Abelian Graphs

Adam Clearwater
BSc(Hons) Student
Department of Mathematics
The University of Auckland

Supervisor: A. Prof. Scott Morrison

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Introduction

For my Summer Research Scholarship I have worked on understanding and extending the results of [CG15]. Let Γ be a connected finite graph, let $n \in \mathbb{N}$ be fixed, and let $v_1, \ldots, v_n$ be vertices of Γ. For any n-tuple of non-negative integers $\mathbf{a} = (a_1, \ldots, a_n)$, the *spider graph* $\Gamma_{\mathbf{a}}$ is defined to be the graph obtained from Γ by adjoining a 2-valent tree of length a_i to vertex v_i. Most of my own work this summer has been on *n-spokes*, which are spider graphs obtained from the graph consisting of a single vertex and no edges, along with the n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ where we suppose that $1 \leq a_1 \leq \cdots \leq a_n$.

Of primary interest is the question of if a given graph is *abelian*. A graph Γ is abelian if $\mathbb{Q}(\lambda^2)$ is an abelian extension of $\mathbb{Q}$, where λ is the Perron-Frobenius eigenvalue of Γ, the largest eigenvalue of the adjacency matrix of Γ. Equivalently, Γ is abelian if λ is a *cyclotomic integer*, which is an algebraic integer that is an element of $\mathbb{Q}(\zeta_N)$ where $\zeta_N = e^{2\pi i/N}$ for some $N \in \mathbb{N}$.

This project consisted of three sections. During an initial reading of [CG15] I undertook a period of background reading on basic graph theory, algebraic number theory and some field theory, in order to understand the definitions and claims made in the papers [ST02, GR01]. My attention then turned to n-spokes, where I reconstructed much of the section of [CG15] on 3-spokes. In doing so, I gained some experience in computing with Mathematica. Finally, a number of lemmas and calculations for 3-spokes were replicated for the case of 4-spokes, making progress towards the goal of finding all abelian 4-spokes.

Motivation

While the category-theoretic motivation behind the study of abelian graphs was not a focus of this project, acknowledging the reason for looking at these graphs is important nonetheless. A thorough treatment of the relevant category theory is both beyond the scope of this report and beyond the extent of my knowledge, so numerous terms are used without definition, and many claims are made with minimal explanation.

Given a finite group G and a vector space V, the set of finite dimensional representations of G (which are homomorphisms from G to $GL(V)$) is a category, the objects of which are representations and the morphisms of which are linear maps between the underlying vector spaces. We call this category Rep$G$, and it is a tensor category.

If a representation maps into a vector space V over $\mathbb{C}$, it is called *simple* if the set of endomorphisms on V is isomorphic to $\mathbb{C}$. Every representation in Rep$G$ can be written as a direct sum of simple representations, so Rep$G$ is *semisimple*.

Such categories can arise from physical systems. The representation category of a physical system's symmetry group is an object of interest, as it can yield information regarding the system's excitations. However, some semisimple tensor categories are not the representation category of any finite group. Other approaches are needed in order to understand these semisimple tensor categories.

Fusion categories, which are semisimple tensor categories with finitely many simple objects, are an active area of research and can be studied by looking at abelian graphs. Given a fusion category $\mathcal{C}$ and an object $X \in \text{obj}(\mathcal{C})$, we can find the *principal graph* for $(\mathcal{C}, X)$ in the following way. The vertices of the principal graph correspond to the simple objects in $\mathcal{C}$. Then for each simple object $Y \in \text{obj}(\mathcal{C})$, write $Y \otimes X$ as a direct sum of simple objects, and for each simple object Z, add an edge from Y to Z for each copy of Z that appears in this direct sum.

In the case that $\mathcal{C}$ is a unitary category, the Perron-Frobenius eigenvalue of the principal graph of $(\mathcal{C}, X)$ is the dimension of X, which is a cyclotomic integer. Therefore the principal graph is abelian. Conversely, given an abelian graph we can try to find $(\mathcal{C}, X)$ for which the principal graph is the given abelian graph. The need to find and understand fusion categories then provides motivation for developing systematic ways of finding abelian graphs.

Background Material

Prior to this project I had never formally studied graph theory, and some additional reading of algebraic number theory was also needed during the course of this project. In this section some of the most important results are mentioned, with a greater emphasis on their applicability to work completed this summer than on their proofs.

The Interlacing Theorem is an important result in the study of graph eigenvalues. A proof can be found in [GR01].

Theorem 1 (Interlacing)**.** *Let A be a real symmetric $n \times n$ matrix with eigenvalues $\theta_1(A) \leq \cdots \leq \theta_n(A)$ and let B be a principal submatrix of A with order $m \times m$ and with eigenvalues $\theta_1(B) \leq \cdots \leq \theta_m(B)$. Then for $i = 1, \ldots, m$, $\theta_{n-m+i}(A) \leq \theta_i(B) \leq \theta_i(A)$.*

In the context of graph theory, this result says that given an undirected graph Γ with adjacency matrix A and a graph Γ' with adjacency matrix B which is obtained from Γ by deleting some vertices and all of their incident edges, the eigenvalues of Γ and Γ' follow the inequalities above. In particular, if we add a vertex to a graph, the Perron-Frobenius eigenvalue of the resulting graph is at least as large as that of the original graph.

In order to find bounds on the parameters $a_1, \ldots a_n$ where $\Gamma_{a_1,\ldots,a_n}$ is an abelian n-spoke, we use properties of the norm of an algebraic number. The norm of an algebraic number λ is defined to be the product of all Galois conjugates of λ. If λ is an algebraic integer, with minimal polynomial $c_0 + c_1x + \cdots + c_{k-1}x^{k-1} + x^k$, then the product of its Galois conjugates is $\pm c_0$, a nonzero integer. This gives us the useful inequality $1 \leq N(\lambda) = \prod_\sigma \sigma\lambda$.

Perhaps the most crucial theorem underpinning the computations involved in this project is an equivalent condition that allows us to check if a given field extension is abelian.

Theorem 2 ([CMS11]). *Let β be an algebraic integer. Then $Q(\beta)$ is abelian if, and only if, β is a cyclotomic integer.*

Every time we checked if a given graph is abelian, it was done not by constructing the Galois group of a field extension, but by checking if the Perron-Frobenius eigenvalue λ is cyclotomic.

These are just a few of the main results that I learned about this summer while working on this project. Speaking more generally, gaining an introduction to graph theory was one of the most important outcomes of this project.

A Summary Of *Abelian Spiders*

This section is included to introduce the results of [CG15], as well as some notation, and some methods used in the section on 3-spokes that we later built upon. The main result of the paper is as follows.

Theorem 3 ([CG15], Thm. 1.1). *Let Γ and n be fixed. Then among the n-spiders $\Gamma_{\mathbf{a}}$ which are not Dynkin diagrams, only finitely many are abelian.*

The next two lemmas are needed for finding which 3-spokes are abelian. See [CG15] for the definition of the function B.

Lemma 1 ([CG15], Thm. 3.3). *Let $L \geq 0$ and let β be a totally real[1] algebraic integer, such that β^2 is not a singularity of B, the largest conjugate $\overline{|\beta|}$ of β satisfies $\overline{|\beta|} < L$, and at most M conjugates of β^2 lie outside $[0, 4]$. Further, suppose that $B(L^2) \leq 0$.*

Then either $[\mathbb{Q}(\beta^2) : \mathbb{Q}] < \dfrac{20}{11} \cdot M \cdot |B(L^2)|$ *or* $\mathcal{M}(\beta) := \dfrac{\mathrm{Tr}_{\mathbb{Q}(\beta^2)/\mathbb{Q}}(\beta^2)}{[\mathbb{Q}(\beta^2) : \mathbb{Q}]} < \dfrac{14}{5}$.

From this point, Γ will always denote the graph of a single vertex, and $\Gamma_{\mathbf{a}}$ will denote the corresponding n-spoke for the n-tuple $\mathbf{a}$. $P_{\mathbf{a}}(x)$ is the characteristic polynomial of the adjacency matrix of $\Gamma_{\mathbf{a}}$. It is frequently necessary to consider $P_{\mathbf{a}}(t + t^{-1})$, which for consistency with [CG15] will be denoted $P_{\mathbf{a}}(t)$. Further, we will simplify some calculations by defining $\overline{P}_{\mathbf{a}}(t) := P_{\mathbf{a}}(t)(t - t^{-1})^n(-1)^{(\sum_{i=1}^{n} a_i)-1}$

Lemma 2 ([CG15], Lemma 7.3). *Let $\mathbf{a} = (a, b, c)$. Then*

$$\overline{P}_{\mathbf{a}}(t) = (t^4 - 2t^2)t^{a+b+c} + t^{a+b-c} + t^{a+c-b} + t^{b+c-a} - t^{a-b-c} \quad (1)$$

$$- t^{b-a-c} - t^{c-a-b} + (2t^{-2} - t^{-4})t^{-a-b-c}. \quad (2)$$

[1] All roots of its minimal polynomial are real.

Let $\rho = \rho(\mathbf{a})$ be the largest root of the polynomial above, and let $\rho_\infty = \lim_{a,b,c\to\infty} \rho$. We see that ρ_∞ is the largest root of $t^4 - 2t^2$, which is $\sqrt{2}$. Accordingly, $\lim_{a,b,c\to\infty} \lambda = \rho_\infty + \rho_\infty^{-1} = 3/\sqrt{2}$.

Using $\lambda = \rho + \rho^{-1}$ and Lemma 1, we can prove that either $\mathcal{M}(\lambda^2 - 2) < 14/5$ or $D \leq 12$, where $D \leq 12$. In order to find bounds on a, b and c, we first make the assumption that $\mathcal{M}(\lambda^2 - 2) \geq 14/5$. This implies that $D \leq 12$, which in turn allows us to impose a bound on $|\lambda^2 - 9/2|$. Lemma 3 is a weaker version of Lemma 7.5 from [CG15], obtained from making trivial estimates.

Lemma 3. *With $a \leq b \leq c$, if $\Gamma_{\mathbf{a}}$ is abelian then $a \leq 36$.*

Proof. Note that $|2x - 9| \leq 9$ on $[0, 4]$. Since λ is an algebraic integer, so is $2\lambda^2 - 9$. Then using the fact that the product of all Galois conjugates of an algebraic integer is a positive integer,

$$1 \leq \prod_{\sigma\lambda^2} |2\sigma\lambda^2 - 9| = |2\lambda^2 - 9| \prod_{\sigma\lambda^2 \neq \lambda^2} |2\sigma\lambda^2 - 9| \leq |2\lambda^2 - 9| \cdot 9^{D-1} \leq |2\lambda^2 - 9| \cdot 9^{11}.$$

The second to last inequality follows from the fact that $\sigma\lambda^2 \in [0, 4]$ whenever $\sigma\lambda^2 \neq \lambda^2$, and that the total number of conjugates of $2\lambda^2 - 9$ (including itself) is D, the degree of the extension. Rearranging, we find that

$$\left|\lambda^2 - \frac{9}{2}\right| \geq \frac{1}{2 \cdot 9^{11}} > 1.59332 \times 10^{-11}.$$

Using Mathematica to calculate λ for triples of the form $\mathbf{a} = (a, a, a)$ we find that this inequality is violated whenever $a \geq 37$. Note that $a = b = c$ is the 'worst case', as we know that λ increases monotonely towards $3/\sqrt{2}$ as b and c increase, by the Interlacing Theorem. It follows that for a particular value of a, among all triples (a, b, c) with $a \leq b \leq c$ the triple which maximises $|\lambda^2 - 9/2|$ is in fact (a, a, a). □

The following is a slightly weaker result than Lemma 7.7 of [CG15].

Lemma 4. *With $a \leq b \leq c$, if $\Gamma_{\mathbf{a}}$ is abelian then $b \leq 84$.*

Proof. For a fixed a, as $b, c \to \infty$, ρ approaches the largest root of $1 - 2t^{2a+2} + t^{2a+4}$. Then since ρ is an algebraic integer, so is $1 - 2\rho^{2a+2} + \rho^{2a+4}$. Note also that $[\mathbb{Q}(\rho^2) : \mathbb{Q}] = [\mathbb{Q}(\rho^2) : \mathbb{Q}(\lambda^2)][\mathbb{Q}(\lambda^2) : \mathbb{Q}] \leq 2 \cdot 12 = 24$. Using the fact that the product of all Galois conjugates of an algebraic integer is a positive integer, we then find that

$$1 \leq \prod_{\sigma\rho} |1 - 2\sigma\rho^{2a+2} + \sigma\rho^{2a+4}|$$

$$
\begin{aligned}
&= |1 - 2\rho^{2a+2} + \rho^{2a+4}| \prod_{\sigma\rho \neq \rho} |1 - 2\sigma\rho^{2a+2} + \sigma\rho^{2a+4}| \\
&\leq |1 - 2\rho^{2a+2} + \rho^{2a+4}| \cdot 4^{D-1} \\
&= |1 - 2\rho^{2a+2} + \rho^{2a+4}| \cdot 4^{23}
\end{aligned}
$$

The second to last inequality follows from the fact that the total number of conjugates of $2\lambda^2 - 9$ (including itself) is D, the degree of the extension. Rearranging, we find that

$$|1 - 2\rho^{2a+2} + \rho^{2a+4}| \geq 4^{-23}$$

For each a such that $1 \leq a \leq 36$, using Mathematica we calculate ρ for triples of the form $\mathbf{a} = (a, b, b)$ and find the first b for which this inequality is violated. We find that the greatest possible value of b is 84. □

The computation required to find the bound $b \leq 84$ took several hours, and so it was apparent that computing a corresponding bound on c would be impractical. Using the same method as in Lemma 4 and the polynomial $t^{2a+2b+4} - 2t^{2a+2b+2} + t^{2b} + t^{2a} - 1$, a bound on c could be found in principle by calculating the largest c for which a certain inequality holds, requiring such a calculation to be carried out for each a and b with $1 \leq a \leq 36$ and $a \leq b \leq 84$. The necessary computation time was not available, and so this result was not replicated. In Lemma 7.8 of [CG15] the bound $c \leq 170$ is found, though the estimates we used were more crude and a slightly larger bound on c would have resulted.

Earlier we had assumed that $\mathcal{M}(\lambda^2 - 2) \geq 14/5$. It can be shown that any 3-spoke such that $\mathcal{M}(\lambda^2 - 2) < 14/5$ must in fact satisfy the bounds on a, b and c we found earlier. Therefore considering all 3-spokes within these bounds will indeed give a complete list of abelian 3-spokes. While we did not replicate this calculation, the result is as follows.

Theorem 4 ([CG15], Thm. 7.1)**.** *$\Gamma_{\mathbf{a}}$ is abelian if, and only if, either $\Gamma_{\mathbf{a}}$ is a Dynkin diagram or* $\mathbf{a} \in \{(2,3,7), (2,4,4), (2,7,11), (2,8,8), (3,3,3), (3,3,7), (3,4,9), (3,5,5), (4,4,4)\}$.

This result was partially verified using Mathematica, whereby for each 3-spoke with $1 \leq a \leq 4$, $a \leq b \leq 8$ and $b \leq c \leq 11$, λ was calculated and whether or not it was cyclotomic was determined[2]. The abelian 3-spokes within these bounds were exactly those listed above. Given time complexity considerations it was not possible to check all 3-spokes up to the bounds calculated in the previous lemmas.

[2]The algorithm used determines if a given algebraic integer is cyclotomic by taking its minimal polynomial modulo various primes. The mathematics upon which this algorithm is built, however, was not a focus of this project.

My Work On Abelian Spiders

The main contributions made in this project are a partial answer to the question of which 4-spokes are abelian, and a detailed method for finding all abelian n spokes for any n. The majority of this section is similar to parts of the last section, though proofs are included here. It is my intention that the method for finding all abelian n-spokes in the same way is made clear.

First, we need a more general version of Lemma 2.

Lemma 5. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$*. Then*

$$\overline{P}_{\mathbf{a}}(t) = \sum_{\mathbf{u}\in\{-1,1\}^n} \left(\prod_{i=1}^{n} u_i\right) \cdot t^{\sum_{i=1}^n u_i a_i} \cdot Q_{\mathbf{u}}(t)$$

where

$$Q_{\mathbf{u}}(t) = \sum_{\mathbf{v}\in\{0,1\}^n} t^{\sum_{i=1}^n (v_i-1)u_i} \cdot P_{\mathbf{v}}(t).$$

Proof. The $n = 1$ case is easily verified. Now consider $\overline{P}_{\mathbf{a}b}(\Gamma) := \overline{P}_{(a_1,\ldots,a_n,b)}(\Gamma)$. Throughout the following calculation we write $\overline{P}$ as a function of Γ rather than a function of t. Γ with its associated subscripts is a fixed graph, while the subscripts on $\overline{P}$ indicate the parameters corresponding to a spider of that graph. The subscripted asterisks indicate where the paths are added, either to $v_1, \ldots, v_n$ (indicated by $*$ as the first character of the subscript), to v_{n+1} (indicated by $*$ as the second character), or $v_1, \ldots, v_{n+1}$ (indicated by $**$). To illustrate, $\overline{P}_{\mathbf{a}}(\Gamma_{*b})$ means we are considering the graph Γ with a path of length b at v_{n+1} adjoined to be a fixed graph, then taking the n-spider of this graph formed by adjoining a path of length a_i at vertex v_i for $1 \le i \le n$. This rather cumbersome notation is necessary because we are simultaneously considering 1-, n- and $n+1$-spiders, requiring a careful treatment of $\overline{P}$. The second line of the calculation follows from Lemma 11 of [MS05].

$$\begin{aligned}
\overline{P}_{\mathbf{a}b}(\Gamma) =&(t-t^{-1})^n(-1)^{a_1+\cdots+a_n-1}\overline{P}_b(\Gamma_{\mathbf{a}*})\\
=&(t-t^{-1})^n(-1)^{a_1+\cdots+a_n-1}\textstyle\sum_{u_{n+1}=\pm1} u_{n+1}(t-t^{-1})^{-1}t^{u_{n+1}b}\big(\overline{P}_1(\Gamma_{\mathbf{a}*})-t^{-u_{n+1}}\overline{P}_0(\Gamma_{\mathbf{a}*})\big)\\
=&(-1)^{a_1+\cdots+a_n-1}\textstyle\sum_{u_{n+1}=\pm1} ut^{u_{n+1}b}\big((-1)^{a_1+\cdots+a_n-1}\overline{P}_{\mathbf{a}}(\Gamma_{*1})-(-1)^{a_1+\cdots+a_n}t^{-u_{n+1}}\overline{P}_{\mathbf{0}}(\Gamma_{*0})\big)\\
=&\textstyle\sum_{u_{n+1}=\pm1} u_{n+1}t^{u_{n+1}b}\big(\overline{P}_{\mathbf{a}}(\Gamma_{*1})+t^{-u_{n+1}}\overline{P}_{\mathbf{0}}(\Gamma_{*0})\big)\\
=&\textstyle\sum_{u_{n+1}=\pm1} u_{n+1}t^{u_{n+1}b}\Big(\sum_{\mathbf{u}\in\{-1,1\}^n}\big(\prod_{i=1}^n u_i\big)\cdot t^{\sum_{i=1}^n u_ia_i}\Big(\sum_{\mathbf{v}\in\{0,1\}^n} t^{\sum_{i=1}^n (v_i-1)u_i}\cdot P_{\mathbf{v}}(\Gamma_{*1})\Big)\\
&\qquad\qquad\textstyle +t^{-u}\sum_{\mathbf{u}\in\{-1,1\}^n}\big(\prod_{i=1}^n u_i\big)\cdot t^{\sum_{i=1}^n u_ia_i}\Big(\sum_{\mathbf{v}\in\{0,1\}^n} t^{\sum_{i=1}^n (v_i-1)u_i}\cdot P_{\mathbf{v}}(\Gamma_{*0})\Big)\Big)\\
=&\textstyle\sum_{\mathbf{u}\in\{-1,1\}^{n+1}}\big(\prod_{i=1}^{n+1} u_i\big)\cdot t^{u_{n+1}b+\sum_{i=1}^n u_ia_i}\sum_{\mathbf{v}\in\{0,1\}^{n+1}} t^{\sum_{i=1}^{n+1}(v_i-1)u_i}\cdot P_{\mathbf{v}}(\Gamma_{**})\\
=&\textstyle\sum_{\mathbf{u}\in\{-1,1\}^{n+1}}\big(\prod_{i=1}^{n+1} u_i\big)\cdot t^{u_{n+1}b+\sum_{i=1}^n u_ia_i}Q_{\mathbf{u}}(\Gamma)
\end{aligned}$$

□

Lemma 5 is sufficiently general to hold for any connected, finite, simple graph Γ, requiring only that every $P_{\mathbf{v}}(\Gamma_{**})$ is calculated for the given choice of Γ and $v_1, \ldots, v_n$. In the particular case of 4-spokes, we find the following polynomial.

Corollary 1. *Let* $\mathbf{a} = (a, b, c, d)$*. Then*

$$\begin{aligned}\overline{P}_{\mathbf{a}}(t) = &-2t^{-1+a-b-c-d} - 2t^{-1-a+b-c-d} - 2t^{-1-a-b+c-d} - 2t^{1+a+b+c-d} \\ &- 2t^{-1-a-b-c+d} - 2t^{1+a+b-c+d} - 2t^{1+a-b+c+d} - 2t^{1-a+b+c+d} \\ &+ t^{a+b-c-d}(t^{-1}+t) + t^{a-b+c-d}(t^{-1}+t) + t^{-a+b+c-d}(t^{-1}+t) \\ &+ t^{a-b-c+d}(t^{-1}+t) + t^{-a+b-c+d}(t^{-1}+t) + t^{-a-b+c+d}(t^{-1}+t) \\ &- t^{a+b+c+d}(t^5 - 3t^3) + t^{-5-a-b-c-d}(-1+3t^2),\end{aligned}$$

and $\rho_\infty = \sqrt{3}$.

Proof. This polynomial follows from Lemma 5. We then observe that ρ_∞ is the largest root of $Q_{++++}(t) = -(t^5 - 3t^3)$ which is $\sqrt{3}$. □

As with 3-spokes, we need to find a bound on the degree D of the extension $[\mathbb{Q}(\lambda^2) : \mathbb{Q}]$ before we can find bounds on a, b, c and d.

Lemma 6. *Either* $\mathcal{M}(\lambda^2 - 2) < 14/5$ *or* $D \leq 26$.

Proof. Let $\beta = \lambda^2 - 2$. The minimal polynomial of λ is a factor of the characteristic polynomial of a graph, all roots of which are real, so λ is a totally real algebraic integer. It follows that β is also a totally real algebraic integer. Then β is a totally real algebraic integer. Noting that at most one conjugate of β lies outside $[-2, 2]$, by Lemma 1 we know that either $\mathcal{M}(\beta) < 14/5$ or $D \leq \frac{20}{11} \cdot 1 \cdot |B(\frac{16}{3})| \approx 26.24$. □

Assume that $\mathcal{M}(\lambda^2 - 2) \geq 14/5$, so that $D \leq 26$. With this assumption we can find a bound on a, using a similar proof to Lemma 3.

Lemma 7. *With* $a \leq b \leq c \leq d$*, if* $\Gamma_{\mathbf{a}}$ *is abelian then* $a \leq 64$.

Proof. Note that $|3x - 16| \leq 16$ on $[0, 4]$. Since λ is an algebraic integer, so is $3\lambda^2 - 16$. Then using the fact that the product of all Galois conjugates of an algebraic integer is a positive integer,

$$1 \leq \prod_{\sigma\lambda^2} |3\sigma\lambda^2 - 16| = |3\lambda^2 - 16| \prod_{\sigma\lambda^2 \neq \lambda^2} |3\sigma\lambda^2 - 16| \leq |3\lambda^2 - 16| \cdot 16^{D-1} \leq |3\lambda^2 - 9| \cdot 16^{25}.$$

This calculation uses that the total number of conjugates of $3\lambda^2 - 16$ (including itself) is D, the degree of the extension. Rearranging, we find that

$$\left|\lambda^2 - \frac{16}{3}\right| \geq \frac{1}{3} \cdot 16^{-25} > 2.62953 \times 10^{-31}.$$

Using Mathematica to calculate λ for 4-tuples of the form $\mathbf{a} = (a, a, a, a)$ we find that this inequality is violated whenever $a \geq 65$. By the Interlacing Theorem the inequality is violated for all (a, b, c, d) where $65 \leq a \leq b \leq c \leq d$. □

Now let a be fixed, and let ρ be such that $\lambda^2 - 2 = \rho^2 + \rho^{-2}$. Then $\rho_\infty = \lim_{b,c,d\to\infty} \rho(\mathbf{a})$ is the largest root of $Q_{++++} + Q_{-+++}$. Then $-2\rho_\infty^{1-a+b+c+d} - \rho_\infty^{a+b+c+d}(\rho_\infty^5 - 3\rho_\infty^3) = 0$, so $\rho_\infty^{2a+4} - 3\rho_\infty^{2a+2} - 2 = 0$. Crucially, this means that $\rho^{2a+4} - 3\rho^{2a+2} - 2 \to 0$ as $b, c \to \infty$. Also,

$$[\mathbb{Q}(\rho^2) : \mathbb{Q}] = [\mathbb{Q}(\rho^2) : \mathbb{Q}(\lambda^2)][\mathbb{Q}(\lambda^2) : \mathbb{Q}] \leq 2D \leq 52.$$

With these results we are ready to compute a bound on b.

Lemma 8. *With $a \leq b \leq c \leq d$, if $\Gamma_{\mathbf{a}}$ is abelian then $b \leq 148$.*

Proof. For a fixed a, as $b, c, d \to \infty$, ρ approaches the largest root of $t^{2a+4} - 3t^{2a+2} + 2$. Then since ρ is an algebraic integer, so is $\rho^{2a+4} - 3\rho^{2a+2} + 2$. Using the fact that the product of all Galois conjugates of an algebraic integer is a positive integer, we then find that

$$\begin{aligned} 1 &\leq \prod_{\sigma\rho} |\sigma\rho^{2a+4} - 3\sigma\rho^{2a+2} + 2| \\ &= |\rho^{2a+4} - 3\rho^{2a+2} + 2| \prod_{\sigma\rho\neq\rho} |\sigma\rho^{2a+4} - 3\sigma\rho^{2a+2} + 2| \\ &\leq |\rho^{2a+4} - 3\rho^{2a+2} + 2| \cdot 6^{D-1} \\ &= |\rho^{2a+4} - 3\rho^{2a+2} + 2| \cdot 6^{51}. \end{aligned}$$

Rearranging, we find that

$$|\rho^{2a+4} - 3\rho^{2a+2} + 2| \geq 6^{-51}$$

For each a such that $1 \leq a \leq 64$, using Mathematica we calculate ρ for triples of the form $\mathbf{a} = (a, b, b)$ and find the first b for which this inequality is violated. We find that the greatest possible value of b is 148. □

As was the case in finding bounds for 3-spokes, finding a bound on c proved to be infeasible. In principle, we could check (a, b, c, c)-spiders with $1 \leq a \leq 64$ and $a \leq b \leq 148$ to find a bound on c. Repeating this process with even larger a calculation, we could go on to find a bound on d. There would remain finitely many 4-tuples (a, b, c, d) within the given bounds, and by calculating the Perron-Frobenius eigenvalue and using the cyclotomic test we could find all abelian 4-spokes.

Even if the necessary bounds could be found, the time required to compute the eigenvalues of many thousands of large matrices is simply too much. We can, however, reach a partial answer by checking all small 4-spokes. Computing the largest eigenvalue and using the cyclotomic integer test for all 4-spokes with $d \leq 15$, the following were found to be abelian:

Theorem 5. *Let*

$$\begin{aligned}(a,b,c,d) \in \{&(1,1,1,1),(1,1,1,2),(1,1,1,3),(1,1,2,2),(1,1,2,3),(1,1,3,3),(1,1,4,4),\\ &(1,2,2,2),(1,2,5,5),(1,3,3,3),(2,2,2,2),(2,3,7,7),(2,4,4,4),(3,3,3,3),\\ &(3,4,9,9),(3,5,5,5),(4,4,4,4)\}.\end{aligned}$$

Then $\Gamma_{a,b,c,d}$ is abelian. Further, there are no other abelian 4-spokes where $d \leq 15$.

Given that no abelian 4-spokes have been found with $10 \leq d \leq 15$, it is at least plausible that there are no abelian 4-spokes with $d > 15$.

The same time-complexity challenges will thwart attempts to find bounds for n-spokes with larger n. However, the method of checking small n-spokes can give us lists of abelian graphs, with the caveat that these lists are not guaranteed to be complete.

Theorem 6. *Let*

$$\begin{aligned}(a,b,c,d,e) \in \{&(1,1,1,1,1),(1,1,1,1,2),(1,1,1,1,3),(1,1,1,2,2),(1,1,1,2,3),\\ &(1,1,1,3,3),(1,1,2,2,2),(1,1,3,3,3),(1,1,4,4,4),(1,2,2,2,2),\\ &(1,2,5,5,5),(1,3,3,3,3),(2,2,2,2,2),(2,3,7,7,7),(2,4,4,4,4),\\ &(3,3,3,3,3),(3,4,9,9,9),(3,5,5,5,5),(4,4,4,4,4)\}.\end{aligned}$$

Then $\Gamma_{a,b,c,d,e}$ is abelian. Further, there are no other abelian 5-spokes where $e \leq 15$.

Theorem 7. *Let*

$$\begin{aligned}(a,b,c,d,e,f) \in \{&(1,1,1,1,1,1),(1,1,1,1,1,2),(1,1,1,1,1,3),(1,1,1,1,2,2),\\ &(1,1,1,1,2,3),(1,1,1,1,2,4),(1,1,1,1,3,3),(1,1,1,2,2,2),\\ &(1,1,1,3,3,3),(1,1,1,3,5,5),(1,1,2,2,2,2),(1,1,3,3,3,3),\\ &(1,1,4,4,4,4),(1,2,2,2,2,2),(1,2,5,5,5,5),(1,3,3,3,3,3),\\ &(1,3,5,5,5,5),(1,3,6,6,6,6),(1,4,4,4,4,4),(2,2,2,2,2,2),\\ &(2,2,2,2,2,3),(2,3,7,7,7,7),(2,4,4,4,4,4),(3,3,3,3,3,3),\\ &(3,5,5,5,5,5),(4,4,4,4,4,4)\}.\end{aligned}$$

Then $\Gamma_{a,b,c,d,e,f}$ is abelian. Further, there are no other abelian 6-spokes where $f \leq 8$.

Conclusion and Future Work

The method outlined for finding which 4-spokes are abelian can, in principle, be used for n-spokes with arbitrarily large n. The more useful outcome would be that a pattern in the occurrence of abelian spoke graphs can be observed from considering n-spokes for small n, from which a complete classification of abelian spoke graphs may be possible. The results of this project fall short of this aspiration, but it is hoped nonetheless that the work done on finding some small spoke graphs is of use. Efforts to optimise how the Perron-Frobenius eigenvalue is found and how the cyclotomic integer test is run might allow some of these computations to be carried out in practice.

Spending this summer studying abelian graphs has been a tremendously rewarding experience. I was drawn to this project in particular because of the opportunity to explore areas that were largely unfamiliar to me, which I certainly have been provided with. I have benefitted greatly from learning some of the fundamentals of graph theory and algebraic number theory, not to mention my first significant encounter with computational mathematics. Given more time it would have been interesting to learn more about fusion categories and subfactors, though I am happy with the balance that was struck between reading mathematics and doing mathematics.

References

[CG15] F. Calegari and Z. Guo, *Abelian Spiders*, arXiv:1502.00035. 2015.

[CMS11] F. Calegari, S. Morrison and N. Snyder, *Cyclotomic integers, fusion categories, and subfactors*,
Commun. Math. Phys. volume 303(3), pages 845-896, 2011.

[ENO11] P. Etingof, D. Nikshych and V. Ostrik, *On Fusion Categories*,
Ann. Math. volume 162(2), pages 581-642, 2011.

[GR01] C. Godsil and G. Royle, *Algebraic Graph Theory*, New York: Springer, 2001.

[MS05] J. McKee and C. Smyth, *Salem Numbers, Pisot Numbers, Mahler Measure, and Graphs*,
Exper. Math. volume 14(2), pages 211-229, 2005.

[ST02] I. Stewart and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem, 3rd Ed.*,
Natick: A. K. Peters, 2002.