial knots, based on a Mathematica package we've written. For several small 2-bridge knots we compute explicitly the entire Kashaev-Reshetikhin knot invariant at a third root of unity and at a fifth root of unity. We're planning to compute the knot invariants evaluated at the finite volume hyperbolic point for a larger collection of 2-bridge knots (again at a third and fifth root of unity).

## 3.2 Representations of $U_q(\mathfrak{g})$

The representation theory of a quantum group forms a planar algebra (equivalently, a spider or pivotal category). For $U_q(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_3)$,$U_q(\mathfrak{so}_5)$ and $U_q(\mathfrak{g}_2)$ there are nice combinatorial models (that is, finite presentations by generators and relations) of the planar algebras. These are the Temperley-Lieb algebra, and Kuperberg's rank 2 spiders.

I've made some progress extending these ideas to treat $U_q(\mathfrak{sl}_n)$ for all $n$. It's easy to find a good set of generators, and hard to find all relations amongst them. The inclusion $SU(n) \subset SU(n+1)$ means that irreducible representations of $U_q(\mathfrak{sl}_{n+1})$ break up as representations of $U_q(\mathfrak{sl}_n)$. This 'branching' can be described combinatorially in terms of the planar algebra, and using this we can lift relations from one level to the next. This method was described in:

**A Diagrammatic Category for the Representation Theory of $U_q(\mathfrak{sl}_n)$**
Ph.D. thesis, available at `http://tqft.net/thesis` and `arXiv:0704.1503`.

Next I hope to prove that

**Conjecture 3.1.** *The diagrammatic relations are complete; that is, they give a 'generators mod relations' presentation of the pivotal category of representations of $U_q(\mathfrak{sl}_n)$.*

My intended method will require finding inductive formulas for all the minimal idempotents in the category, analogous to the Jones-Wenzl idempotents in the case $n = 2$.

Further, in discovering the above relations for $SU(n)$, I developed computer algebra packages for dealing with representations of quantum groups via diagrams. I plan to make use of these programs to look for generators and relations in other quantum groups, particularly for $SO(n)$, where they are not known.

These diagrammatic presentations of quantum groups have a close connection with 'foam' models for $SU(n)$ Khovanov homology (see [3], [7] and my paper [8] for the $SU(3)$ case, and [6] for the $SU(n)$ case). Ideally one would see that the 'foam' models are a categorification of the representation theory. I'm hoping to contribute some of the many details which remain to be understood in the $SU(n)$ case.

## References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