Miller Fellow *1071 Evans Hall, University of California, Berkeley, CA 94720*
+1 805 453-7347 scott@tqft.net `http://tqft.net/`

Scott Morrison - Research statement

1 TQFT and Khovanov homology

During the 1980s and 1990s, surprising and deep connections were found between topology and algebra, starting from the Jones polynomial for knots, leading on through the representation theory of a quantum group (a braided tensor category) and culminating in topological quantum field theory (TQFT) invariants of 3-manifolds.

In 1999, Khovanov [Kho00] came up with something entirely new. He associated to each knot diagram a chain complex whose homology was a knot invariant. Some aspects of this construction were familiar; the Euler characteristic of his complex is the Jones polynomial, and just as we had learnt to understand the Jones polynomial in terms of a braided tensor category, Khovanov homology can now be understood in terms of a certain braided tensor 2-category. But other aspects are more mysterious. The categories associated to quantum knot invariants are essentially *semisimple*, while those arising in Khovanov homology are far from it. Instead they are *triangulated*, and *exact triangles* play a central role in both definitions and calculations.

My research on Khovanov homology attempts to understand the 4-dimensional geometry of Khovanov homology (§1.1). Modulo a conjecture on the functoriality of Khovanov homology in S^3, we have a construction of an invariant of a link in the boundary of an arbitrary 4-manifold, generalizing the usual invariant in the boundary of the standard 4-ball.

While Khovanov homology and its variations provide a 'categorification' of quantum knot invariants, to date there has been no corresponding categorification of TQFT 3-manifold invariants. The failure of standard TQFT methods in Khovanov homology has led me and my coauthor Kevin Walker to define 'blob homology', a certain generalisation of a TQFT which promises to be particularly useful for extracting topological information from non-semisimple categories. (See §1.2.)

1.1 Khovanov homology for 4-manifolds

Khovanov homology as originally defined was in invariant of unoriented link in B^3, associating (doubly-graded) vector spaces to knots, and linear maps (only well-defined up to sign) to cobordisms between links.

Motivated by ideas from the representation theory of $U_q(\mathfrak{sl}_2)$, I gave a new variation of Khovanov homology in my paper

Fixing the functoriality of Khovanov homology
Joint with David Clark and Kevin Walker, available at `arXiv:math.GT/0701339`, accepted by *Geometry and Topology*.

This associates a vector space to an *oriented* link in B^3, and the linear maps associated to cobordisms are now well-defined. The construction uses 'disorientations'.

However, the situation is still unsatisfactory. In particular, because we don't know how to extend this invariant to disoriented links, it is not even possible to ask whether the exact triangle in Khovanov homology is natural with respect to cobordisms.

We're working on several different approaches to such an extension. We hope that at least one of these approaches will give a natural exact triangle, which will in turn allow a proof of the following:

Conjecture 1.1. *Khovanov homology gives linear maps for cobordisms in S^3, not just B^3.*

Along with a description (partially appearing in [CMW09]) of Khovanov homology as a 4-category with duals, this conjecture allows us to define an invariant of a pair $(W, L \subset \partial W)$, with W a 4-manifold, and L a link in its boundary. For $L \subset \partial B^4$, this recovers the usual Khovanov homology invariant. It also applies to 4-manifolds without boundary, potentially giving non-trivial invariants.

In this direction, I have already proposed an application of Khovanov homology to potential counterexamples to the smooth 4-dimensional Poincaré conjecture,

Man and machine thinking about the smooth 4-dimensional Poincaré conjecture
Joint with Michael Freedman, Robert Gompf and Kevin Walker, *Quantum Topology*, Volume 1, Issue 2 (2010), pp. 171208. Available at arXiv:0906.5177.

This application relied on particular special properties of the potential counterexamples, and besides required a computer calculation at the limits of what is presently possible. I am working on gluing formulas for the new invariant, which may make it amenable to calculation in a wide class of examples. This would allow applications to 4-manifold topology. As an example, the lower bounds for the slice genus of a knot in S^3 obtained in [Ras] naturally generalize in this context to lower bounds for the genus of knot in the boundary of an arbitrary 4-manifold, in each relative second homology class.

1.2 Blob homology

With Kevin Walker, I've defined the "blob complex" $\mathcal{B}_*(M, \mathcal{C})$ associated to an n-manifold M and a (suitable) n-category $\mathcal{C}$.

The blob complex
Joint with Kevin Walker. Submitted to *Geometry & Topology*, available at arXiv:1009.5025.

This is a simultaneous generalisation of two interesting gadgets. When $n = 1$, $M = S^1$ and $\mathcal{C}$ is an algebra, the homology of the blob complex is the Hochschild homology of the algebra. On the other hand, the 0-th homology of the blob complex is the usual TQFT skein module of "pictures from $\mathcal{C}$ drawn on M". In this sense the blob complex is a "derived" version of a TQFT.

We expect that the blob complex is related to Lurie's topological chiral homology [Lur09], at least in the case of an n-category which is trivial at all levels below the topmost one.

We can prove several interesting properties of the blob complex. It is a functorial construction, and homeomorphisms of the manifold act on the complex. Moreover, there's an action of chains of homeomorphisms as well. Thus for example on the torus we get not only an action of the mapping class group, but also a compatible action of rotations along rational slopes. There is a gluing formula, expressed in terms of A_∞ bimodules, which in principle allows computations based on handle decompositions.

As an application, we can prove a considerable generalization of Deligne's conjecture that the little discs operad acts of Hochschild cohomology. We define blob cochains in terms of maps of the A_∞ modules associated to manifolds with boundary of arbitrary dimension, and then describe an action of the 'surgery cylinder operad' on blob cochains. This specialises in the one-dimensional case to give an entirely geometric proof of Deligne's conjecture.

We hope to apply blob homology to tight contact structures (for $n = 3$) and Khovanov homology (for $n = 4$). In both theories exact triangles play an important role. These exact triangles don't interact well with the gluing structure of the usual TQFTs, however. One of our motivations for considering blob homology is to work around these difficulties: for the Khovanov homology of a knot in the boundary of a 4-manifold (described in the previous section), there is a spectral sequence relating the invariants of link with a particular crossing and its resolutions.

2 Fusion categories and subfactors

I have just completed the classification of subfactors with index at most 5, in joint work with several authors, obtaining the following

Theorem 2.1. *There are exactly five subfactors (up to complex conjugation and duality) other than Temperley-Lieb with index between* 4 *and* 5*, namely the Haagerup subfactor, the extended Haagerup subfactor, the Asaeda-Haagerup subfactor, the '3311' GHJ subfactor and the '2221' Izumi subfactor.*

The details are appearing in a series of four papers

Subfactors of index less than 5, part 1: the principal graph odometer
Joint with Noah Snyder, available at `arXiv:1007.1730`.

Subfactors of index less than 5, part 2: triple points
Joint with David Penneys, Emily Peters and Noah Snyder, available at `arXiv:1007.2240`.

Subfactors of index less than 5, part 3: quadruple points
Joint with Masaki Izumi, Vaughan Jones and Noah Snyder, to appear.

Subfactors of index less than 5, part 4: cyclotomicity
By David Penneys and James Tener, available at `arXiv:1010.3797`.

(The first and second parts of this series will be updated soon, to reflect further progress which has completed the result.)

The results of this classification have been quite surprising; instances of small index subfactors are far sparser than anyone expected, making the particular known examples more interesting.

Prior to these results, the classification up to index $3 + \sqrt{3}$ was known: this was begun by Haagerup [Haa94], who listed the possible principal graphs and continued by Asaeda-Haagerup [AH99] who constructed subfactors (the Haagerup and Asaeda-Haagerup subfactors) corresponding to two of the graphs, and Bisch [Bis98] and Asaeda-Yasuda [Asa07, AY09] who ruled out all other possibilities except one. This last possibility was long suspected to in fact exist, and I constructed this recently, in

Constructing the extended Haagerup planar algebra

Joint with Stephen Bigelow, Emily Peters and Noah Snyder. To appear *Acta Mathematica*, available at arXiv:0909.4099.

As an example application of the new understanding of small examples made possible by this classification work, we have recently answered a question of Etingof, Nikshych and Ostrik [ENO05] "Are all fusion categories definable over a cyclotomic field?" in the negative.

Non-cyclotomic fusion categories

Joint with Noah Snyder. To appear *Transactions of the American Mathematical Society*, available at arXiv:1002.0168.

This result establishes that certain fusion categories coming from the Haagerup and extended Haagerup subfactors are 'exotic' relative to all previously known examples coming from group theory or quantum groups.

On the other hand, it is known that the dimension of any object in a fusion category is always a cyclotomic integer. In

Cyclotomic integers, fusion categories, and subfactors

Joint with Frank Calegari and Noah Snyder, with an appendix by Victor Ostrik. To appear *Communications in Mathematical Physics*, available at arXiv:1004.0665.

we used this to obtain strong restrictions on the possible dimensions of objects between 2 and 76/33, as well as to give a uniform arithmetic approach for ruling out certain infinite families of principal graphs of subfactors, generalizing the previous work of Asaeda-Yasuda mentioned above.

There is plenty of scope for further progress understanding small index subfactors and fusion categories with small objects. A recently awarded DARPA grant will support parts of this work.

It seems likely that the classification at index exactly 5 will be completed soon. At integer index, slightly different techniques are required (many examples here come from group-theoretical constructions, and proving the known examples are unique may require a combination of group theory and subfactor factor).

On the other hand complete classification results above index 5 may not be tractable with present techniques. Nevertheless, computer searches up to index 6 produce small lists of candidate examples, as well as lower bounds on the complexity of any remaining cases. It will be interesting to attempt to construct these potential new subfactors; just as the exceptional Lie algebras were discovered as part of the Killing-Cartan classification program, the exotic small index subfactors have all been discovered as a result of classification projects.

The techniques which we have so far mostly applied to the classification of subfactors will also work for finding small examples of fusion categories, and I anticipate that there will be many results in this direction in the future. Direct results on the classification of fusion categories will be of interest to physicists working on the fractional quantum Hall effect and conformal field theory. Indeed efforts are underway to compare the results we have obtained so far with classification results for rational conformal field theories.

Further, I plan to work on the classification of fusion categories with small global dimension. There has been some work in this direction for fusion categories with integer global dimension, but not much in the general case. This will require a wide variety of techniques, including number theory, combinators, subfactor theory, and the representation theory of Hopf algebras. With Rowell and Snyder I have recently proposed an AIM workshop on the subject.

3 Diagrammatic representation theory

3.1 Kashaev-Reshetikhin knot invariants

With Noah Snyder I'm working on computations and a paper about the Kashaev-Reshetikhin knot invariants.

We'll build on the papers of Kashaev and Reshetikhin to give a fully rigorous construction of their new knot invariants. This invariant is a function on the space $Hom(G_m(K), SL_2)/SL_2$ where $G_m(K)$ is the generalized knot group, and the action of SL_2 is by conjugation. We prove several basic results, for example, that the value of the function on the trivial point is $|J_m(\zeta_m)|^2$.

Further, we'll give the first computations of this invariant for nontrivial knots, based on a Mathematica package we've written. For several small 2-bridge knots we compute explicitly the entire Kashaev-Reshetikhin knot invariant at a third root of unity and at a fifth root of unity. We're planning to compute the knot invariants evaluated at the finite volume hyperbolic point for a larger collection of 2-bridge knots (again at a third and fifth root of unity).

3.2 Representations of $U_q(\mathfrak{g})$

The representation theory of a quantum group forms a planar algebra (equivalently, a spider or pivotal category). For $U_q(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_3)$,$U_q(\mathfrak{so}_5)$ and $U_q(\mathfrak{g}_2)$ there are nice combinatorial models (that is, finite presentations by generators and relations) of the planar algebras. These are the Temperley-Lieb algebra, and Kuperberg's rank 2 spiders.

I've made some progress extending these ideas to treat $U_q(\mathfrak{sl}_n)$ for all n. It's easy to find a good set of generators, and hard to find all relations amongst them. The inclusion $SU(n) \subset SU(n+1)$ means that irreducible representations of $U_q(\mathfrak{sl}_{n+1})$ break up as representations of $U_q(\mathfrak{sl}_n)$. This 'branching' can be described combinatorially in terms of the planar algebra, and using this we can lift relations from one level to the next. This method was described in:

A Diagrammatic Category for the Representation Theory of $U_q(\mathfrak{sl}_n)$
Ph.D. thesis, available at `http://tqft.net/thesis` and `arXiv:0704.1503`.

Next I hope to prove that

Conjecture 3.1. *The diagrammatic relations are complete; that is, they give a 'generators mod relations' presentation of the pivotal category of representations of $U_q(\mathfrak{sl}_n)$.*

My intended method will require finding inductive formulas for all the minimal idempotents in the category, analogous to the Jones-Wenzl idempotents in the case $n = 2$.

Further, in discovering the above relations for $SU(n)$, I developed computer algebra packages for dealing with representations of quantum groups via diagrams. I plan to make use of these programs to look for generators and relations in other quantum groups, particularly for $SO(n)$, where they are not known.

These diagrammatic presentations of quantum groups have a close connection with 'foam' models for $SU(n)$ Khovanov homology (see [Kho04], [MV07] and my paper [MN08] for the $SU(3)$ case, and [MSV] for the $SU(n)$ case). Ideally one would see that the 'foam' models are a categorification of the representation theory. I'm hoping to contribute some of the many details which remain to be understood in the $SU(n)$ case.

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