Presentations of the Mapping Class Group

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Declaration

The work in this thesis is my own except where otherwise stated.

Adele Jackson

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Abstract

The mapping class group is an important algebraic invariant of a surface. Presentations of this group have wide applications to low-dimensional topology. We explicitly construct Hatcher and Thurston's finite presentation with Dehn twist generators for genus one and two surfaces. We then extend Bene's chord slide presentation from surfaces with connected boundary to those with disconnected boundary. This presentation arises from studying a cell decomposition of Teichmüller space whose vertices are fatgraph decorations of surfaces. We can convert the resulting fatgraph presentation of the mapping class group to one with chord slide generators. This chord slide presentation has potential applications to computing bordered Heegaard Floer invariants for open books with disconnected binding.

Contents

Notation and terminology

We will assume all surfaces are orientable and connected with no punctures, though they may have boundary components. In the following, S is an orientable surface, ∂S is its boundary, and x is a point in S. We use g for the genus of a surface, and n for the number of boundary components. A *closed* surface is one with no boundary components.

All groups of maps from S to S come with the compact-open topology (Definition 2.2), and if F is such a group of maps containing the identity, we take $\pi_1(F)$ to be based at the identity.

We will identify the following objects, which are defined in Chapters 4-6:

1. an edge e of a marked bordered fatgraph,
2. the graph dual of e, which we can view as an element of the fundamental path groupoid, and
3. if e is in the canonical generating set for the fundamental path groupoid, the image of e under branch reduction. This image is an edge in a chord diagram that has the same dual as e.

Notation

$\preceq$	If c_1 and c_2 are oriented chords, we write $c_1 \preceq c_2$ if the initial point of c_1 immediately precedes that of c_2.
$[a, b]$	The commutator of a and b, $aba^{-1}b^{-1}$.
$\{p_i\}$	On $\Sigma_{g,n}$, this is a collection of n marked *dual points*, disjoint from the n marked points, with one on each boundary component.
$\overline{\mathbf{e}}$	If $\mathbf{e}$ is an oriented edge in a graph or chord diagram, $\overline{\mathbf{e}}$ is the same edge with the opposite orientation.
$\pi_n(S, x)$	The n^{th} homotopy group of S with basepoint x.

$\pi_1(S, X, x)$	The relative fundamental group. For $x \in X \subset S$, this is homotopy classes of maps $(D^1, S^0, s_0) \to (S, X, x)$ where D^1 has endpoints s_0 and s_1.
$\Pi_1(S, \{p_i\})$	Let $\Pi_1(S)$ be the fundamental path groupoid of S, that is, the groupoid consisting of homotopy classes of paths in S. Then $\Pi_1(S, \{p_i\})$ is the full subgroupoid of $\Pi_1(S)$ containing all paths that start and end at points in $\{p_i\}$..
$\Sigma_{g,n}$	A fixed orientable genus g surface with n boundary components, with an ordering on the boundary components. Comes equipped with n marked points, one on each boundary component.
D^n	The n-dimensional closed disk.
$\mathrm{Diff}(S)$	The group of diffeomorphisms $S \to S$.
$\mathrm{Diff}^+(S)$	The group of orientation-preserving diffeomorphisms $S \to S$.
$\mathrm{Diff}(S, \partial S)$	The group of diffeomorphisms $S \to S$ that fix the boundary of S, ∂S, pointwise.
$\mathrm{Diff}(S, \partial S, x)$	The group of self-diffeomorphisms of S that fix ∂S pointwise and send x to itself.
$\mathrm{Diff}(S \,\mathrm{rel}\, X)$	The group of self-diffeomorphisms of S that fix $X \subset S$ as a set.
$\mathrm{Homeo}(S)$	The group of orientation-preserving homeomorphisms $S \to S$. This can be modified in the same way as the group of diffeomorphisms (for example, $\mathrm{Homeo}(S, \partial S)$).
$Mod(S)$	The mapping class group of S. This is the group of orientation-preserving homeomorphisms (or diffeomorphisms) of our surface S, that fix ∂S pointwise, up to isotopy also fixing ∂S. See Definition 2.3.
P_n	The pure braid group on n strands. See Definition 2.15.
$P(X; n)$	Configuration space of n points in X. See Definition 2.14.
S^n	The n-dimensional sphere.
$\pm Sym_n$	The signed permutation group on n elements. Each element of this group is a permutation σ of $\{\pm 1, \pm 2, \ldots, \pm n$ such that $-\sigma(k) = \sigma(-k)$ for all k.

T_γ	A Dehn twist about γ.
$T(e)$	For e an edge in the maximal tree of a bordered fatgraph, this is the subset of the maximal tree that is disconnected from the tail within the tree if we remove e.
$W(e)$	A Whitehead move on the edge e.

Chapter 1

Introduction

The mapping class group $Mod(S)$ is the group of isotopy classes of homeomorphisms from a surface to itself. It is an important algebraic invariant of a surface, and is deeply connected to Teichmüller space and the moduli space of Riemann surfaces homeomorphic to S. One natural problem is to find a "nice" presentation of the mapping class group. Such a presentation allows us to easily work with and investigate the group.

We first consider a presentation developed by Hatcher and Thurston. Its generators are Dehn twists (Definition 2.5), which are a type of mapping class that arises naturally in the 2-manifold context. A classical result of Dehn and Lickorish shows that $Mod(S)$ is finitely generated by Dehn twists [Lic64]. More recently, Hatcher and Thurston showed that for a certain choice of Dehn twist generators of $Mod(S)$ for closed surfaces S, one could construct a finite generating set of relations [HT80].

After giving some background in Chapter 2, in Chapter 3 we give an exposition of this Dehn twist presentation of $Mod(S)$. Hatcher and Thurston show that one can construct a finite presentation whose generating set is Dehn twists, but leave an explicit execution of it to the reader. They examine the action of $Mod(S)$ on a certain combinatorial complex, the cut system complex (Definition 3.2). They prove that a series of fibrations of diffeomorphism spaces determines a presentation of the subgroup of $Mod(S)$ that fixes a vertex of this complex, then build the full presentation from this. While Wajnryb gave a very explicit exposition of their work, his intentional choice of elementary techniques mean his approach is substantially removed from the fibrations and exact sequences of homotopy groups used in the original paper [Waj99]. Chapter 3 bridges this gap in techniques, using Hatcher and Thurston's fibrations and the Birman exact sequence (Theorem 2.22). We derive an explicit presentation of the subgroup of $Mod(S)$ that fixes a vertex for genus one and two surfaces, then discuss how to extend this to a presentation of the whole group.

There are many other presentations of $Mod(S)$. We consider another one in Chapters 4-6. Though Dehn twists are natural in the two-dimensional topology context, in different contexts other generators arise. Penner, among others, considers cell decompositions of Teichmüller space that have an action of $Mod(S)$ [Pen87; Pen04]. In this setting, the edges of this cell complex provide a natural set of generators. One cell decomposition of Teichmüller space has graphs embedded in S as vertices (Definition 5.4) and Whitehead moves on these graphs as edges (Definition 5.3). We naturally have a presentation of the fundamental path groupoid whose generators are the Whitehead moves, and whose relations are the faces in the complex. The Whitehead move presentation has a much more combinatorial flavour than Hatcher and Thurston's more algebraic topology-influenced work.

As in Hatcher and Thurston's presentation, the action of $Mod(S)$ on the complex can be used to find a presentation of the group. In this case, the result is an infinite presentation of $Mod(S)$ whose generators are Whitehead moves. For surfaces with one boundary component, Bene showed that this Whitehead move presentation of the fundamental path groupoid of the complex can be translated to one whose generators are chord slides [Ben10] using the branch reduction algorithm [ABP09]. This chord slide presentation of the fundamental path groupoid is finite, and by restriction gives an infinite presentation of $Mod(S)$. Bene builds on work lifting various representations of $Mod(S)$ to fundamental path groupoids in combinatorial complexes, which has been particularly successful in the connected boundary case (for example, see [ABP09] and [MP08]).

A chord slide presentation of $Mod(S)$ has applications in bordered Heegaard Floer homology, which is an invariant of a three-manifold with boundary that generalises the Heegaard Floer invariants for closed 3-manifolds. Heegaard Floer homology has been one of the most important developments in low-dimensional topology this century. It gives rich and complicated new structures, and has been instrumental in proofs of previously open problems, such as computing knot genus. The fundamental Heegaard Floer invariant for a closed three-manifold Y is $\widehat{HF}(Y)$. The bordered theory computes $\widehat{HF}(Y)$ as an iterated Hom space of modules associated to a standard handlebody and a set of generators of $Mod(S)$. In the Heegaard Floer context, the morphisms associated to the chord slide generators of $Mod(S)$ are called arc slides, and these have been computed for the surfaces covered by Bene's work.

In Chapters 4-6, we extend Bene's chord slide presentation from surfaces with one boundary component to those with n boundary components for any $n \geq 1$. The results in these chapters have potential applications in low-dimensional topology. As noted in Section 1.6 of [LOT14], the current bordered Heegaard Floer techniques can be used to compute $\widehat{CF}(Y)$ for a 3-manifold presented as an open book with a connected binding. Extending this approach to open books with disconnected binding requires a chord slide generating set for surfaces with multiple boundary components. Furthermore, chord slides appear in [GL18] in the form of handleslides occurring in an S^1-family of Morse functions on a surface. A presentation of Mod(S) in terms of these generators would be a significant step towards a classification of the Morse structures compatible with a fixed contact structure.

Our extension of Bene's work proceeds as follows. First, in Chapter 4, we define a marked bordered fatgraph as in [God07] and discuss its dual, a quasi-triangulation, to show that the mapping class group acts freely on the set of marked bordered fatgraphs. In Chapter 5, we define the marked bordered fatgraph complex, which is homotopy equivalent to Teichmüller space [God07; Har86]. The action of the mapping class group on this complex gives an infinite presentation of $Mod(S)$ whose generators are Whitehead moves (Corollary 5.12). In Chapter 6, we generalise Bene's chord slide presentation by extending the branch reduction algorithm [ABP09] to the n boundary component context. The branch reduction algorithm takes a fatgraph to a chord diagram, which we show is determined by a canonical set of generators of the fundamental path groupoid of the surface that is associated to the marking of a fatgraph. We show that, under this algorithm, the images of Whitehead moves on fatgraphs are generated by chord slides (Theorem 6.20). We find a set of chord slide generators for $Mod(S)$ for surfaces with boundary, and show one can construct a finite but large relation set. We give a simpler candidate relation set (Conjecture A). The proof that the relations in Conjecture A generate all relations is not complete, as we have not checked all the necessary cases.

The marked bordered fatgraphs defined in Chapter 4 have been previously explored, and Sections 4.1 and 4.2 are an exposition of previous work. Section 4.3, which discusses the dual of a marked bordered fatgraph, is a topological approach to Harer's arc-system [Har86] and Penner's quasi-triangulation [Pen04]. The marked bordered fatgraph complex (Section 5.1) is well-known, and by taking the dual of its vertices is homeomorphic to Harer's arc-system complex. For example, this complex can be used to compute the homology of the mapping class group for punctured or bordered surfaces [God07]. However, to the author's knowledge, marked bordered fatgraphs have not been applied to find a presentation of the mapping class group aside from in the connected boundary case.

From Section 5.2 onwards, we generalise the work in [ABP09] and [Ben10], extending the objects defined there from the one-boundary to the n-boundary case. One particular difficulty in the n-boundary case is that the effects of Whitehead moves are not local (Theorem 5.13). Constructions that particularly differ from the one boundary case include the following: (1) the definition of a chord diagram (Definition 6.1), (2) the method to associate a generating set for the fundamental path groupoid to a marked bordered fatgraph (Proposition 4.30), and (3) the proof that chord diagrams are determined by this associated generating set, which involves showing that reduced words in the fundamental path groupoid are unique (Proposition 6.19). Most of the remaining proofs in Chapters 4-6 follow Bene's ideas, but are more involved since we have more complicated cases of Whitehead moves.

Chapter 2

Background

In this chapter we introduce the mapping class group of a surface and provide some background material for the rest of this thesis. See [FM12] for more details.

Let S be an orientable genus g surface with n boundary components. It is a convention of a field to say that S is a *closed* surface if it has no boundary components. Let $\mathrm{Homeo}(S, \partial S)$ be the group of boundary-fixing orientation-preserving homeomorphisms of S, and $\mathrm{Diff}^+(S, \partial S)$ be the boundary-fixing orientation-preserving diffeomorphisms of S.

2.1 Some results on the mapping class group

First, we define the fundamental group and fundamental path groupoid.

Definition 2.1. Let S be a topological space. The *fundamental group* of S based at $x \in S$, $\pi_1(S, x)$, is the group of homotopy classes of paths that begin and end at x with composition.

The *fundamental (path) groupoid* of S, $\Pi_1(S)$, is the groupoid of homotopy classes of paths in S, which may begin and end anywhere in S. There is a composition operation on pairs of paths f, g such that f ends at some $x \in S$ and g starts at x.

By convention, the composition of paths fg is the map $I \to S$ that sends $t \in [0, 1/2]$ to $f(2t)$ and $t \in [1/2, 1]$ to $g(2t - 1)$.

Note that in the fundamental group, the identity is the trivial path $I \to \{x\}$ and the inverse of a path is the same path with the other orientation. See [Hat01, Chapter 1] for a verification of these claims. There is a natural inclusion of the fundamental group in the fundamental groupoid.

We define a topology on a space of maps $S \to S$.

Definition 2.2. Let S be a surface, and F some space of maps $S \to S$. The *compact-*

open topology on F is generated by

$$V_{U,K} = \{f \in F \mid f(K) \subseteq U\}$$

for $U \subset S$ open and $K \subset S$ compact.

When we write $\text{Diff}(S)$, or any other space of maps of a surface, we will assume it has the compact-open topology. This will allow us to consider the homotopy groups of the spaces of diffeomorphisms or homeomorphisms. With this topology, as our surfaces are Hausdorff and compact, the connected components of $\text{Homeo}(S, \partial S)$ are isotopy classes of homeomorphisms, and paths are isotopies. We will take as a convention that $\pi_1(\text{Homeo}(S, \partial S))$ is based at the identity.

Definition 2.3. The *mapping class group* $Mod(S)$ of S is the group of (boundary-fixing) isotopy classes of $\text{Homeo}^+(S, \partial S)$.

Note that the set of isotopy classes of $Homeo(S, \partial S)$ is precisely $\pi_0(Homeo(S, \partial S))$. This definition is equivalent to several others.

Theorem 2.4 (Theorem 6.4 [Eps66], [Bae27], [Bae28], Theorem 6.3 [Mun60])**.** *The following groups are isomorphic:*

$$\begin{aligned} Mod(S) &= \pi_0(\text{Homeo}(S, \partial S)) \\ &\cong \text{Homeo}(S, \partial S)/\textit{homotopy} \\ &\cong \pi_0(\text{Diff}^+(S, \partial S)) \\ &\cong \text{Diff}^+(S, \partial S)/\textit{smooth isotopy.} \end{aligned}$$

We call an element of $Mod(S)$ a *mapping class*. Note that a mapping class acts on the set of simple closed curves and on the fundamental group(oid).

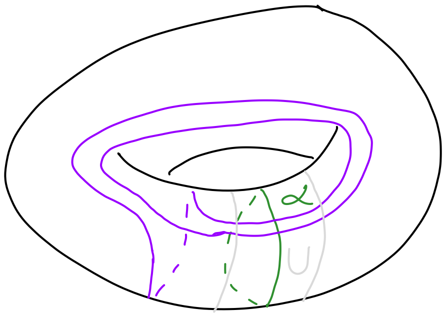
Some particularly important mapping classes are the Dehn twists.

Definition 2.5. Let α be a simple closed curve in S. The *Dehn twist about* α, T_α, is defined as follows.

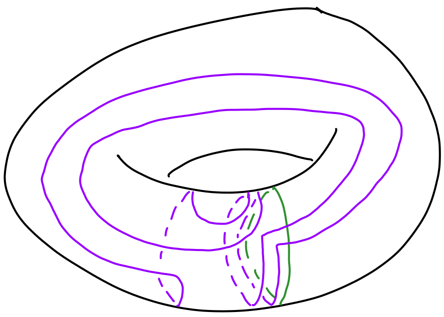
Let $A = S^1 \times I$ be the annulus, with orientation from some embedding in the plane. Let the twist about A, $T : A \to A$, be the map sending $(\theta, t) \mapsto (\theta + 2\pi t, t)$. Note that T is orientation-preserving and fixes the boundary of the annulus pointwise.

Let U be a neighbourhood of α, with $\varphi : U \to A$ an orientation-preserving homeomorphism. Then T_α acts as follows:

$$T_\alpha(x) = \begin{cases} x & x \notin N \\ \varphi^{-1} \circ T \circ \varphi(x) & x \in N. \end{cases}$$



(a) Before the Dehn twist. We will twist around α in the neighbourhood U.



(b) The resulting curve after the Dehn twist.

Figure 2.1: The effect of a Dehn twist on α on the purple curve.

For example, see Figure 2.1.

We give some basic results on Dehn twists. Let a and b be homotopy classes of simple closed curves with representatives α and β respectively. Let $i(a, b)$ be the algebraic intersection number of a and b. This is the minimal number of intersections between any two representatives of a and b.

Now, T_α acts on β as follows. Replace each piece of β that crosses α by a piece that turns left as it approaches α, does one circuit of α, and then turns right to follow β again.

We give some special cases for T_a and T_b depending on the intersection number of a and b.

Lemma 2.6. *If α and β are simple closed curves of homotopy classes a and b respectively, with $i(a, b) = 0$, then T_α acts trivially on β.*

Proposition 2.7 (Braid relation, Prop. 3.11 [FM12])**.** *If $i(a, b) = 1$, then*

$$T_a T_b T_a = T_b T_a T_b.$$

It is a classical result that the Dehn twists generate the mapping class group, and that in fact one can pick a finite generating set.

Theorem 2.8 (Dehn (1938), Lickorish (1964), [Lic64])**.** *If S has no boundary components, $Mod(S)$ is generated by finitely many Dehn twists about non-separating simple closed curves.*

Theorem 2.9 (Section 4.4.4 [FM12])**.** *For a surface S, possibly with boundary, $Mod(S)$ is generated by finitely many Dehn twists about (possibly separating) simple closed curves.*

2.2 Additional useful results

We give some results we will reference in later chapters.

Isotopies of curves in S can be extended to isotopies of S.

Lemma 2.10 (Proposition 1.11, [FM12]). *Let $F : S^1 \times I \to S$ be a smooth isotopy of simple closed curves. Then there is an (ambient) isotopy $\widetilde{F} : S \times I \to S$ such that for $x \in S$, $\widetilde{F}(x \times 0) = x$, and for $y = F(\theta, 0)$, $\widetilde{F}(y,t) = F(\theta,t)$.*

By observing that the mapping class group of the disk is trivial, we have the following lemma.

Lemma 2.11 (Alexander Trick). *Let $f : D^2 \to D^2$ be a map fixing the boundary pointwise. Then f is isotopic to the identity by an isotopy fixing the boundary pointwise.*

We can describe diffeomorphism classes of a disjoint union of circles as a permutation group.

Definition 2.12. The *signed permutation group* on g elements, $\pm Sym_g$, is the group of permutations σ of $\{\pm 1, \dots, \pm g\}$ such that $\sigma(-k) = -\sigma(k)$.

Lemma 2.13. *The smooth isotopy classes of diffeomorphisms of circles, $\pi_0\left(\text{Diff}\left(\bigcup_{i=1}^n S^1\right)\right)$, is isomorphic to $\pm Sym_n$.*

Proof. Let α_n be the n^{th} copy of S^1. Define a homomorphism

$$\phi : \pi_0\left(\text{Diff}\left(\bigcup_{i=1}^{g} S^1\right)\right) \to \pm Sym_g$$

as follows. For $[f]$ an isotopy class of diffeomorphisms with a representative f and $n > 0$, let $|\phi([f])(n)|$ be the integer such that $f(\alpha_n) = \alpha_{|\phi([f])(n)|}$. This is well-defined, as, since f is a diffeomorphism, it sends α_n surjectively to exactly one α_j. Further, f is injective, so for $n \neq m$, $|\phi([f])(n)| \neq |\phi([f])(m)|$.

Now, consider $f|_{\alpha_n}$. This is a map $S^1 \to S^1$, so $f|_{\alpha_n}$ is isotopic to ω_k, the winding number k map. As it is a diffeomorphism, it is injective, so is isotopic to $\omega_{\pm 1}$. Let the sign of $\phi([f])(n)$ be $+$ if $f|_{\alpha_n}$ is isotopic to ω_1, and $-$ if $f|_{\alpha_n}$ is isotopic to ω_{-1}.

Then ϕ is an isomorphism. □

The braid group appears naturally as the fundamental group of configuration space. We briefly discuss this space. For more details, see Chapter 9, [FM12], particularly Section 9.3.

Let $D(S^{\times n})$ be the points in $S^{\times n}$ where at least two coordinates are equal.

Definition 2.14. The *configuration space* of n ordered points in a surface S is $S^{\times n} - D(S)$.

Definition 2.15. Let ϕ be a homomorphism from the braid group on n strands, B_n, to the symmetric group on n elements, Sym_n, that acts as follows. For $b \in B_n$, b induces a permutation on the n strands. Let $\phi(b)$ be this permutation.

The *pure braid group* on n strands, P_n, is the kernel of ϕ.

Equivalently, the pure braid group is the subgroup of B_n that induces the trivial permutation on the strands.

Lemma 2.16 (p.249 [FM12])**.** *The pure braid group P_n is isomorphic to $\pi_1(P(D^2; n))$ where D^2 is the disk.*

This theorem is a consequence of a generalised version of the Birman exact sequence (Theorem 2.22) from one marked point to n marked points. We describe the pure braid group in terms of generators and relations.

Proposition 2.17 (Theorem 16, [Art47])**.** *The braid group on n strands has a finite presentation with generators σ_i for $1 \leq i < n$, which takes strand i over strand $i+1$, and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i < n-1$.*

Proposition 2.18 (p. 251 [FM12], Theorem 18 [Art47])**.** *The pure braid group P_n has the following presentation.*

For $1 \leq i < j \leq n$, let $A_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{j-1} \cdots \sigma_{i+1})^{-1}$.

Then P_n is generated by all such $A_{i,j}$, with relations, for all $1 \leq i < j < k < l \leq n$, as follows:

$$
\begin{aligned}
[A_{i,j}, A_{k,l}] &= 1 && j < k \\
[A_{i,l}, A_{j,k}] &= 1 && i < j < k < l \\
[A_{k,l}A_{i,k}A_{k,l}^{-1}, A_{j,l}] &= 1 && i < j < k < l
\end{aligned}
$$

and the relation that the three cyclic permutations of $A_{i,k}A_{j,k}A_{i,j}$ are equal.

2.3 Birman exact sequence

The Birman exact sequence relates the mapping class group of a surface to the mapping class group of the same surface with a fixed point. The exactness of this sequence is proven in Theorem 4.6, [FM12], but the authors only outline the main steps of the proof. For example, they do not explicitly show the maps in the sequence are $Push$ and $Forget$.

Let S be a surface with $x \in S$. This theorem will give us a well-defined map $Push : \pi_1(S, x) \to Mod(S, x)$, which we will use in Chapter 3.

Definition 2.19. The group $\text{Homeo}^+(S, x, \partial S)$ is the orientation-preserving homeomorphisms of S fixing x and the boundary ∂S pointwise.

The mapping class group of S with a fixed point x, $Mod(S,x)$, is $\text{Homeo}^+(S,x,\partial S)/\sim$ where for $f,g \in \text{Homeo}^+(S,x,\partial S)$, $f \sim g$ if there is some isotopy fixing x from f to g.

We first give a lemma on the centre of $\pi_1(S)$.

Lemma 2.20. *Let S be a compact surface with $\chi(S) < 0$. Then $\pi_1(S)$ is centreless.*

Proof. (This proof follows ideas from Keeley Hoek and Chris Hone.)

As $\chi(S) < 0$, we have $2g+b > 2$ where g is the genus and b the number of boundary components of S.

If $b > 0$, the surface S deformation retracts to a wedge sum of circles. The fundamental group is thus free, so trivially centreless.

Otherwise, suppose $b = 0$ so $g > 1$. Then

$$\pi_1(S) \cong \{a_1, b_1, \ldots, a_g, b_g \,|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g^{-1} b_g^{-1}\}.$$

Consider the quotient $\pi_1(S)/\langle b_1, \ldots, b_g\rangle$. This is $\{a_1, \ldots, a_g \,|\, a_1 a_1^{-1} \cdots a_g a_g^{-1}\}$. Note that $a_1 a_1^{-1} \cdots a_g a_g^{-1} = 1$ trivially, so $\pi_1(S)/\langle b_1, \ldots, b_g\rangle$ is the free group on g generators, for $g > 1$, so is centreless. Also, the kernel of this quotient map is similarly the free group on $\{b_1, \ldots, b_g\}$ which is also centreless.

Suppose $x \in \pi_1(S)$ such that x commutes with all elements of $\pi_1(S)$. We must have $x \mapsto 1$ in $\pi_1(S)/\langle b_1, \ldots, b_g\rangle$, as this group has trivial centre. Thus x must be in the centre of the kernel, which is trivial. Therefore x is trivial, so $\pi_1(S)$ has trivial centre. □

Before proving the Birman exact sequence, we will define the maps in it. The map $Forget : Mod(S,x) \to Mod(S)$ acts by inclusion. The equivalence class of an element of $Mod(S,x)$ up to isotopy not fixing x is an element of $Mod(S)$.

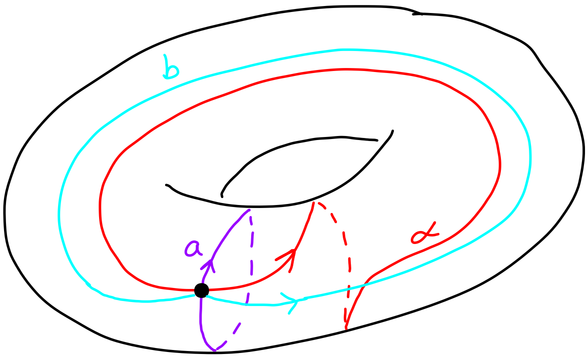
Definition 2.21. The map $Push : \pi_1(S,x) \to Mod(S,x)$ is defined as follows.

Let $\alpha : I \to S$ be a representative of an element of $\pi_1(S,x)$. We can view α as an isotopy of points $\{*\} \times I \to S$. We can consider α^{-1} as an isotopy of curves by

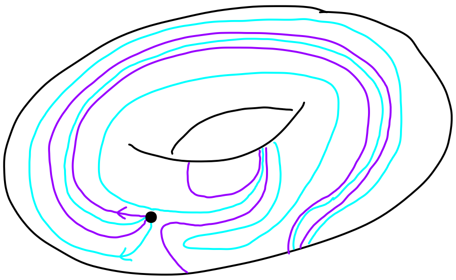
$$I \times I \xrightarrow{\text{projection}} \{*\} \times I \xrightarrow{\alpha^{-1}} S.$$

We can extend this to an isotopy of the whole surface S, $F : S \times I \to S$, such that $F|_{S\times 0}$ is the identity on S (Lemma 2.10). Let ϕ be the homeomorphism at the end of this ambient isotopy, that is, $F(-,1)$. Then $Push(\alpha)$ is the equivalence class of ϕ in $Mod(S,x)$.

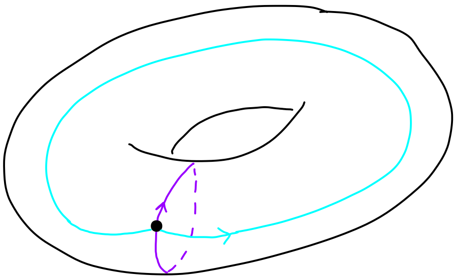
(We use α^{-1} here as by convention composition in π_1 is left to right, but in Mod is right to left. We must consider the inverse to get a homomorphism rather than an anti-homomorphism.)



(a) The curve α (red), and the fundamental group generators a (purple) and b (blue).



(b) The image of a (purple) and b (blue) under $Push([\alpha])$



(c) The images of a and b after homotopy, showing the action is trivial.

Figure 2.2: The action of $Push([\alpha])$ on the fundamental group of the torus.

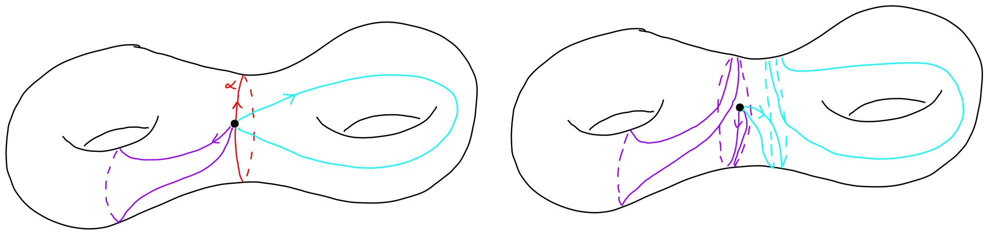
Note that it is unclear that homeomorphism is well-defined, as there are many extensions of α to an isotopy of the whole of S. However, if it is, $Push([\alpha])$ induces an automorphism on $\pi_1(S,x)$ taking $[\beta] \mapsto [\alpha^{-1}\beta\alpha]$.

We examine the action of $Push([\alpha])$ in a few cases. Figure 2.2 shows the action of $Push([\alpha])$ on the torus for a certain choice of α. We have picked generators a and b of the fundamental group, and drawn a curve $\alpha = ab$. Then $Push([\alpha])$ acts on the fundamental group by sending

$$a \mapsto \alpha^{-1} \cdot a \cdot \alpha = (ab)^{-1}a(ab) = a$$
$$b \mapsto \alpha^{-1} \cdot b \cdot \alpha = (ab)^{-1}b(ab) = b.$$

As the fundamental group is abelian, conjugation is trivial, and so $Push([\alpha])$ is in fact trivial for all choices of α.

However, on higher genus surfaces, $Push([\alpha])$ is in general not trivial. For example, see Figure 2.3. As the Euler characteristic of the genus two surface is negative, its fundamental group is centreless (Lemma 2.20). Thus for any non-trivial α, conjugation



(a) The curves α (red), a (purple) and b (blue).

(b) The image of a (purple) and b (blue) under $Push([\alpha])$

Figure 2.3: A non-trivial action of $Push([\alpha])$ on the genus two surface.

by α is non-trivial.

Theorem 2.22 (Birman exact sequence, Theorem 1 [Bir69])**.** *Let S be a surface with $\chi(S) < 0$. Fix some $x \in S$. Then the following sequence is exact:*

$$0 \to \pi_1(S, x) \xrightarrow{Push} Mod(S, x) \xrightarrow{Forget} Mod(S) \to 0.$$

Remark 2.23. We can directly see that $Forget$ is surjective. Since S is path-connected, any homeomorphism of $(S, \partial S)$ is isotopic to one fixing x.

If we assume that $Push$ is well-defined, we can also directly show that it is injective. As $\pi_1(S)$ is centreless, for $[\alpha]$ non-trivial, there is some $[\beta] \in \pi_1(S, x)$ such that $[\alpha^{-1}\beta\alpha] \neq [\beta]$.

For the proof of the Birman exact sequence, we will first construct the sequence and then show that the maps in it are indeed $Push$ and $Forget$.

First, we describe $\pi_1(\text{Homeo}^+(S))$. Let $\text{Homeo}_0(S, \partial S)$ be homeomorphisms of S fixing the boundary pointwise that are isotopic to the identity.

Theorem 2.24 (Theorem [Ham62], Theorem 3 [Ham65], Theorem 5.1 and 5.2 [Ham66])**.** *Let S be a compact surface. If S is not homeomorphic to S^2, $\mathbb{R}^2$, D^2, the torus or the annulus, then* $\text{Homeo}_0(S, \partial S)$ *is contractible.*

Corollary 2.25. *Let S be a compact surface, possibly with a finite number of punctures, such that $\chi(S) < 0$. Then* $\pi_1(\text{Homeo}^+(S, \partial S)) = 0$.

Proof. Note that as $\chi(S) < 0$, S is not S^2, $\mathbb{R}^2$, D^2, T^2, the annulus, $D^2 - \{*\}$ or $\mathbb{R}^2 - \{*\}$. Now, the connected component of $\text{Homeo}^+(S, \partial S)$ that contains the identity is precisely $\text{Homeo}_0(S, \partial S)$. By Theorem 2.24, any loop in $\pi_1(\text{Homeo}^+(S, \partial S))$, which is based at $\text{id} \in \text{Homeo}_0(S, \partial S)$, is trivial. □

Proof of Birman exact sequence. We will show that we have the following fibre bundle, where ev_x takes $\phi \in \mathrm{Homeo}^+(S, \partial S)$ to $\phi(x)$.

$$\begin{array}{ccc} \mathrm{Homeo}^+(S, x, \partial S) & \hookrightarrow & \mathrm{Homeo}^+(S, \partial S) \\ & & \downarrow{\scriptstyle ev_x} \\ & & S - \partial S \end{array}$$

The inclusion $\mathrm{Homeo}^+(S, x, \partial S) \to \mathrm{Homeo}^+(S, \partial S)$ is well-defined since any homeomorphism of S that fixes x is a homeomorphism of S.

To show this is a fibre bundle, we provide a local trivialisation. Pick an open neighbourhood U of x that is homeomorphic to the open complex unit disk $D_1(0) \subset \mathbb{C}$ by a homeomorphism $p : U \to D_1(0)$ such that $p(x) = 0$. Note p exists as the manifold is locally Euclidean. For each point $\xi \in D_1(0)$, let $\gamma_\xi : D_1(0) \to D_1(0)$ be the function

$$z \mapsto \frac{z - \xi}{1 - \bar{\xi} z}.$$

Note that γ_ξ fixes the unit circle in $\mathbb{C}$ as a set, so is a homeomorphism, and sends $\xi \mapsto 0$. For a point $u \in U$, let $\phi_u = p^{-1} \circ \gamma_{p(u)}^{-1} \circ p$, which is a homeomorphism of U and sends $x \mapsto u$. Then ϕ_u varies continuously as a function of u.

We now have a homeomorphism $U \times \mathrm{Homeo}^+(S, x) \to ev_x^{-1}(U)$ by the map

$$(u, \psi) \mapsto \phi_u \circ \psi$$

with inverse

$$\alpha \mapsto (\alpha(x), \phi_{\alpha(x)}^{-1} \circ \alpha).$$

To get a local trivialisation at an arbitrary point $y \in S$, pick $\alpha \in \mathrm{Homeo}^+(S)$ with $\alpha(x) = y$. Then $\alpha : U \to \alpha(U)$ induces a homeomorphism $\alpha(U) \times \mathrm{Homeo}^+(S, x) \to ev^{-1}(\alpha(U))$. This homeomorphism is given explicitly by

$$(z, \phi) \mapsto \alpha \circ \psi_{\alpha^{-1}(z)} \circ \phi$$

with inverse

$$\beta \mapsto (\beta(x), \psi^{-1}_{\alpha^{-1} \circ \beta(x)} \circ \alpha^{-1} \circ \beta).$$

Now, as we have a fibre bundle, we have an induced long exact sequence in the homotopy groups [Hat01, Theorem 4.41] that ends with

$$\pi_1(\mathrm{Homeo}^+(S, \partial S), \mathrm{id}) \to \pi_1(S - \partial S, x) \to \pi_0(\mathrm{Homeo}^+(S, x, \partial S), \mathrm{id})$$

$$\to \pi_0(\mathrm{Homeo}^+(S, \partial S), \mathrm{id}) \to \pi_0(S - \partial S, x).$$

As S is connected, $\pi_0(S - \partial S)$ is trivial. As $x \notin \partial S$, we can homotope any loop in S to be disjoint from the boundary. Thus $\pi_1(S - \partial S, x) = \pi_1(S, x)$. Recall that $\pi_0(\mathrm{Homeo}^+(S, \partial S))$ is the homeomorphisms of S up to isotopy, which is precisely

$Mod(S)$. Similarly, $\pi_0(\text{Homeo}^+(S, x, \partial S)) \cong Mod(S, x)$. Finally, by Lemma 2.25, $\pi_1(\text{Homeo}^+(S, \partial S))$ is trivial. Thus we can rewrite this sequence as

$$0 \to \pi_1(S, x) \to Mod(S, x) \to Mod(S) \to 0.$$

Now, it remains to show that the maps between these groups are $Push$ and $Forget$.

First, we show $f : \pi_1(S, x) \to Mod(S, x)$ is $Push$. Let α be a loop in S based at x, representing $[\alpha] \in \pi_1(S, x)$. As we have a fibre bundle, the induced map from projection $(ev_x)_* : \pi_1(\text{Homeo}^+(S, \partial S), \text{Homeo}^+(S, x, \partial S), \text{id}) \to \pi_1(S - \partial S, x)$ is an isomorphism (Theorem 4.41, [Hat01]).

Now $(ev_x)_*^{-1}([\alpha])$ is the homotopy class of a map

$$(D^1, S^0, s_0) \to (\text{Homeo}^+(S, \partial S), \text{Homeo}^+(S, x, \partial S), \text{id}).$$

From the construction of the long exact sequence, $f([\alpha])$ is the restriction of $(ev_x)_*^{-1}([\alpha])$ to the homotopy class of a map $(S^0, s_0) \to (\text{Homeo}^+(S, x, \partial S), \text{id})$. Let ϕ be a representative of $(ev_x)_*^{-1}([\alpha])$. Then $\phi(t)(x) = ev_x(\phi(t)) = \alpha(t)$, so ϕ is an isotopy of the surface extending the isotopy of points given by α as in the definition of $Push$. Let the point in S^0 that is not s_0 be s_1. The map $f([\alpha])$ is determined by where s_1 is sent, up to isotopy fixing x. Thus, $f([\alpha])$ is the isotopy class of $\phi(s_1)$ in $\text{Homeo}^+(S, x, \partial S)$, as in the definition of $Push([\alpha])$, so $Push$ is well-defined.

Second, the map $Mod(S, x) \to Mod(S)$ is induced by the inclusion

$$\text{Homeo}^+(S, x, \partial S) \hookrightarrow \text{Homeo}^+(S, \partial S).$$

This map sends an element of $Mod(S, x)$ to its isotopy class in $\text{Homeo}^+(S, \partial S)$ (rather than in $\text{Homeo}^+(S, x, \partial S)$), which is precisely the $Forget$ map. □

Chapter 3

The Hatcher-Thurston presentation of $Mod(S)$

In [HT80], the authors give a finite presentation of the mapping class group of a closed surface (that is, with $n = 0$) in terms of Dehn twists. It is a result from the 1930s that a finite number of Dehn twists generate the mapping class group (Theorem 2.8). Hatcher and Thurston showed that one could construct a Dehn twist presentation with a finite number of relations.

The authors do not explicitly demonstrate their method to construct a full presentation. We give a complete description of the construction outlined by Hatcher and Thurston, and then apply it to find an explicit presentation for genus one and two surfaces.

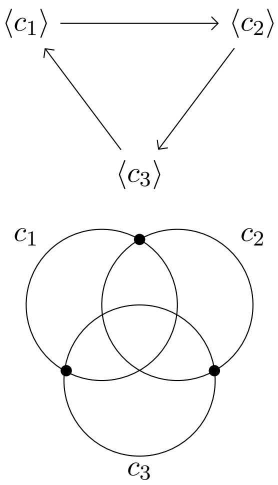
3.1 The cut system complex

To construct the presentation, Hatcher and Thurston consider the action of the mapping class group on a cut system complex.

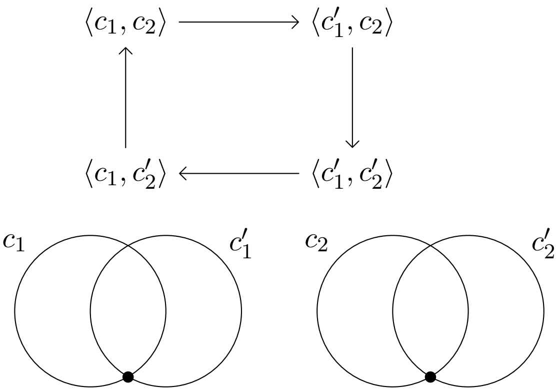
Definition 3.1. A *cut system* $\langle \alpha_1, \ldots, \alpha_g \rangle$ on a closed genus g surface S is an unordered collection of g disjoint smoothly embedded circles such that their complement is a sphere with $2g$ punctures. A cut system is defined up to isotopy of the surface.

Definition 3.2. The *cut system complex* is constructed as follows. Its vertices are the cut systems of S. Let $\langle \alpha_1, \ldots, \alpha_g \rangle$ be a cut system. Suppose that α_1' is a circle that intersects α_1 once and is disjoint from the rest of the circles, such that $\langle \alpha_1', \alpha_2, \ldots, \alpha_g \rangle$ is a cut system. Then there is an edge from $\langle \alpha_1, \alpha_2, \ldots, \alpha_g \rangle$ to $\langle \alpha_1', \alpha_2, \ldots, \alpha_g \rangle$. The 2-cells are the triangle, commutativity and pentagon relations shown in Figure 3.1.

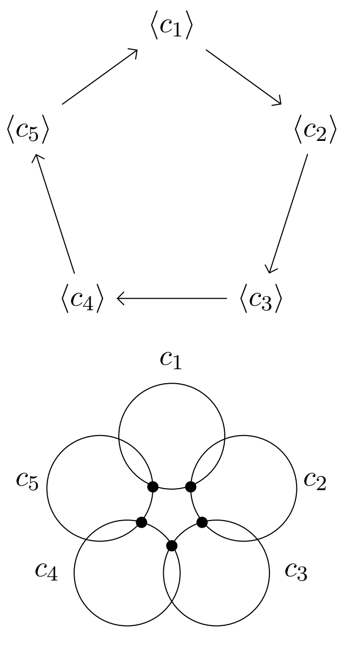
There is a natural action of $Mod(S)$ on this complex induced by the $Mod(S)$-action on simple closed curves of S.



(a) The triangle face.



(b) The commutativity face.



(c) The pentagon face

Figure 3.1: The faces of the cut system complex. Black dots correspond to intersection points between circles. Each face corresponds to a commutative diagram in moves on cut systems, exchanging a circle with another circle that intersects it once.

Theorem 3.3 (Theorem 1.1 [HT80]). *The cut system complex is connected and simply connected.*

Hatcher and Thurston show this result using Cerf theory, which we will not discuss in this thesis.

The *standard cut system*, C, is the cut system consisting of a meridian of each handle of the surface. Let H be the subgroup of the mapping class group that fixes C. Equivalently, this is the subgroup that fixes the corresponding vertex of the cut system complex.

We will construct a mapping class that flips an edge incident to C. Let a be the meridian of the first handle, and b be a fixed longitude of the first handle. Let C' be the cut system consisting of a meridian of each handle except for the first handle, and a longitude of the first handle. That is, $C' = C - \{a\} \cup \{b\}$. Note that there is an edge between C and C'.

Definition 3.4. The mapping class σ is $T_a T_b^{-1} T_a$.

This is the identity outside a neighbourhood of the first handle, so fixes the meridians of all other handles. On the first handle, one can verify that σ takes the curve a to b and takes b to a. Thus, $\sigma(C) = C'$ and $\sigma(C') = C$.

Given a finite presentation of H and such an element σ, which allows us to flip an edge in the complex, we can give a finite presentation of the mapping class group (Theorem 2.2, [HT80]). We will discuss this further in Section 3.3.

The authors give the construction of the full presentation quite explicitly. However, for developing the presentation of H, they give only the essential steps, and leave the construction of the presentation itself to the reader. In [Waj99], the author gives an explicit presentation of H following [HT80], but this employs only elementary techniques and does not directly use the fibrations used to construct the exact sequence in [HT80]. We will show how to apply an exact sequence to construct an explicit presentation for H genus one and two surfaces by directly following Hatcher and Thurston's argument.

3.2 Building a presentation of H

Using the $Push$ map (Definition 2.21), we describe how to construct a presentation for H in the general case, following [HT80], and then apply this to give explicit presentations for genus one and two.

3.2.1 The key steps in the H presentation

Before considering the genus one and two cases in detail, we first outline the key steps in the construction. Let S be a (closed) genus g surface, with $\langle \alpha_i \rangle$ a fixed cut system.

There is a fibration [HT80, Proof of Prop. 2.1]

$$\begin{array}{ccc} \mathrm{Diff}^+(S\,\mathrm{rel}\{\alpha_i\}) & \hookrightarrow & \mathrm{Diff}^+(S) \\ & & \downarrow p \;\text{restriction} \\ & & \mathrm{Diff}(\{\alpha_i\}) \end{array}$$

which induces a long exact sequence in homotopy groups [Hat01, Theorem 4.41] ending with

$$\pi_1(\mathrm{Diff}(\{\alpha_i\})) \to \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\{\alpha_i\})) \to \pi_0(\mathrm{Diff}^+(S,\{\alpha_i\})) \to \pi_0(\mathrm{Diff}(\{\alpha_i\})) \to 0. \tag{3.1}$$

Lemma 2.4 [HT80] shows that $\pi_0(\mathrm{Diff}^+(S,\alpha))$ is isomorphic to H. Proposition 2.1 [HT80] states this exact sequence is equivalent to

$$\mathbb{Z} \to \mathbb{Z}^g \oplus P_{2g-1} \to H \to \pm\Sigma_g \to 0. \tag{3.2}$$

We describe how to extract a presentation from an exact sequence.

Lemma 3.5 (Lemma 28 [Waj99]). *Let $0 \to A \to B \to C \to 0$ be an exact sequence, with A and C finitely presented groups. Let $\langle a_1,\dots,a_n \,|\, r_1,\dots,r_m\rangle$ be a presentation of A, and $\langle c_1,\dots,c_h \,|\, s_1,\dots,s_g\rangle$ a presentation of C. Then we get a presentation of B as follows.*

Let α_i be the image of a_i in B. Let γ_j be a fixed choice of element of B that is sent to c_j in C. For each relation r_k of A, which is a word in the a_i, let t_k be the word in the α_i that is the image of the word r_k under the map $a_i \mapsto \alpha_i$. For each relation s_ℓ of C, let u_ℓ be the image of s_ℓ under the map $c_j \mapsto \gamma_j$. As u_ℓ is sent to 0 in C, it is in the image of A in B. Thus we have $u_\ell = v_\ell$ where v_ℓ is some word in the α_i. For each α_i and γ_j, $\gamma_j\alpha_i\gamma_j^{-1}$ is also sent to 0 in C, so is equal to w_{ij} where w_{ij} is some word in the α_i.

Then we have a presentation of B as follows:

$$B = \langle \alpha_i, \gamma_j \,|\, t_k, u_\ell v_\ell^{-1}, \gamma_j\alpha_i\gamma_j^{-1}w_{ij}^{-1}\rangle$$

where the variables range as follows: $1 \le i \le n$, $1 \le j \le m$, $1 \le k \le h$ and $1 \le \ell \le g$.

3.2.2 Genus one case

We construct a presentation of H for genus one, following [HT80]. Let S be a torus.

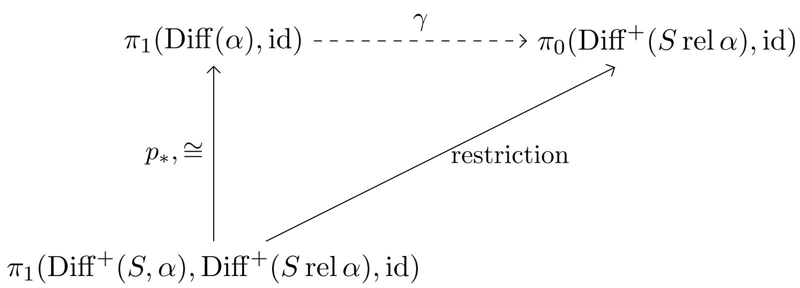
We have a long exact sequence, from Equation 3.1, ending with

$$\pi_1(\mathrm{Diff}(\alpha)) \xrightarrow{\gamma} \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha)) \to \pi_0(\mathrm{Diff}^+(S,\alpha)) \to \pi_0(\mathrm{Diff}(\alpha)) \to 0.$$

To understand these groups, first, we consider $\gamma : \pi_1(\mathrm{Diff}(\alpha)) \to \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha))$. In the construction of the long exact sequence, this map is defined as follows. The induced map from the relative homotopy group of the fibration to the homotopy group of the base space,

$$p_* : \pi_1(\mathrm{Diff}^+(S,\alpha), \mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha), \mathrm{id}) \to \pi_1(\mathrm{Diff}(\alpha), \mathrm{id}),$$

which comes from the projection in the fibration, is an isomorphism [Hat01, Theorem 4.41]. Now γ is defined as the induced map in the following diagram.



One may easily verify that $\pi_1(\mathrm{Diff}(\alpha), \mathrm{id})$ is isomorphic to $\mathbb{Z}$, and generated by a rotation by 2π of $\alpha \cong S^1$. Then $p_*^{-1}(1)$ is the map

$$(D^1, S^0, s_0) \to (\mathrm{Diff}^+(S,\alpha), \mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha), \mathrm{id})$$

which has a representative sending s_0 to id and s_1 to $Push(\alpha)$ (as in Definition 2.21). Then the restriction map sends this map to the image of s_1, which is $Push(\alpha)$. Thus, γ acts by sending n to the map induced by pushing a point on α around the curve n times.

Note that in particular, this map is isotopic to the identity if we do not fix α pointwise. Thus it is in the kernel of the induced map from inclusion $\pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha)) \to \pi_0(\mathrm{Diff}^+(S,\alpha))$, as we expect from the exact sequence.

By Lemma 2.4 [HT80], $\pi_0(\mathrm{Diff}^+(S,\alpha)) \cong H$. The group $\pi_0(\mathrm{Diff}(\alpha))$ is the isotopy classes of diffeomorphisms of $\alpha \cong S^1$. Any map $S^1 \to S^1$ is homotopic to the n^{th} winding map for some n, and this map is injective only for $n = \pm 1$. Thus $\pi_0(\mathrm{Diff}(\alpha)) \cong \mathbb{Z}/2 \cong \pm\Sigma_1$.

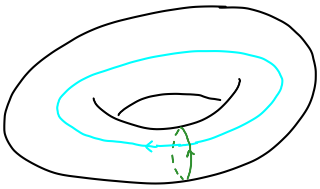
We can now write our exact sequence as

$$\mathbb{Z} \xrightarrow{\gamma} \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha)) \to H \to \mathbb{Z}/2 \to 0.$$

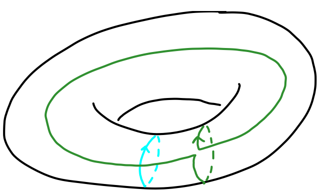
It remains to interpret $\pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha))$.

Suppose we cut S along α. The resulting surface is an annulus, which we can write as $D^2 - \mathring{D}_1$ for D_1 a disk in the interior of the D^2. This induces a diffeomorphism $\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha) \to \mathrm{Diff}(D^2\,\mathrm{rel}\{D_1, \partial D^2\})$. Thus

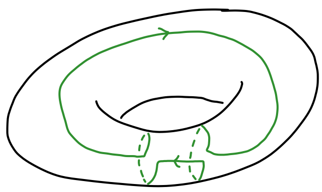
$$\pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha)) \cong \pi_0(\mathrm{Diff}(D^2\,\mathrm{rel}\{D_1, \partial D^2\})).$$



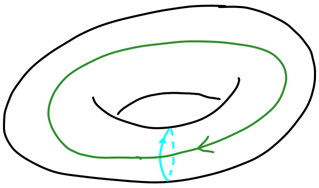
(a)



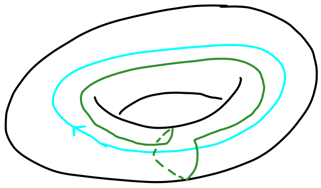
(b) T_β



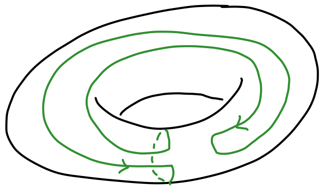
(c) $T_\alpha T_\beta$



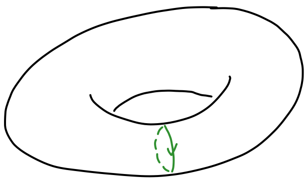
(d) is homotopic to



(e) $T_\alpha T_\beta T_\alpha T_\beta$



(f) $T_\beta T_\alpha T_\beta T_\alpha T_\beta$



(g) is homotopic to

Figure 3.2: Verifying that $r(\alpha) = \alpha^{-1}$. The green curve is the image of the given series of Dehn twists; the blue is the curve we will twist about next.

The mapping class group $Mod(D^2 - \mathring{D}_1)$ is isomorphic to $\mathbb{Z}$ and generated by a Dehn twist about the curve parallel to one of the boundary components. Thus, $\text{Diff}^+(S \operatorname{rel} \alpha) \cong \mathbb{Z}$. When we re-glue, the generator in $\pi_0(\text{Diff}(D^2 \operatorname{rel}\{D_1, \partial D^2\}))$ is a Dehn twist about α.

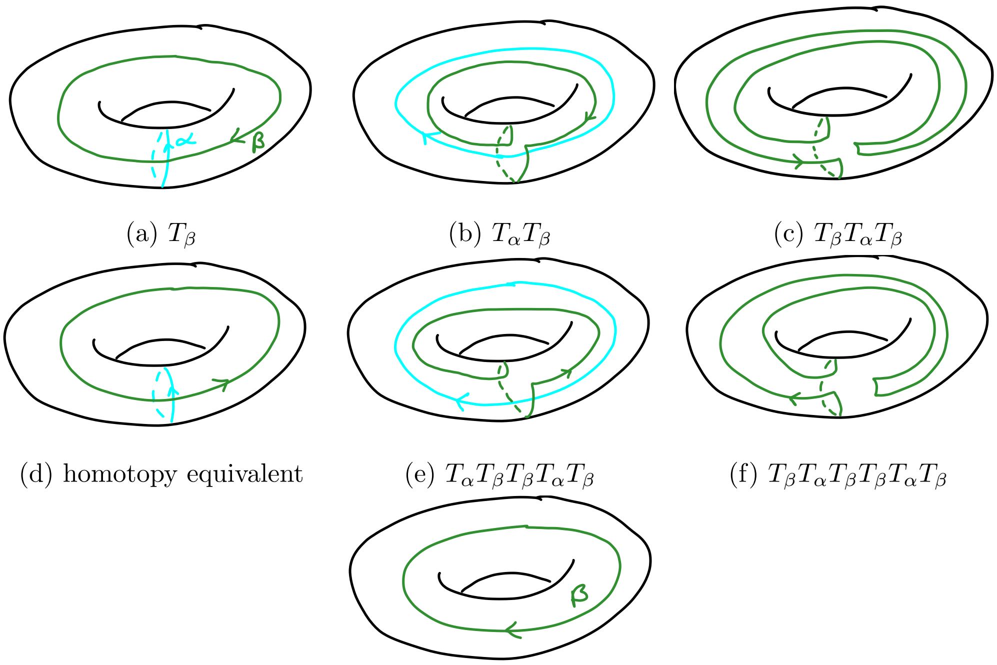
Consider the map $\gamma : \pi_1(\text{Diff}(\alpha)) \to \pi_0(\text{Diff}^+(S \operatorname{rel} \alpha))$. We have the generator of $\pi_1(\text{Diff}(\alpha))$ going to $Push(\alpha)$. But on the torus, $Push(\alpha)$ is isotopic to the identity, even if α is fixed. (We can also see this from the exact sequence, as in terms of Dehn twists about α, the generator of $\pi_1(\text{Diff}(\alpha))$ is sent to $T_\alpha T_\alpha^{-1}$ which is trivial.) Thus $\gamma : \pi_1(\text{Diff}(\alpha)) \to \pi_0(\text{Diff}^+(S \operatorname{rel} \alpha))$ is in fact the zero map.

Thus, we have a short exact sequence

$$0 \to \mathbb{Z} \to H \to \mathbb{Z}/2 \to 0.$$

From this, we can build a presentation for H as described in Lemma 3.5. The generators are r, which is some element in the preimage of $1 \in \mathbb{Z}/2$, and t, which is the image of $1 \in \mathbb{Z}$ in H. We have no relations from $\mathbb{Z}$. We have $r^2 \mapsto 0 \in \mathbb{Z}/2$, so r^2 is some word in the generators of $\mathbb{Z}$. Similarly, $rtr^{-1} \mapsto 0 \in \mathbb{Z}/2$, so rtr^{-1} is also some word in the generators of $\mathbb{Z}$.

Let T_γ be the Dehn twist about γ (Definition 2.5). Let t be a Dehn twist about



(a) T_β (b) $T_\alpha T_\beta$ (c) $T_\beta T_\alpha T_\beta$

(d) homotopy equivalent (e) $T_\alpha T_\beta T_\beta T_\alpha T_\beta$ (f) $T_\beta T_\alpha T_\beta T_\beta T_\alpha T_\beta$

(g) homotopy equivalent

Figure 3.3: Verifying that r sends $\beta \mapsto \beta^{-1}$. The green curve is the image of the given series of Dehn twists; the blue is the curve we will twist about next.

α. Let r be $(T_\alpha T_\beta T_\alpha)^2$, for β a curve that intersects α once. By the braid relation (Prop. 2.7), since β and α intersect once we have $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$. So

$$r = T_\alpha T_\beta T_\alpha T_\beta T_\alpha T_\beta = T_\beta T_\alpha T_\beta T_\alpha T_\beta T_\alpha.$$

Since $T_\alpha(\alpha) = \alpha$, as illustrated in Figure 3.2, we can verify this map takes $\alpha \mapsto \alpha^{-1}$. Thus r does indeed map to $1 \in \mathbb{Z}/2$.

Since r^2 fixes α, and all other curves on the torus intersect α at least once, r^2 is some number of Dehn twists about α. To determine r^2 in terms of Dehn twists it suffices to calculate its action on β. As illustrated in Figure 3.3, r sends $\beta \mapsto \beta^{-1}$. Thus r^2 is the identity.

Now, consider rtr^{-1}. This fixes α so is also some word in the Dehn twists about α. As $t = T_\alpha$, using the braid relation we have

$$\begin{aligned} rtr^{-1} &= (T_\alpha T_\beta T_\alpha)^2 T_\alpha (T_\alpha T_\beta T_\alpha)^{-2} \\ &= (T_\alpha T_\beta T_\alpha T_\beta T_\alpha T_\beta) T_\beta^{-1} T_\alpha^{-1} T_\beta^{-1} T_\alpha^{-1} T_\beta^{-1} \\ &= T_\alpha \end{aligned}$$

so $rtr^{-1} = t$, which is to say, t and r commute. Thus,

$$H \cong \langle r, t \mid r^2, rtr^{-1}t^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

To verify this, we can describe H explicitly without using Equation 3.2, as the mapping class group of the torus is easy to describe.

It is a classical fact that $Mod(S) \cong SL(2,\mathbb{Z})$. Fix a meridian and longitude. Then this group is generated by the Dehn twist about the meridian, which is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and the Dehn twist about the longitude, which is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ [FM12, Theorem 2.5]. The fundamental group of the torus is generated by the meridian and longitude. If we identify them with $(1,0)$ and $(0,1)$ respectively, the mapping class group acts on them as $SL(2,\mathbb{Z})$ acts on these vectors.

Let the fixed cut system $\langle \alpha \rangle$ be the meridian. Note that an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mod(S)$ takes $(1,0) \mapsto (a,c)$. Then the subgroup H of $Mod(S)$ that fixes $(1,0)$ is the set that sends $(1,0) \mapsto (\pm 1, 0)$ (as $(-1,0)$ is $(1,0)$ with the opposite orientation, and the cut system is defined up to orientation and ordering). This is the subgroup

$$\begin{aligned} H &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{Z}) \,|\, ad - bc = 1, a = \pm 1, c = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{Z}) \,|\, a = \pm 1, d = a, c = 0 \right\}. \end{aligned}$$

All elements of H look like

$$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$$

for $b \in \mathbb{Z}$, and we can verify that this set is closed under multiplication and inverses, since for example

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & b' \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & b' - b \\ 0 & -1 \end{pmatrix}.$$

Relating this back to the presentation of H from Equation 3.2, the correspondence is

$$r = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which reverses the orientation of both α and β, and

$$t = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

which is a Dehn twist about α.

3.2.3 Genus two case

Next, we consider the genus two case. With genus one, the problem is substantially simpler: the pure braid group does not make an appearance, and the zero map γ :

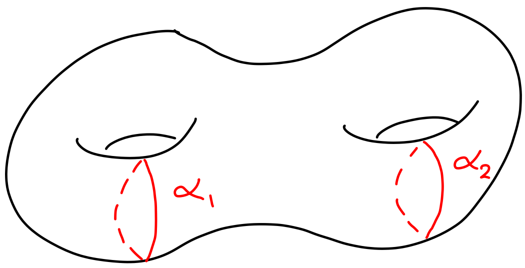


Figure 3.4: Illustrating α_1 and α_2 on a genus two surface S.

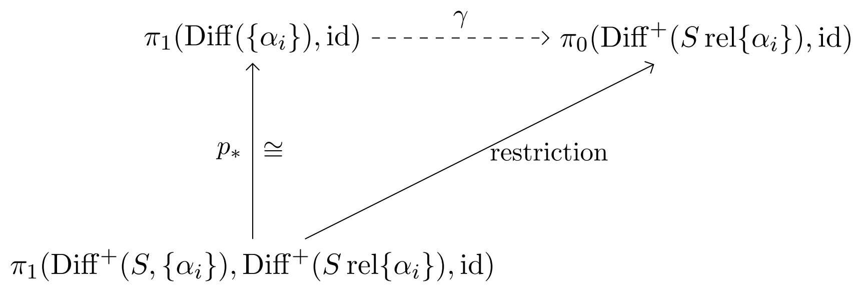
$\pi_1(\mathrm{Diff}(\alpha)) \to \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\,\alpha))$ makes the exact sequence much simpler. Genus two is more indicative of the complexity of the general case.

Let S be the genus two surface, and α_1 and α_2 be the curves marked in Figure 3.4. As with genus one, from Equation 3.1 we have a fibration inducing a long exact sequence ending with

$$\pi_1(\mathrm{Diff}(\{\alpha_i\})) \xrightarrow{\gamma} \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\{\alpha_i\})) \xrightarrow{\phi} \pi_0(\mathrm{Diff}^+(S,\{\alpha_i\})) \xrightarrow{\psi} \pi_0(\mathrm{Diff}(\{\alpha_i\})) \to 0. \quad (3.3)$$

Recall that by Lemma 2.4 [HT80], $\pi_0(\mathrm{Diff}^+(S,\{\alpha_i\})) \cong H$.

Consider the map $\gamma : \pi_1(\mathrm{Diff}(\{\alpha_i\})) \to \pi_0(\mathrm{Diff}^+(S\,\mathrm{rel}\{\alpha_i\}))$ which is defined as in the genus one case by the following diagram.



One may easily verify that $\pi_1(\mathrm{Diff}(\{\alpha_i\}), \mathrm{id}) \cong \mathbb{Z}^2$ with generators the rotations by 2π about α_1 and α_2. Then $p_*^{-1}(n,m)$ has a representative

$$\phi_{n,m} : (D^1, S^0, s_0) \to (\mathrm{Diff}^+(S,\{\alpha_i\}, \mathrm{Diff}^+(S\,\mathrm{rel}\{\alpha_i\}), \mathrm{id})$$

sending s_0 to id and s_1 to $Push(\alpha_1)^m \circ Push(\alpha_2)^n$. Thus γ sends $(1,0)$ to $Push(\alpha_1)$ and $(0,1)$ to $Push(\alpha_2)$. Note that these $Push$ maps are trivial if we fix $\{\alpha_i\}$ as a set, rather than pointwise. Thus, as we would expect, $\mathrm{im}\,\gamma$ is sent to 0 in H by ϕ. Also note that γ is injective, which allows us to rewrite Equation 3.3 as

$$0 \to \pi_1(\mathrm{Diff}(\{\alpha_i\})) \xrightarrow{\gamma} \pi_0(\,\mathrm{Diff}^+(S\,\mathrm{rel}\{\alpha_i\})) \xrightarrow{\phi} H \xrightarrow{\psi} \pi_0(\mathrm{Diff}(\{\alpha_i\})) \to 0. \quad (3.4)$$

Now, consider $\pi_0(\text{Diff}^+(S \operatorname{rel}\{\alpha_i\}))$. Suppose we cut S along $\{\alpha_i\}$. We get a sphere with four boundary components, which we can write as $D^2 - \{\mathring{D}_1, \mathring{D}_2, \mathring{D}_3\}$ where the D_i are disks embedded in D^2. Note that any diffeomorphism of S that fixes the α_i pointwise can be interpreted as a diffeomorphism of $D^2 - \{\mathring{D}_i\}$, so

$$\pi_0(\text{Diff}^+(S \operatorname{rel}\{\alpha_i\})) \cong \pi_0(\text{Diff}(D^2 \operatorname{rel}\{D_1, D_2, D_3, \partial D^2\})).$$

To find $\pi_0(\text{Diff}(D^2 \operatorname{rel}\{D_1, D_2, D_3, \partial D^2\}))$, note that there is a fibration

$$\begin{array}{ccc} \text{Diff}(D^2 \operatorname{rel}\{D_1, D_2, D_3, \partial D^2\}) & \hookrightarrow & \text{Diff}(D^2 \operatorname{rel} \partial D^2) \\ & & \big\downarrow g \\ & & B \end{array}$$

where B is the space of orientation-preserving embeddings of three disjoint disks in $\mathring{D}^2$ [HT80, Proof of Lemma 2.5]. The map $g : \text{Diff}(D^2 \operatorname{rel} \partial D^2) \to B$ is given by sending a diffeomorphism ϕ to the embedding sending the first disk to $\phi(D_1)$, the second to $\phi(D_2)$ and the third to $\phi(D_3)$. From the fibration, we get a long exact sequence in homotopy groups:

$$\begin{aligned} \pi_1(\text{Diff}(D^2 \operatorname{rel} \partial D^2)) \to \pi_1(B) \xrightarrow{\xi} \pi_0(\text{Diff}(D^2 \operatorname{rel}\{D_1, D_2, D_3, \partial D^2\})) \\ \to \pi_0(\text{Diff}(D^2 \operatorname{rel} \partial D^2)). \end{aligned} \quad (3.5)$$

We use two statements from the proof of Lemma 2.5 [HT80]. First, $\text{Diff}(D^2 \operatorname{rel} \partial D^2)$ is contractible. Thus the isotopy classes of diffeomorphisms and the fundamental group of this space are trivial, so, in Equation 3.5, ξ is an isomorphism. Thus

$$\pi_1(B) \cong \pi_0(\text{Diff}(D^2 \operatorname{rel}\{D_1, D_2 D_3, \partial D^2\})).$$

Second, B is weakly homotopy equivalent to $[SL(2,\mathbb{R})]^3 \times P(\mathring{D}^2; 3)$ by the following map. Let $f \in B$ be an embedding. The weak homotopy equivalence sends f to $(d_1, d_2, d_3, \{c_1, c_2, c_i\})$, defined as follows. A point c_i is the image of the centre of the i^{th} disk under f. A matrix d_i is the gradient of f at the centre of the i^{th} disk.

From these statements, we have

$$\begin{aligned} \pi_0(\text{Diff}(D^2 \operatorname{rel}\{D_1, D_2, D_3, \partial D^2\})) &\cong \pi_1(B) \\ &\cong \pi_1([SL(2,\mathbb{R})]^3 \times P(\mathring{D}^2; 3)) \\ &\cong \mathbb{Z}^3 \times P_3. \end{aligned}$$

With these results, we can rewrite Equation 3.4 as

$$0 \to \mathbb{Z}^2 \xrightarrow{\gamma} \mathbb{Z}^3 \times P_3 \xrightarrow{\phi} H \xrightarrow{\psi} \pi_0(\text{Diff}(\{\alpha_i\})) \to 0. \quad (3.6)$$

Consider the image of the generators of $\pi_1(B)$ under ξ. The fundamental group of $SL(2,\mathbb{R})$ is isomorphic to $\mathbb{Z}$, and is generated by the loop $g : S^1 \to SL(2,\mathbb{R})$ sending θ to the matrix rotating the plane by θ. In B, $\xi(1,0,0,e)$ then corresponds to a full rotation of the embedding of the first disk. Thus $\xi(1,0,0,e)$ is a Dehn twist about the boundary of first disk. Similarly, $\xi(0,1,0,e)$ and $\xi(0,0,1,e)$ are Dehn twists about the boundaries of the second and third disks respectively. When we glue up $D^2 - \{\mathring{D}_1, \mathring{D}_2, \mathring{D}_3\}$ to make the genus two surface, we get $\xi(1,0,0,e)$ mapping to T_{α_1}, $\xi(0,1,0,e)$ mapping to $T_{\alpha_1}^{-1}$ and $\xi(0,0,1,e)$ mapping to T_{α_2}. The Dehn twist about the boundary of the disk goes to $T_{\alpha_2}^{-1}$. See Figure 3.5 for some examples of how the disk glues up.

From the presentation of P_3 in Proposition 2.18, P_3 is generated by $A_{12} = \sigma_1^2$, $A_{23} = \sigma_2^2$, and $A_{13} = \sigma_2\sigma_1^2\sigma_2^{-1}$. The two relations are

$$A_{13}A_{23}A_{12} = A_{23}A_{12}A_{13} \text{ and}$$
$$A_{13}A_{23}A_{12} = A_{12}A_{13}A_{23}.$$

Consider the image of A_{12} under ξ. In configuration space, this loop is a rotation of two points around each other. In $\pi_1(B)$, this is a rotation of the first and second disks around each other, as illustrated. This is a Dehn twist about the circle separating the first and second disks from the third and the boundary. Now, $\xi(0,0,0,A_{12})$ is a Dehn twist about the loop illustrated in red on the right in Figure 3.5a. Similarly, $\xi(0,0,0,A_{23})$ and $\xi(0,0,0,A_{13})$ are Dehn twists about the red loops in Figure 3.5b and c.

Now, recall $\gamma : \pi_1(\text{Diff}(\{\alpha_i\})) \to \pi_0(\text{Diff}^+(S\,\text{rel}\{\alpha_i\}))$ sends $(1,0)$ to $Push(\alpha_1)$ and $(0,1)$ to $Push(\alpha_2)$. As α_1 is a simple closed curve, $Push(\alpha_1)$ is $t_+t_-^{-1}$, where t_+ is the Dehn twist about α_1 pushed off to the right, and t_- is the Dehn twist about α_1 pushed off to the left. When we cut along α_1 and α_2, t_+ is the Dehn twist about D_1 and t_- is the Dehn twist about D_2. Thus $\gamma(1,0)$ is $T_{D_1}T_{D_2}^{-1}$ on D^2 with an embedding of three disjoint disks. Similarly, $\gamma(0,1)$ is $T_{\partial D^2}T_{D_3}^{-1}$. Thus in $\mathbb{Z}^3 \times P_3$, $\gamma(1,0) = (1,-1,0,e)$, and $\gamma(0,1) = (0,0,1,A_{23}^{-1}A_{12})$ (since $A_{23}^{-1}A_{12}$ is the Dehn twist about ∂D^2).

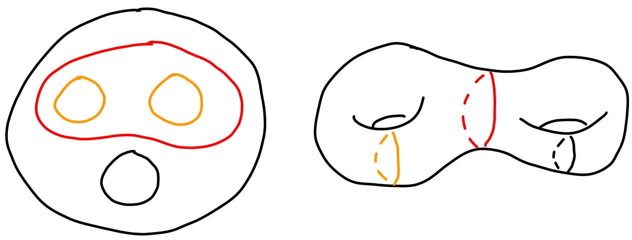
We can take the quotient of the groups in Equation 3.6 by $(1,0) \in \pi_1(\text{Diff}(\{\alpha_i\}))$ and its image. This is only $(1,-1,0,e) \in \pi_0(\text{Diff}^+(S\,\text{rel}\{\alpha_i\}))$, as by exactness $(1,-1,0,e) \mapsto 0 \in H$. Then $\mathbb{Z}^2/\langle(1,0)\rangle \cong \mathbb{Z}$ and $\mathbb{Z}^3 \times P_3/\langle(1,-1,0,e) \cong \mathbb{Z}^2 \times P_3$. Thus from Equation 3.6, we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}^2 \times P_3 \xrightarrow{\phi} H \xrightarrow{\psi} \pi_0(\text{Diff}(\{\alpha_i\})) \to 0 \tag{3.7}$$

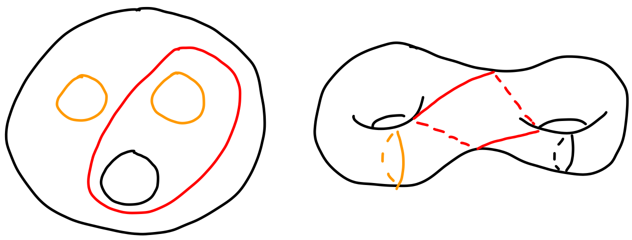
with the map $\gamma : \mathbb{Z} \to \mathbb{Z}^2 \times P_3$ taking $1 \mapsto (0,1,A_{23}^{-1}A_{12})$.

As shown in Lemma 2.13, $\pi_0(\text{Diff}(\{\alpha_i\})) \cong \pm\Sigma_2$, the group of symmetries of a square. A presentation of this group is

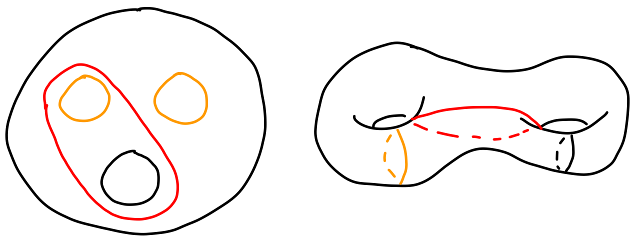
$$\langle r, f \mid f^2, frfr \rangle.$$



(a) The action of $\xi(0,0,0,A_{12})$.



(b) The action of $\xi(0,0,0,A_{23})$.



(c) The action of $\xi(0,0,0,A_{13})$.

Figure 3.5: The actions of the generators of the permutation group under ξ. On the left is $D^2 - \{D_1, D_2, D_3\}$. The element $\xi(0,0,0,A_{ij})$ is a Dehn twist about the red curve. On the right, we show how this glues to make the genus two surface.

On the square, f is a flip and r a rotation by $\pi/2$. As elements of $\pm Sym_2$, these act as f sending $1 \mapsto -1$ and $2 \mapsto 2$, and r sending $1 \mapsto 2$ and $2 \mapsto -1$. The map $\psi : H \to \pi_0(\mathrm{Diff}(\{\alpha_i\}))$ is induced by restriction of S to $\{\alpha_i\}$.

From these results, we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}^2 \times P_3 \xrightarrow{\phi} H \xrightarrow{\psi} \pm Sym_2 \to 0 \tag{3.8}$$

from Equation 3.7, as we expect from Equation 3.2.

Now, we build a presentation of H using Lemma 3.5. Note that for a sequence $K \to A \to B \to C \to 0$, the generators of K give us additional words in the generators of A that are trivial in B. It suffices to add these relations to the relation set from Lemma 3.5.

We have generators

$$g_{T_{D_1}}, g_{T_{D_3}}, g_{A_{12}}, g_{A_{13}}, g_{A_{23}}, g_f \text{ and } g_r$$

where the first five are the images of the generators of $\mathbb{Z}^3 \times P_3$ under ϕ, and the last two are some choice of elements of H that get sent to f and r respectively by ψ.

From the relations in $\mathbb{Z}^2 \times P_3$, we have relations

$$\begin{aligned}
g_{A_{13}} g_{A_{23}} g_{A_{12}} g_{A_{13}}^{-1} g_{A_{12}}^{-1} g_{A_{23}}^{-1} \\
g_{A_{13}} g_{A_{23}} g_{A_{12}} g_{A_{23}}^{-1} g_{A_{13}}^{-1} g_{A_{12}}^{-1} \\
[g_{T_{D_1}}, g_{A_{12}}] \\
[g_{T_{D_1}}, g_{A_{23}}] \\
[g_{T_{D_1}}, g_{A_{13}}] \\
[g_{T_{D_3}}, g_{A_{12}}] \\
[g_{T_{D_3}}, g_{A_{23}}] \\
[g_{T_{D_3}}, g_{A_{13}}] \\
[g_{T_{D_1}}, g_{T_{D_3}}].
\end{aligned}$$

We know the map $\gamma : \mathbb{Z} \to \mathbb{Z}^2 \times P_3$ takes $1 \mapsto (0, 1, A_{23}^{-1} A_{12})$. Thus the image of this map is generated by $(0, 1, A_{23}^{-1} A_{12})$, so by exactness this gives us the relation

$$g_{T_{D_3}}^{-1} g_{A_{23}}^{-1} g_{A_{12}} = \text{id}$$

in H, that is, $g_{T_{D_3}} = g_{A_{23}}^{-1} g_{A_{12}}$. Thus we can discard $g_{T_{D_3}}$ as a generator. We are then reduced to the relations

$$\begin{aligned}
g_{A_{13}} g_{A_{23}} g_{A_{12}} g_{A_{13}}^{-1} g_{A_{12}}^{-1} g_{A_{23}}^{-1} \\
g_{A_{13}} g_{A_{23}} g_{A_{12}} g_{A_{23}}^{-1} g_{A_{13}}^{-1} g_{A_{12}}^{-1} \\
[g_{T_{D_1}}, g_{A_{12}}] \\
[g_{T_{D_1}}, g_{A_{23}}] \\
[g_{T_{D_1}}, g_{A_{13}}] \\
[g_{A_{12}}^{-1} g_{A_{23}}, g_{A_{12}}] \\
[g_{A_{12}}^{-1} g_{A_{23}}, g_{A_{23}}] \\
[g_{A_{12}}^{-1} g_{A_{23}}, g_{A_{13}}].
\end{aligned} \tag{3.9}$$

Let the curves a, b, c and d be as in Figure 3.6. Note that $g_{T_{D_1}}$ is T_a, and $g_{T_{D_3}}$ is T_b. We choose g_f and g_r to be

$$\begin{aligned}
g_r &= T_d T_b^{-1} T_a T_d \\
g_f &= T_c T_a^2 T_c.
\end{aligned}$$

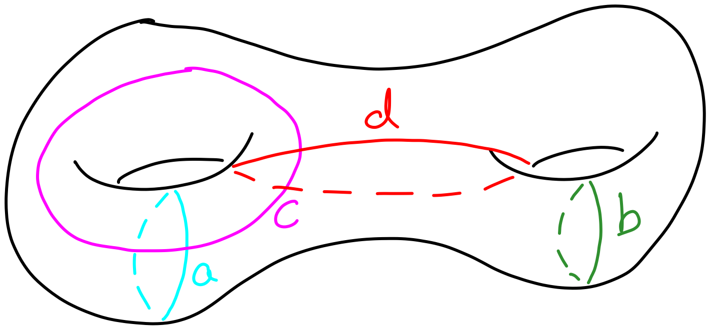


Figure 3.6: The curves a (blue), b (green), c (pink) and d (green) on the genus two surface.

One can check that these act on α_1 and α_2 as we expect, with g_r sending α_1 to α_2 and α_2 to α_1^{-1}, and g_f reversing the orientation of α_1 and fixing α_2.

From Lemma 3.5, each of

$$\begin{aligned}&g_f^2, g_f g_r g_f g_r, g_f g_{T_{D_1}} g_f^{-1}, g_f g_{A_{12}} g_f^{-1}, g_f g_{A_{13}} g_f^{-1}, g_f g_{A_{23}} g_f^{-1},\\ &g_r g_{T_{D_1}} g_r^{-1}, g_r g_{A_{12}} g_r^{-1}, g_r g_{A_{13}} g_r^{-1} \text{ and } g_r g_{A_{23}} g_r^{-1}\end{aligned} \tag{3.10}$$

is equal to some words in the generators corresponding to generators of $\mathbb{Z}^2 \times P_3$, giving us another ten relations. We will not compute these relations, as it is a tedious and uninsightful computation.

Wajnryb's calculation [Waj99, Prop. 27] also gives an explicit set of relations for H, though with a slightly different set of generators. To give a sense of what our final ten relations would be, we express those of his relations that correspond to these ten words in our generators. They are the following:

$$\begin{aligned}
g_r^2 &= g_{A_{13}} g_{A_{23}} g_{A_{12}} (g_{A_{12}}^{-1} g_{A_{23}})^{-2} g_{T_{D_1}}^{-2}\\
g_f^2 &= g_{A_{23}} g_{T_{D_1}}^{-4}\\
g_f g_r g_f g_r &= g_{A_{23}}^2 g_{T_{D_1}}^{-4} (g_{A_{12}}^{-1} g_{A_{23}})^4 (g_{A_{13}} g_{A_{23}} g_{A_{12}})^{-2}\\
g_f g_{T_{D_1}} g_f^{-1} &= g_{T_{D_1}}\\
g_f g_{A_{12}} g_f^{-1} &= g_{A_{13}}\\
g_f g_{A_{13}} g_f^{-1} &= g_{A_{23}}^2 g_{A_{12}} g_{A_{23}}^{-2}\\
g_f g_{A_{23}} g_f^{-1} &= g_{A_{23}} g_{T_{D_1}}^{-3}\\
g_r g_{T_{D_1}} g_r^{-1} &= g_{A_{12}}^{-1} g_{A_{23}}\\
g_r g_{A_{12}} g_r^{-1} &= g_{A_{12}}\\
g_r g_{A_{13}} g_r^{-1} &= g_{A_{12}}^{-1} g_{A_{23}}^{-1} g_{A_{13}}^{-1} g_{A_{12}}^{-1} g_{A_{13}} g_{A_{23}} g_{A_{12}} g_{A_{13}}\\
g_r g_{A_{23}} g_r^{-1} &= g_r g_{A_{12}}^{-1} g_{A_{13}}^{-1} g_{A_{23}}^{-1} g_r^{-1} g_{A_{23}} g_r (g_r g_{A_{12}}^{-1} g_{A_{13}}^{-1} g_{A_{23}}^{-1})^{-1}.
\end{aligned}$$

In summary, we have the mapping class group of the genus two surface generated by

$$g_{T_{D_1}}, g_{A_{12}}, g_{A_{13}}, g_{A_{23}}, g_f \text{ and } g_r$$

with relations those listed in Equation 3.9 and the ten relations coming from Equation 3.10.

3.3 Constructing a presentation of the mapping class group

We now discuss how to use these presentations of a designated subgroup of the mapping class group to construct a presentation for the whole group. Let σ be the mapping class that takes C to C' from Definition 3.4.

Theorem 3.6 (Theorem 2.2 [HT80]). *Let ϕ be a mapping class. Then ϕ can be written as a word in elements p_i of H and σ, of the form $\phi = p_1\sigma \cdots \sigma p_k \sigma p_{k+1}$.*

To prove this, we first show the mapping class group acts transitively on the complex.

Lemma 3.7. *The mapping class group acts transitively on the vertices of the cut system, and H acts transitively on the edges incident to C.*

Proof. The mapping class group acts transitively on the vertices of the cut systems. We can see this as follows. Cutting along the cut system gives a sphere with $2g$ holes. This is homeomorphic to any other such sphere, and we can pick how this homeomorphism acts on the boundary components.

Also, H is transitive on the edges incident to the vertex C in the complex. Let w be a vertex at the end of an edge incident to C. So w is a cut system with all the curves in the cut system the same as those in C, except for one, β, which intersects $\alpha_i \in C$ once. Let z be another such vertex, which has a curve γ that intersects $\alpha_j \in C$ once.

Cut the surface S open along $C \cup \{\beta\}$. We get a sphere with $2g-1$ holes. Each α_k becomes the boundary of two holes, except for α_i, where we get a hole with boundary $\alpha_i\beta\alpha_i^{-1}\beta^{-1}$. We get an analogous picture if we cut along $C \cup \{\gamma\}$. Then there is a homeomorphism between the two spheres with boundary that takes the two boundary components corresponding to α_k to each other, for all $k \neq i, j$, and takes the boundary $\alpha_i\beta\alpha_i^{-1}\beta^{-1}$ to $\alpha_j\gamma\alpha_j^{-1}\gamma^{-1}$. The remaining boundary components are those corresponding to α_j after we cut $C \cup \{\beta\}$, and those corresponding to α_i after we cut $C \cup \{\gamma\}$. Map these to each other.

This homeomorphism fixes C as a set, though it exchanges α_i and α_j. Thus it is in H. It also takes β to γ and vice versa so takes the edge from C to w to the edge from C to z, as required. □

Proof of theorem. We have fixed our standard cut system C. Recall that $\sigma(C) = C'$.

As the cut system complex is connected, there is an edge path $(C = v_0, v_1, \ldots, v_n = \phi(C))$ where the v_i are pairwise-adjacent vertices. We build the word for ϕ in elements of H and σ as follows. First, as H is transitive on the edges incident to C, there is some $p_1 \in H$ taking C' to v_1 and fixing C. Then $p_1\sigma$ takes C to v_1 and C' to C.

We have $(p_1\sigma)^{-1}$ taking v_2 to some v_1', which is connected by an edge to $(p_1\sigma)^{-1}(v_1) = C$. Then pick some $p_2 \in H$ taking C' to v_1', so $p_1\sigma p_2\sigma$ takes C to v_2. We can repeat this process to get some word $p_1\sigma \cdots p_k\sigma p_{k+1}$ such that $p_1\sigma \cdots p_j\sigma$ takes C to v_j.

Note that $H(p_1\sigma \cdots p_k\sigma p_{k+1})^{-1}$ fixes C, so is some element f of H. Thus $H = fp_1\sigma \cdots p_k\sigma p_{k+1}^{-1}$ with $fp_1 \in H$. □

Thus, the mapping class group is finitely generated by the generators of H and σ. It remains to find the relations between σ and the generators of H. Hatcher and Thurston give explicit classes of relations between σ and the generators of H by analysing the five types of ways in which two of these words in H and σ may represent the same mapping class. These are as follows, where the square, triangle, and pentagon relations are as in Figure 3.1:

1. The two words are different words corresponding to the same path.
2. The two words correspond to paths that differ by an involution. For example, if one path of edges is $e_1e_2e_3$, the other might be $e_1e_2e_4e_4^{-1}e_3$.
3. The two words differ by a triangle relation.
4. The two words differ by a square relation.
5. The two words differ by a pentagon relation.

The corresponding five classes of relations are as follows.

Theorem 3.8 (Theorem 2.2 [HT80])**.** *All relations between words of the form $p_1\sigma \cdots p_k\sigma p_{k+1}$ as mapping classes follow from the following five classes of relations. These relations correspond to the five ways in which two of these words may represent the same mapping class.*

1. *The letter σ commutes with the subgroup of $Mod(S)$ that fix both C and $C' = \sigma(C)$, which one can show is finitely generated.*
2. *The element σ^2 is in H.*
3. *For a finite set of p_γ in H, $(p_\gamma\sigma)^3 \in H$.*
4. *For a single $p \in H$, such that p is a particular mapping class taking the meridian of the first handle to the meridian of the second, and the longitude of the first handle to the longitude of the second, σ and $p^{-1}\sigma p$ commute.*

5. *For a finite number of choices of* $p_0, p_1, \ldots, p_4$*,* $\sigma p_4 \sigma p_3 \cdots \sigma p_0$ *is in* H*.*

By this theorem, the mapping class group has a finite set of relations. A complete and entirely explicit presentation of the mapping class group in Hatcher and Thurston's generators can be found in Theorems 3 and 31 of [Waj99].

Chapter 4

Marked Bordered Fatgraphs and Their Duals

A marked bordered fatgraph (Definition 4.16) is an embedding of a certain class of graph in a surface such that its complement consists of one contractible piece for each boundary component. The dual graph to this is embedding is a *quasi-triangulation*. Quasi-triangulations consist of monogons (each of which contains one boundary component) and interior triangles. As in Chapter 3, we can consider the mapping class group action on a surface S decorated with one of these objects. In this case, the action is free (Proposition 4.26).

In this chapter, we will define a marked bordered fatgraph and prove some results on its dual. Principally, we will show that the mapping class group acts freely on marked bordered fatgraphs, and we will give an algorithm to associate a minimal generating set for the fundamental path groupoid to a marked bordered fatgraph. In Chapter 5, this will allow us to define a marked bordered fatgraph complex (Theorem 5.5) with a natural $Mod(S)$-action, giving us an infinite presentation of a mapping class groupoid in terms of moves on bordered fatgraphs.

We will use this presentation in Chapter 6 to give a finitely generated presentation of the same groupoid in terms of chord slides on chord diagrams whose chords are the minimal set of generators for the fatgraph. This will induce an infinite presentation of the mapping class group. Sections 4.1-2 follow [God07].

4.1 Defining n-bordered Fatgraphs

4.1.1 Fatgraphs

First, we establish a convention for formally describing graphs.

Definition 4.1. A *combinatorial graph* G consists of two sets, $V(G)$ and $\xi_{or}(G)$, and

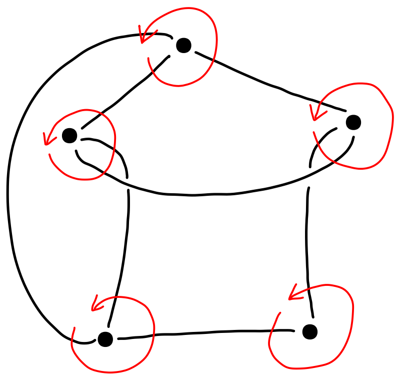


Figure 4.1: An example of a fatgraph.

two maps, $v : \xi_{or}(G) \to V(G)$ and an involution $i : \xi_{or}(G) \to \xi_{or}(G)$ that fixes no elements.

We interpret the set $V(G)$ as the vertices of the graph, and $\xi_{or}(G)$ as the oriented edges. Then v sends an oriented edge e to the vertex it points to, and i sends an oriented edge to the same edge pointing in the opposite direction. As shorthand, we use the notation $\overline{e} = i(e)$. To avoid ambiguity, $\overline{ac}$, for instance, is $i(a)i(c)$ (that is, the order does not reverse as it does with inverses).

We assume a basic knowledge of graph theory. For full definitions, see [Die00]. Our definition of a graph is equivalent to Diestel's definition of a graph, by sending one of our edges e to the edge $(v(e), v(\overline{e}))$. Our notation, however, allows us to easily refer to an edge with a particular orientation.

Definition 4.2. A *morphism of combinatorial graphs*, $\psi : G \to \widetilde{G}$, is a map of sets $\xi_{or}(G) \cup V(G) \to \xi_{or}(G) \cup V(G)$ such that for any oriented edge e of $\widetilde{G}$, $\psi^{-1}(e)$ is a single oriented edge of G, and for any vertex v of $\widetilde{G}$, $\psi^{-1}(v)$ is a tree.

Definition 4.3. A *fatgraph* Γ is a connected combinatorial graph with a permutation σ of the oriented edges $\xi_{or}(\Gamma)$ that restricts to a cyclic ordering of the edges pointing to each vertex.

Given an oriented edge e, we say $i \circ \sigma(e)$ is the consecutive edge. This is the next edge in the ordering around $v(e)$ but pointing away from the vertex rather than towards it.

Definition 4.4. A *fatgraph morphism* is a morphism of combinatorial graphs that respects the cyclic ordering on edges at a vertex.

We can visualise the edge orderings in a fatgraph by immersing the graph into the plane such that going anticlockwise around a vertex in the plane agrees with σ. For

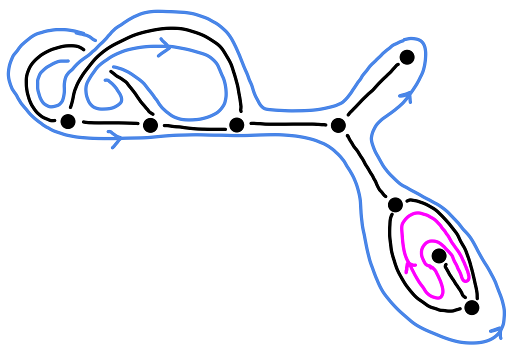


Figure 4.2: A fatgraph with two boundary cycles, each coloured.

example, see Figure 4.1. When we draw fatgraphs, we will draw them such that this is their cyclic ordering.

Definition 4.5. A *boundary cycle* of a fatgraph Γ is a cycle of consecutive oriented edges, that is, a cycle of the permutation $i \circ \sigma$.

For example, see Figure 4.2, which has boundary cycles outlined in blue and pink.

4.1.2 Bordered Fatgraphs

We wish to consider a subclass of fatgraphs that arises as the dual of quasi-triangulations.

Definition 4.6. A *bordered fatgraph* is a fatgraph with an ordering on the boundary cycles satisfying the following conditions.

- First, there is precisely one univalent vertex in each boundary cycle, called the *tail vertex*. The single edge incident to it is called the *tail*.
- All vertices that are not tail vertices are trivalent.

A bordered fatgraph is *n-bordered* if it has n boundary cycles.

Figure 4.2 is an example of a twice-bordered fatgraph. In Theorem 5.5, we will show that it suffices to consider these n-bordered fatgraphs to get a presentation of $Mod(S)$. To prove this, we will need a generalisation of a bordered fatgraph, where some edges have been collapsed.

Definition 4.7. A *collapsed* n-bordered fatgraph is a fatgraph satisfying the conditions for an n-bordered fatgraph, save that it may have vertices with valency more than three.

Given a collapsed bordered fatgraph, we can define a preferred orientation on the edges. This will mean the dual graph, described in Section 4.3 is a directed graph, and in particular is a subset of the fundamental path groupoid. We define this orientation as follows.

Definition 4.8. Let Γ be a collapsed n-bordered fatgraph. Suppose e is an (unoriented) edge of Γ. The *preferred orientation* of e is the orientation in which we first traverse it in the following procedure.

Starting at the first tail vertex, follow the first boundary cycle around the graph until you return to this vertex. Then repeat this for the second through n^{th} tail vertices in order.

Lemma 4.9. *There is at most one fatgraph isomorphism between any two collapsed bordered fatgraphs that preserves the ordering of the boundary cycles.*

Proof. Note the inverse of such a fatgraph morphism also preserves the ordering of the boundary cycles.

First, we show that the only automorphism of a collapsed bordered fatgraph fixing the boundary cycle ordering is the identity. Such an automorphism must preserve the valency of the vertices (as every edge is sent to exactly one edge), so in particular sends the set of tail vertices to the set of tail vertices.

The i^{th} boundary cycle is a cycle of $i \circ \sigma$. The elements of this cycle are mapped bijectively to another cycle of $i \circ \sigma$, which must be the same cycle by the preservation of the ordering on boundary cycles. Then once we find the image of one element in the cycle, we have determined the images of all the elements in the cycle (as the relationship of consecutive edges is preserved). Now, the unique tail vertex in this boundary cycle must be sent to a tail vertex, so in particular is sent to the tail vertex of this boundary cycle. Thus, the map is the identity on the i^{th} boundary cycle for all i, so is the identity on the entire fatgraph.

Now, suppose there were two isomorphisms fixing the boundary cycles, ϕ_1 and ϕ_2, from Γ_1 to Γ_2. Then $\phi^{-1} \circ \psi : \Gamma_1 \to \Gamma_1$ is an automorphism that is not the identity, a contradiction. □

We now define a morphism of collapsed bordered fatgraphs as a forest collapse.

Definition 4.10. Let Γ be an n-bordered fatgraph. Let $F \subset \Gamma$ be the union of a pairwise disjoint set of trees of the graph (a *forest*), such that F does not contain any of the tails. The *forest collapse* of F acts by collapsing each connected component of F to a vertex.

An example of a forest collapse is shown in Figure 4.3.

Definition 4.11. A *morphism of collapsed bordered fatgraphs* $\psi : \Gamma_1 \to \Gamma_2$ is a composition of forest collapses and boundary cycle-preserving isomorphisms.

We define an equivalence relation on these morphisms as follows.

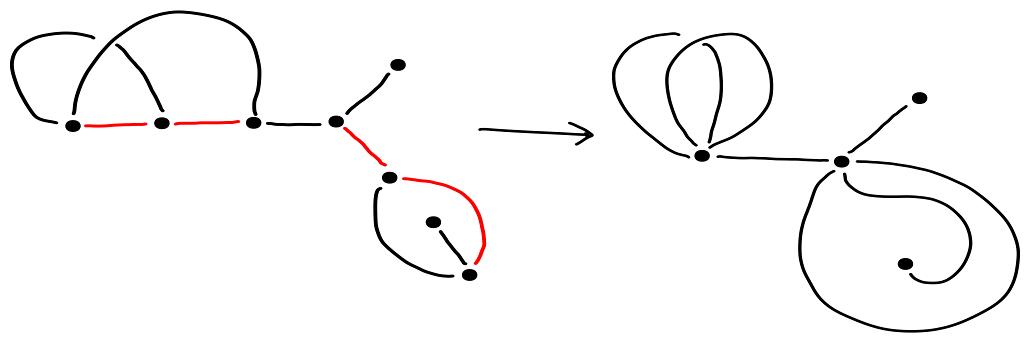


Figure 4.3: A morphism of collapsed twice-bordered fatgraphs, collapsing the forest in red.

Definition 4.12. We say $\psi_1 : \Gamma_1 \to \widetilde{\Gamma}_1$ is *equivalent* to $\psi_2 : \Gamma_2 \to \widetilde{\Gamma}_2$ if there exist isomorphisms $\theta : \Gamma_1 \to \Gamma_2$ and $\widetilde{\theta} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ such that the following diagram commutes.

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\psi_1} & \widetilde{\Gamma}_1 \\ \downarrow \theta & & \downarrow \widetilde{\theta} \\ \Gamma_2 & \xrightarrow{\psi_2} & \widetilde{\Gamma}_2 \end{array}$$

From here, all isomorphisms between collapsed bordered fatgraphs will be boundary cycle order-preserving isomorphisms. We will say a collapsed bordered fatgraph to mean an equivalence class of collapsed bordered fatgraphs, up to one of these isomorphisms.

Remark 4.13. We can more generally define a collapsed bordered fatgraph morphism as a fatgraph morphism preserving the boundary cycle ordering. On equivalence classes of these fatgraphs, this is equivalent to our definition of these fatgraph morphisms being forest collapses [God07, Lemma 3].

In the following lemma, we show we can compose (equivalence classes of) morphisms of collapsed bordered fatgraphs in a well-defined manner.

Let $\psi : \Gamma_1 \to \Gamma_2$ and $\phi : \Gamma_3 \to \Gamma_4$ be collapsed bordered fatgraph morphisms, with Γ_2 and Γ_3 in the same equivalence class. Then there is a (unique) isomorphism θ between Γ_2 and Γ_3. We define the composition of the equivalence classes of these morphisms, $[\phi] \circ [\psi]$, to be the equivalence class of the composition

$$\Gamma_1 \xrightarrow{\psi} \Gamma_2 \xrightarrow{\theta} \Gamma_3 \xrightarrow{\phi} \Gamma_4.$$

Lemma 4.14. *This composition is well-defined.*

Proof. We give the case of two equivalent morphisms $\Gamma_1 \to \Gamma_2$, as the case for $\Gamma_3 \to \Gamma_4$ is very similar. Let $\psi : \Gamma_1 \to \Gamma_2$ and $\widetilde{\psi} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ be equivalent morphisms, with $\theta_i : \Gamma_i \to \widetilde{\Gamma}_i$ the unique isomorphisms showing equivalence.

Suppose $\psi : \Gamma_3 \to \Gamma_4$ with Γ_3 and Γ_2 isomorphic so composition is well-defined. As $\Gamma_2 \cong \widetilde{\Gamma}_2$, there is a (unique) isomorphism $\widetilde{\theta} : \widetilde{\Gamma}_2 \to \Gamma_3$. Then we wish to show that the following diagram commutes.

$$\begin{array}{ccccccc}
\Gamma_1 & \xrightarrow{\psi} & \Gamma_2 & \xrightarrow[\cong]{\theta} & \Gamma_3 & \xrightarrow{\phi} & \Gamma_4 \\
{\scriptstyle \theta_1}\downarrow{\scriptstyle \cong} & & {\scriptstyle \theta_2}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\widetilde{\Gamma}_1 & \xrightarrow{\widetilde{\psi}} & \widetilde{\Gamma}_2 & \xrightarrow[\cong]{\widetilde{\theta}} & \Gamma_3 & \xrightarrow{\phi} & \Gamma_4
\end{array}$$

The rightmost square commutes trivially, and the leftmost commutes as ψ and $\widetilde{\psi}$ are equivalent morphisms. Then as $\widetilde{\theta}$ is the unique isomorphism $\widetilde{\Gamma}_2 \to \Gamma_3$, we have $\widetilde{\theta} = \theta \circ \theta_2^{-1}$ since this is also an isomorphism between these fatgraphs. Thus, the middle square commutes, so $\phi \circ \theta \circ \psi$ and $\phi \circ \widetilde{\theta} \circ \widetilde{\psi}$ are equivalent morphisms. □

4.2 Marked bordered fatgraphs

Let $\Sigma_{g,n}$ be the genus g orientable surface with n boundary components, with a fixed ordering on the boundary components and a marked point on each boundary component. We will now view n-bordered fatgraphs as embedded in $\Sigma_{g,n}$ for some g, and use this to construct a category of these embedded fatgraphs, $EFat_{g,n}$, with a natural action of $Mod(\Sigma_{g,n})$.

Suppose we embed an n-bordered fatgraph as a spine in a surface with n boundary components, which we will formally define in Definition 4.16. Let the number of vertices of Γ be V, and the number of edges be E. We can view the surface as a thickened version of the fatgraph, motivating a definition of the genus of the fatgraph. We can construct the surface as a neighbourhood of the fatgraph as follows. We thicken each edge of the fatgraph to a rectangle with four edges, two of which are on the boundary, and each vertex to a triangle with three edges and three vertices. We can then calculate

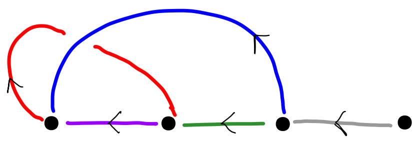
$$2 - 2g - n = \chi(\Sigma_\Gamma) = 3V - (3V + 2E) + (V + E) = V - E = \chi(\Gamma).$$

This motivates the following definition.

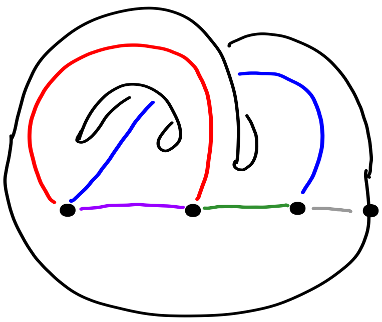
Definition 4.15. The *genus* of an n-bordered fatgraph Γ is the number g such that

$$\chi(\Gamma) = 2 - 2g - n.$$

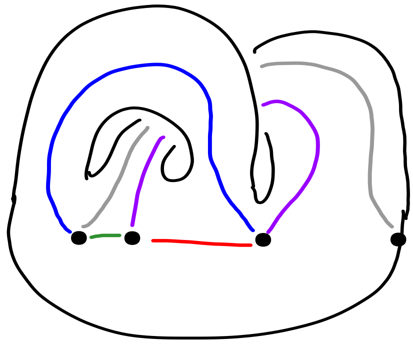
We will restrict our attention to the set of (isomorphism classes of) collapsed n-bordered fatgraphs of genus g, $Fat_{g,n}$. Then these are precisely the collapsed bordered fatgraphs that embed into $\Sigma_{g,n}$ as a spine, that is so that the complement of the embedding is n disjoint contractible pieces.



(a) A genus one once-bordered fatgraph, Γ.



(b) Marking 1 of Γ.



(c) Marking 2 of Γ.

Figure 4.4: Two different markings of the same once-bordered fatgraph.

Definition 4.16. A *marking* of a collapsed bordered fatgraph $\Gamma \in Fat_{g,n}$ is a boundary-fixing isotopy class of embeddings p of Γ in $\Sigma_{g,n}$ satisfying the following conditions. First, the embedding sends the i^{th} tail vertex to the marked point on the i^{th} boundary component. Second, the embedding sends the remainder of the graph to the interior of $\Sigma_{g,n}$. Finally, the complement of the image of a representative of p is a set of n disjoint contractible pieces.

A *marked bordered fatgraph* is (Γ, ϕ) where Γ is a bordered fatgraph and ϕ is a marking of Γ.

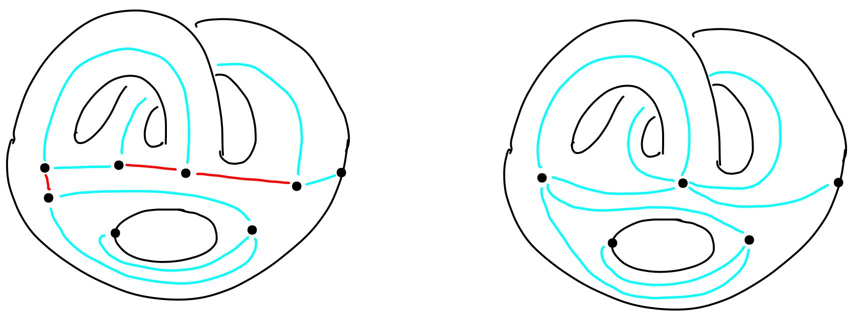
Figure 4.4 shows two different markings of the same once-bordered fatgraph.

Note that $Mod(\Sigma_{g,n})$ acts on marked collapsed bordered fatgraphs by composition with the marking. This fixes the bordered fatgraph. (The markings are defined up to isotopy, as is $Mod(\Sigma_{g,n})$, so this action is well-defined.) The morphisms of these objects are defined as follows.

Definition 4.17. A *morphism* of marked collapsed bordered fatgraphs is defined as follows.

Let (Γ_1, p_1) be a marked collapsed bordered fatgraph. Let $[\phi] : \Gamma_1 \to \Gamma_2$ be a morphism of collapsed bordered fatgraphs. Let ϕ be a representative of this morphism as a forest collapse. Then the associated marked collapsed bordered fatgraph morphism $\widetilde{\phi} : (\Gamma_1, p_1) \to (\Gamma_2, p_2)$ is defined as taking Γ_1 to Γ_2 by ϕ, and sending the marking p_1 to the marking p_2 such that

$$\Gamma_1 \xrightarrow{\phi} \Gamma_2 \xrightarrow{p_2} \Sigma_{g,n}$$



(a) Before the forest collapse. (b) After the forest collapse.

Figure 4.5: A morphism of marked bordered fatgraphs, collapsing the forest in red.

is homotopy equivalent to p_1.

Figure 4.5 shows an example.

We now define the marked bordered fatgraph category for $\Sigma_{g,n}$.

Definition 4.18. The marked bordered fatgraph category for $\Sigma_{g,n}$, $EFat_{g,n}$, is defined as follows. The objects of this category are (isomorphism classes) of n-bordered genus g fatgraphs with markings (up to isotopy). The morphisms are marked bordered fatgraph morphisms, as defined above.

We have shown composition of these morphisms is well-defined on the bordered fatgraphs, and as the action on the marking is entirely characterised by the action on the bordered fatgraph, composition is well-defined on marked bordered fatgraphs. Thus we have the following result.

Lemma 4.19. *The category $EFat_{g,n}$ is well-defined.*

In Chapter 5 we will further consider this category and the action of $Mod(\Sigma_{g,n})$ on it. This will allow us to construct a presentation of $Mod(\Sigma_{g,n})$ generated by moves on a fatgraph. However, we will first discuss the dual of a marked bordered fatgraph, as we will need some results on it to convert the fatgraph presentation to one in chord slides.

4.3 Duals and the fundamental path groupoid

Viewed as an embedded graph in $\Sigma_{g,n}$, a marked n-bordered fatgraph admits a dual graph. Before we discuss the complex we get from the marked bordered fatgraph category, we prove some results involving this dual. This will give us a canonical way of picking a generating set for the fundamental path groupoid from each fatgraph, and a simple proof that $Mod(\Sigma_{g,n})$ acts freely on marked bordered fatgraphs.

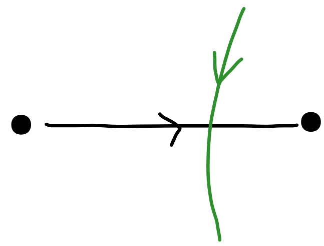


Figure 4.6: The orientation of the dual edges relative to the orientation of the graph edges. The black edge is in the graph and the green edge is its dual.

Definition 4.20. Let G be a directed graph embedded in a surface S, such that $S - G$ is a disjoint union of contractible components. Then the *dual graph* of G is the graph defined as follows. For each component of the complement (a face of the graph), place a vertex in this component. For each edge e of G, we add an edge between the vertices of the faces that e separates such that this edge passes once through e and through no other edges of G, and this edge has the orientation relative to e shown in Figure 4.6.

Now, we consider the dual graph of a marked bordered fatgraph Γ. We view Γ as embedded in $\Sigma_{g,n}$, with its edges equipped with their preferred orientation (as defined in Definition 4.8). This allows us to easily distinguish different markings.

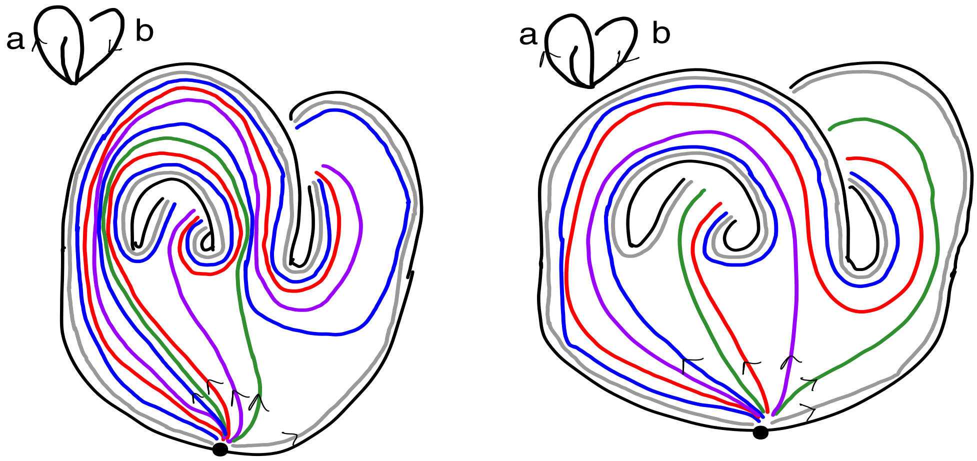
For example, Figure 4.7 shows the duals of the two markings in Figure 4.4. We will use these duals to demonstrate that the markings are not the same. Let a and b be the generators of the fundamental group as drawn in Figure 4.7. Around the boundary cycle, the order of the edges is grey, blue, red, purple then green. Then the marking in Figure 4.4b has dual edges (in this same order) $b^{-1}aba^{-1}$, $ab^{-1}aba^{-1}$, aba^{-1}, ba^{-1} and a^{-1}. In contrast, the dual edges of the marking in Figure 4.4c are $b^{-1}aba^{-1}$, aba^{-1}, ba^{-1}, a^{-1} and b^{-1}.

4.3.1 Characterising the dual

Let (Γ, p) be a marked n-bordered fatgraph. Recall that each vertex in an n-bordered fatgraph is either univalent or trivalent.

As this fatgraph has n boundary components, which give n components in its complement in $\Sigma_{g,n}$, its dual is a graph with n vertices. Each component of the complement of Γ contains exactly one boundary component. Thus, without loss of generality, we may place each vertex of the dual graph on a boundary component. For example, Figure 4.8 shows the dual of a once-bordered fatgraph which has one vertex, while Figure 4.9 shows the dual of a twice-bordered fatgraph which has two vertices.

As all the faces of the surface from the original fatgraph are contractible, the dual faces are contractible or contain a boundary component. The dual faces come in two



(a) The dual of the marking in Figure 4.4b. (b) The dual of the marking in Figure 4.4c.

Figure 4.7: The duals of the two markings in Figure 4.4.

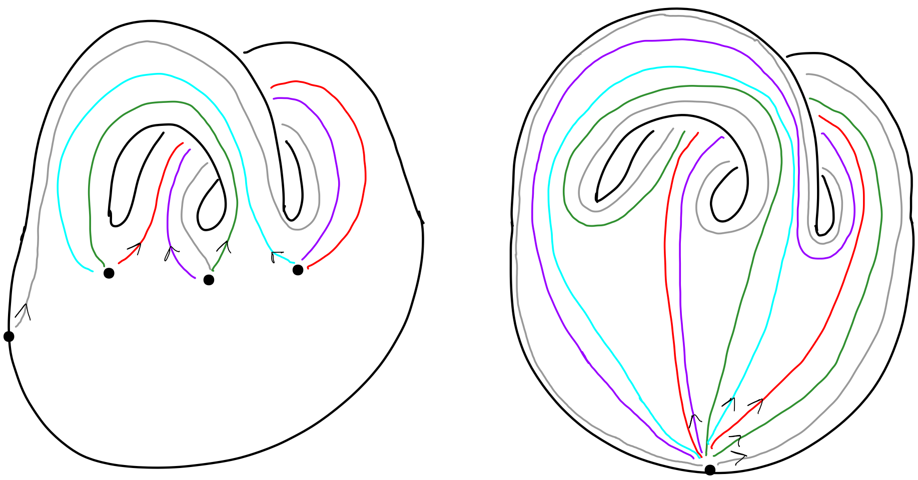
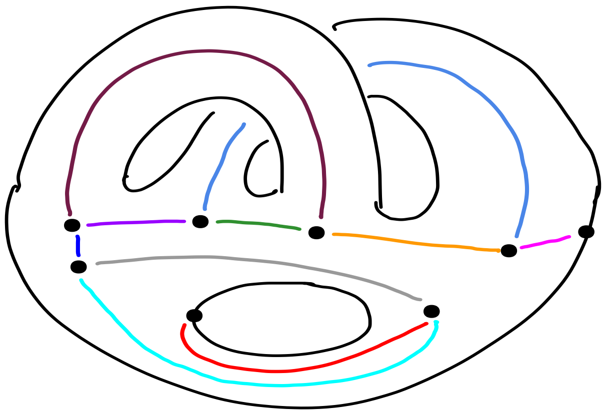


Figure 4.8: On the right, a quasi-triangulation of $\Sigma_{1,1}$ from taking the dual of the marked once-bordered fatgraph on the left. Preferred orientations of edges are marked with arrows, giving us the illustrated orientations of the edges of the dual graph.



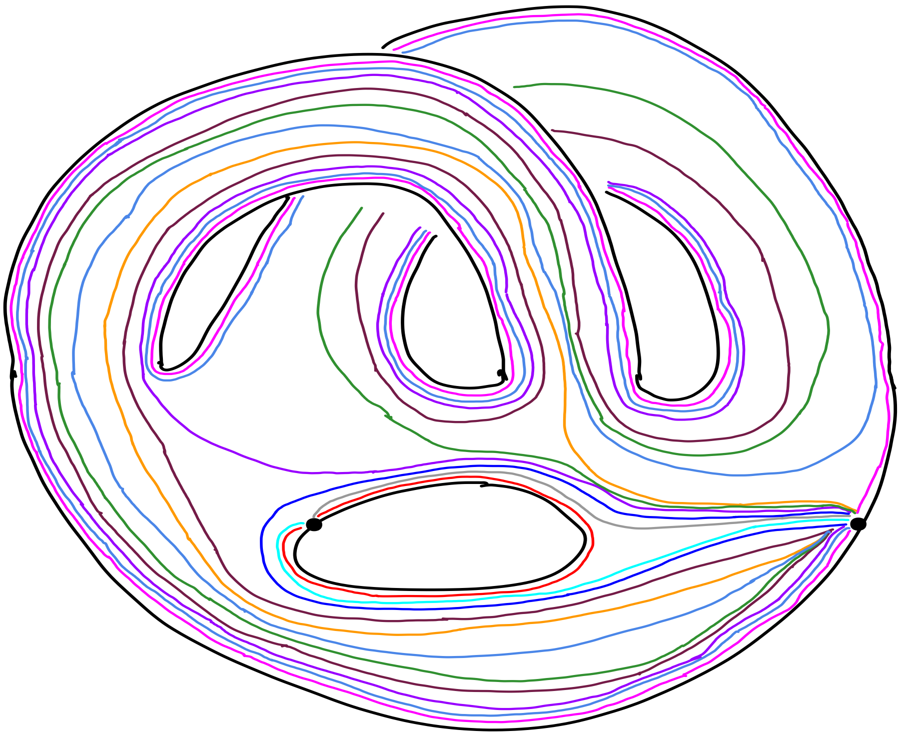


Figure 4.9: Top: a marking of a twice-bordered genus one fatgraph. Bottom: a quasi-triangulation of $\Sigma_{1,2}$ from taking the dual of this fatgraph. Note the dual has two vertices and the resulting faces are all triangles, as we expect. We have not marked orientations.

types. First, for every trivalent vertex we have a contractible triangle. Second, for each tail vertex we have a monogon containing one boundary component.

We will use the term *quasi-triangulation* of $\Sigma_{g,n}$ to refer to a graph cutting $\Sigma_{g,n}$ into (a) contractible triangles and (b) monogons containing one boundary component that deformation retract to the boundary. The dual graph of a marked bordered fatgraph in $\Sigma_{g,n}$ is a quasi-triangulation.

We already have that $\Sigma_{g,n}$ is equipped with n marked points. Let $\{p_i\}$ be an additional n distinct marked points, the *dual points*, with one on each boundary component. Note that for any marking, there is one dual point in each of the n faces of the surface. When we take the dual, let the point from the i^{th} face of the marking be the dual point on the i^{th} boundary component.

Definition 4.21. The groupoid $\Pi_1(\Sigma_{g,n}, \{p_i\})$ is the full subgroupoid of the fundamental path groupoid consisting of paths f such that both endpoints of f are in $\{p_i\}$.

Remark 4.22. As the duals of the edges are each homotopy classes of paths between the n marked dual points, we view the dual edges of the graph as elements of the fundamental path groupoid.

Definition 4.23. Let (Γ, p) be a marked bordered fatgraph. The *dual marking* of (Γ, p) is a map $\Pi : \xi_{or}(\Gamma) \to \Pi_1(\Sigma_{g,n}, \{p_i\})$ taking each oriented edge, viewed as embedded in $\Sigma_{g,n}$, to the edge dual to it in the dual graph, viewed as an element of the fundamental path groupoid.

We give some properties of the map $\Pi : \xi_{or}(\Gamma) \to \Pi_1(\Sigma_{g,n}, \{p_i\})$.

Lemma 4.24. *A dual marking $\Pi : \xi_{or}(\Gamma) \to \Pi_1(\Sigma_{g,n}, \{p_i\})$ satisfies the following properties.*

1. (orientation respecting) *For $e \in \xi_{or}(\Gamma)$, $\Pi(e)\Pi(\bar{e})$ is the identity at the initial point of $\Pi(e)$.*
2. (face contractability) *For each trivalent vertex of Γ, v, with $e \in \xi_{or}(\Gamma)$ such that $v(e) = v$, $\Pi(e)\Pi(\sigma(e))\Pi(\sigma^2(e))$ is the identity at the initial point of $\Pi(e)$.*
3. (fundamental path groupoid generation) *The image of Π generates $\Pi_1(\Sigma_{g,n}, \{p_i\})$.*
4. (tail-fixing) *The map Π sends the i^{th} tail to the homotopy class of the i^{th} boundary component based at the i^{th} marked dual point.*

Proof. The first property follows since reversing the orientation of an edge reverses the orientation of the edge dual to it.

The second condition is equivalent to saying that the duals of the edges incident to v bound a contractible face in the surface. As all the faces of the surface from the original fatgraph are contractible, the dual faces are contractible or contain a boundary component. By construction, the faces containing a boundary component are precisely the duals of the tail vertices, so since v is not a tail vertex, we have the result.

For the third condition, we wish to show that $\Pi(\xi_{or}(\Gamma))$ generates $\Pi_1(\Sigma_{g,n}, \{p_i\})$. All the triangular faces of the quasi-triangulation are contractible, so any path through one of these is homotopic to a path along the boundary of the face. Although the monogons are not contractible, any path through a monogon is homotopic to a path entering and leaving the dual point on its boundary. The piece of this path within the monogon is then homotopic to a path making some number of circuits around the edge of the monogon.

For the final property, we have required that any marking sends all the graph but the vertices of the tails to the interior of the surface. Now, the dual of the i^{th} tail is a homotopy class of paths characterised by the following conditions.

1. It passes through the i^{th} tail.
2. It intersects no other edges of graph.
3. It begins and ends at a fixed point on the i^{th} boundary component.

As the i^{th} boundary component satisfies these conditions, the dual edge of the tail is the homotopy class of this boundary component. □

Remark 4.25. As Π is injective, we identify an edge $e \in \xi_{or}(\Gamma)$ with $\Pi(e)$.

By considering the action of $Mod(\Sigma_{g,n})$ on the dual of the marking, we have the following result.

Proposition 4.26. *The group $Mod(\Sigma_{g,n})$ acts freely on marked genus g n-bordered fatgraphs.*

Proof. Let $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$ be a mapping class, and Γ a marked genus g n-bordered fatgraph. Suppose that ϕ fixes the dual of Γ, up to homotopy, as a set. We wish to show that ϕ is the identity.

As the isotopy class ϕ fixes the boundary components pointwise, ϕ has some representative that fixes a neighbourhood of the boundary components. We assume ϕ fixes each pointwise of the monogons with one boundary component in the dual pointwise. In particular, ϕ fixes the duals of each of the tails pointwise.

Let e be an oriented edge in the dual graph that is fixed by ϕ, where we furthermore know that ϕ sends one of the faces e borders to itself. Let this face be U, and the other face e borders be L.

As ϕ is a homeomorphism and sends the set of edges to itself, it sends each face from the quasi-triangulation to another face. Since ϕ is continuous, it sends L to either

U or L (since the image of L must have the image of e, which is e, as one of its edges). Then as ϕ is injective, it sends L to L. Furthermore, it fixes the orientation of L since e is sent to itself as an oriented edge. As L is sent to itself, up to orientation, and e is sent to itself, the edge following e around the boundary of L is sent to itself as an oriented edge. Thus the edges of L are fixed. Now, $\phi|_L$ is a homeomorphism of L that is isotopic to the identity on the boundary. Thus by the Alexander trick (Lemma 2.11), ϕ is isotopic to the identity on L.

By induction, ϕ is isotopic to the identity on all faces. We are inducting in from the tails of the fatgraph. At each step, we take a edge where we know ϕ fixes its dual edge and the dual of one of its adjacent vertices. By the inductive step, we have that ϕ fixes the dual of its other adjacent vertex, as well as the duals of the other edges incident to that vertex. As the fatgraph is connected, when this terminates we will have shown that ϕ fixes the duals of the whole graph.

Thus, ϕ is isotopic to the identity on all faces of the quasi-triangulation, so is the identity in the mapping class group. □

4.3.2 Adding a boundary component to a surface

In [Ben10], Bene works with marked once-bordered fatgraphs (which he calls bordered fatgraphs). To give a feel for how these differ from our bordered fatgraphs, we define a local move that takes a marked n-bordered fatgraph to a marked $(n+1)$-bordered fatgraph. This move will add a boundary cycle and one tail within the boundary cycle. The move corresponds to adding a boundary component at a given location on the surface. We illustrate the move in Figures 4.10-4.12.

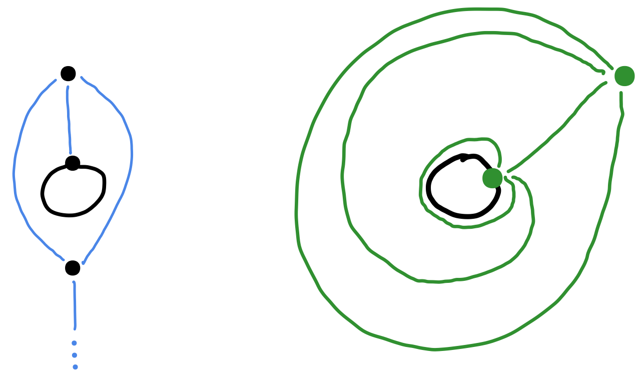
If we fix the subgraph corresponding to this new boundary cycle to the one in Figure 4.10, there are six choices in how we add this subgraph to the fatgraph. The vertex we add this subgraph to is fixed, as the new boundary component is in a certain face of the dual marking that corresponds to a certain vertex of the fatgraph.

To add the subgraph, we first pick where out of three options we will add it in the cyclic ordering of edges around the vertex, as shown in Figure 4.11. This creates a four-valent vertex, which we can expand in two distinct ways, as in Figure 4.12. For an illustration of this, see Figure 5.1.

4.3.3 Picking a minimal generating set for $\Pi_1(\Sigma_{g,n}, \{p_i\})$

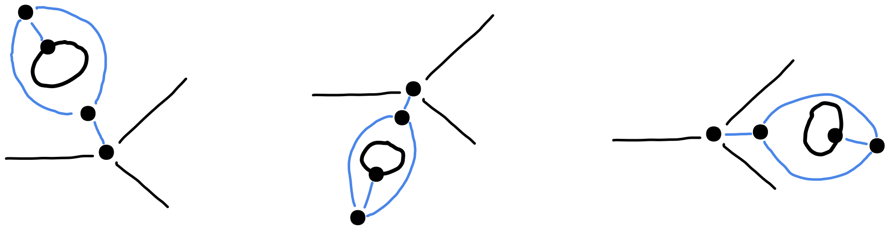
One sees easily that a dual marking determines a generating set for $\Pi_1(\Sigma_{g,n}, \{p_i\})$. In this section, we give a method to canonically choose a minimal generating subset. This minimal set of generators will later determine a canonical form of the graph as a chord diagram (Theorem 6.18).

First, we give the number of vertices and edges of any n-bordered genus g fatgraph.



(a) The subgraph. (b) The dual of the subgraph.

Figure 4.10: A subgraph corresponding to adding a puncture.



(a) Option 1 (b) Option 2 (c) Option 3

Figure 4.11: The three options in the cyclic ordering around the vertex where we can glue on the subgraph.

Lemma 4.27. *Let* Γ *be an* n*-bordered genus* g *fatgraph. Then* Γ *has* V *vertices and* E *edges, where*

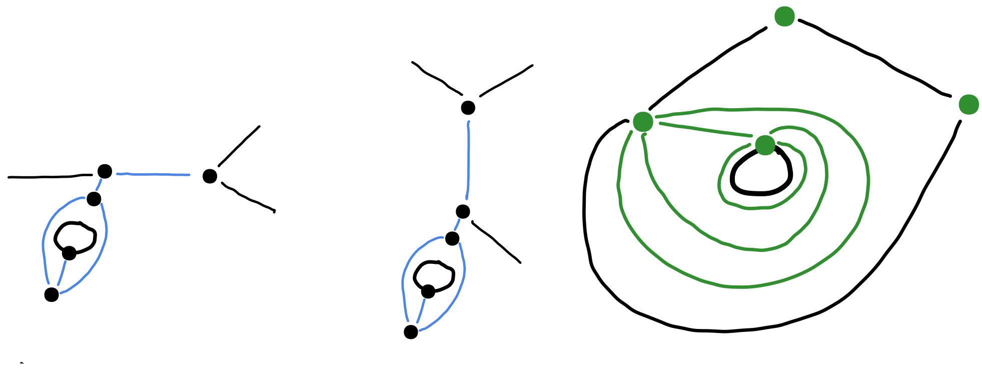
$$V = 4n + 4g - 4 \text{ and}$$
$$E = 6g + 5n - 6.$$

Proof. From the definition of genus (Definition 4.15), $V - E = \chi(\Gamma) = 2 - 2g - n$. Also, there are precisely n univalent vertices, and $V - n$ trivalent ones. Thus $E = \frac{3V-2n}{2}$. The results follow. □

Now, we give a procedure for picking a minimal generating set for the fundamental path groupoid, extending [ABP09, §3].

Definition 4.28. Let Γ be an n-bordered fatgraph. The *greedy maximal tree* of Γ, T_Γ, is the tree in Γ defined as follows.

Start at the first tail vertex. From this vertex, follow the first boundary cycle. At each edge e, if $\{e\} \cup T_\Gamma$ does not contain a cycle, add e to T_Γ. Then repeat this process



(a) Option 1 (b) Option 2 (c) The dual of option 1

Figure 4.12: The two ways of expanding the four-valent vertex from Figure 4.11.

for the second tail, and then the third, and so on. Now, remove all tails but the first from T_Γ.

Remark 4.29. Before we remove the tails, T_Γ is a maximal tree in G, so is connected. As all the tails were leaves of T_Γ, T_Γ is still connected after removing tails, and contains an edge touching every vertex except for the tail vertices.

For p a marking of Γ, let $X_{(\Gamma,p)}$ be the set of duals of the edges of Γ not in T_Γ.

Proposition 4.30. *The subset of the dual edges, $X_{(\Gamma,p)}$, is a minimal generating set for the fundamental path groupoid.*

Proof. First, note that the fundamental group of $\Sigma_{g,n}$ is the free group on $2g+n-1$ generators. One choice of minimal generating set for this group is a generating set for the fundamental group at the first basepoint, together with edges from the first basepoint to each other basepoint. This is of size $(2g+n-1)+(n-1) = 2g+2n-2$.

Now, the edge set of a maximal tree in Γ is of size $V-1 = 4n+4g-5$ (Lemma 4.27). Let $|E_T|$ be the number of edges in T_Γ. As $n-1$ tails were removed from a maximal tree to make T_Γ, we have

$$|E_T| = (4n+4g-5)-(n-1) = 3n+4g-4.$$

Then $X_{(\Gamma,p)}$ contains all edges but those in T_Γ, so (from Lemma 4.27)

$$|X_{(\Gamma,p)}| = E - |E_T| = 2g+2n-2.$$

As $X_{(\Gamma,p)}$ is the size of a minimal generating set for $\Pi_1(\Sigma_{g,n},\{p_i\})$, it suffices to show that $X_{(\Gamma,p)}$ in fact generates this groupoid. Viewing the dual edges of G as elements of $\Pi_1(\Sigma_{g,n},\{p_i\})$, as $\Pi(\xi_{or}(\Gamma))$ generates $\Pi_1(\Sigma_{g,n},\{p_i\})$, it suffices to show that $X_{(\Gamma,p)}$

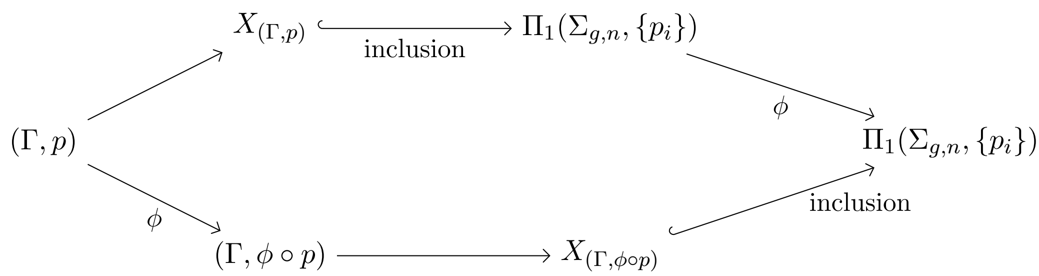
generates the dual edges of Γ. To show this it suffices to show that $X_{(\Gamma,p)}$ generates the dual edges of T_Γ.

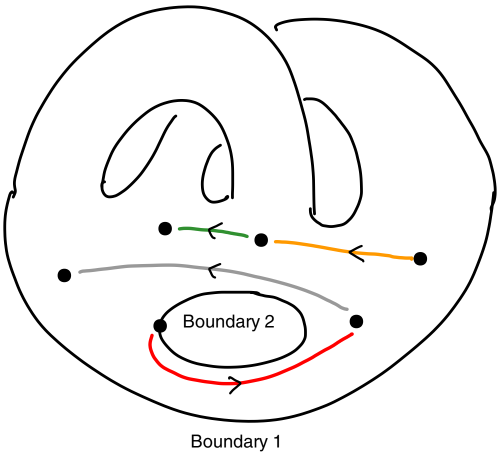
We show this by induction. View T_Γ as being rooted at the first tail. Let e be a leaf of T_Γ. As the only tail in T_Γ is the first one, one end of e is at a vertex which has two non-tree edges ending at it. Let these edges, whose duals are in $X_{(\Gamma,p)}$, be x_1 and x_2. Recall we identify an edge of the graph with its dual. Now, by face-contractability, for some indexing of x_1 and x_2, ex_1x_2 is the identity at the starting point of the dual of e. Therefore $e = x_2^{-1}x_1^{-1}$. Thus, e is in the set generated by X_G.

By induction, all of T_Γ is in the set, as required. □

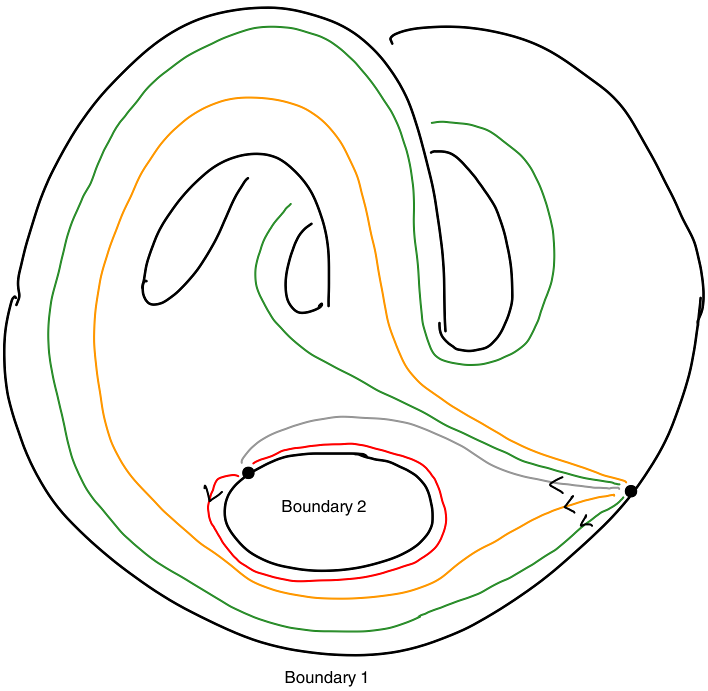
Thus a marked bordered fatgraph Γ has an associated generating set, $X_{(\Gamma,p)}$, for $\Pi_1(\Sigma_{g,n}, \{p_i\})$. For example, Figure 4.13a shows $\Gamma - T_\Gamma$ for the fatgraph in Figure 4.9, which is the edges whose dual edges form the generating set. Figure 4.13b shows $X_{(\Gamma,p)}$ for this marked fatgraph, the dual edges of $\Gamma - T_\Gamma$.

Note that acting by $Mod(\Sigma_{g,n})$ on the marked fatgraph does not alter which edges correspond to generators. Consider the action of $Mod(\Sigma_{g,n})$ on minimal generating sets of the fundamental path groupoid. Then this action is the same as that induced by the action of $Mod(\Sigma_{g,n})$ on marked n-bordered fatgraphs. This equivalence is shown in the following commutative diagram for $\phi \in Mod(\Sigma_{g,n})$.





(a) The subgraph of the marked bordered fatgraph Γ in Figure 4.9 that is $\Gamma - T_\Gamma$.



(b) The dual edges of a, X_Γ.

Figure 4.13: The canonical generating set for Π_1 from Figure 4.9.

Chapter 5

A presentation of the mapping class group in terms of fatgraphs

We have a natural action of $Mod(\Sigma_{g,n})$ on $EFat_{g,n}$, and in this chapter will derive from it an infinite presentation of the mapping class group (Corollary 5.12). The generators of this presentation will be moves on (unmarked) n-bordered fatgraphs.

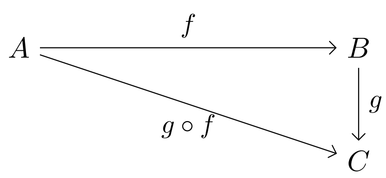
Having defined a marked bordered fatgraph, we now define the marked bordered fatgraph complex. This is the classifying space of $EFat_{g,n}$ (Definition 4.18). Similarly to Chapter 3, we will examine the action of $Mod(\Sigma_{g,n})$ on this complex to derive a presentation whose generators are paths in the complex.

5.1 The marked bordered fatgraph complex

Definition 5.1. Let C be a category. The *classifying space* $|C|$ of C is the geometric realisation of the nerve of C.

As a CW-complex, $|C|$ has one vertex for every object of C. If A, B are objects, there is an edge between A and B for every morphism $A \to B$.

Now we describe the higher dimensional faces. Suppose $f : A \to B$ and $g : B \to C$ are composable morphisms. For every such pair, there is a face $[A, B, C]$ corresponding to the following triangle.



Similarly, suppose $f_0, \ldots, f_n$ are morphisms such that $f_n \circ \cdots \circ f_1 \circ f_0$ is well-defined. Then there is an n-simplex with edges (in order) $f_0, f_1, \ldots, f_n, f_n \circ \cdots \circ f_0$.

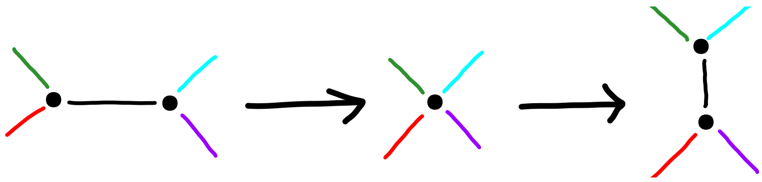


Figure 5.1: A Whitehead move.

Note that as defined, we have many degenerate simplices. For example $[f, \mathrm{id}, f]$ is a 2-simplex for any f. As an abstract collection of simplices rather than a complex, this is a simplicial set. The category theoretic concepts in Definition 5.1 are discussed formally in [Rez18]. In particular, see Remark 3.3 for a discussion of the nerve as we use it here.

From Godin and Harer, we have the following theorem on the homotopy class of $|EFat_{g,n}|$.

Theorem 5.2 ([God07; Har86])**.** *The complex $|EFat_{g,n}|$ is homotopy equivalent to Techmüller space, which is homeomorphic to a ball. Therefore its 2-skeleton is connected and simply connected.*

Since we wish to consider paths between vertices in this complex, we will only consider its 2-skeleton. We will show that it suffices to consider vertices corresponding to fatgraphs with all vertices trivalent or univalent (Theorem 5.5).

First, we define an elementary move on a bordered fatgraph.

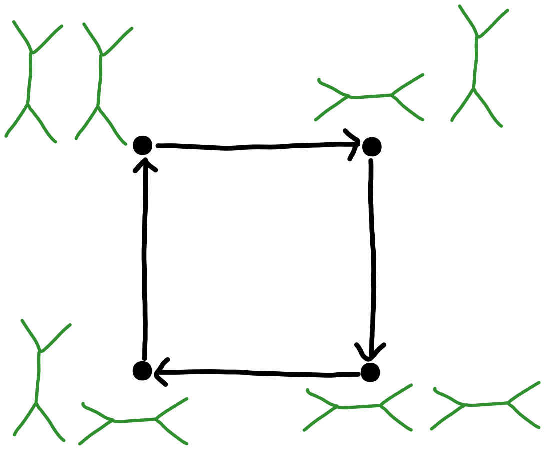
Definition 5.3. Let Γ be a bordered fatgraph. Let e be a non-tail edge of Γ. The *Whitehead move* on e, $W(e)$, is the collapse of e, forming a four-valent vertex, followed by the expansion of this vertex into the unique distinct edge.

Figure 5.1 depicts an example of a Whitehead move. By performing this move in the surface, a Whitehead move induces a map on the marking. Thus $W(e)$ takes a marked bordered fatgraph (Γ, p) to some marking of a (generally distinct) bordered fatgraph (Γ', p'). Note that Whitehead moves do not affect the homotopy type of the graph in $\Sigma_{g,n}$. Thus for $e' \in \xi_{or}(\Gamma)$ with $e' \neq e$, $W(e)$ preserves the dual edge of e'.

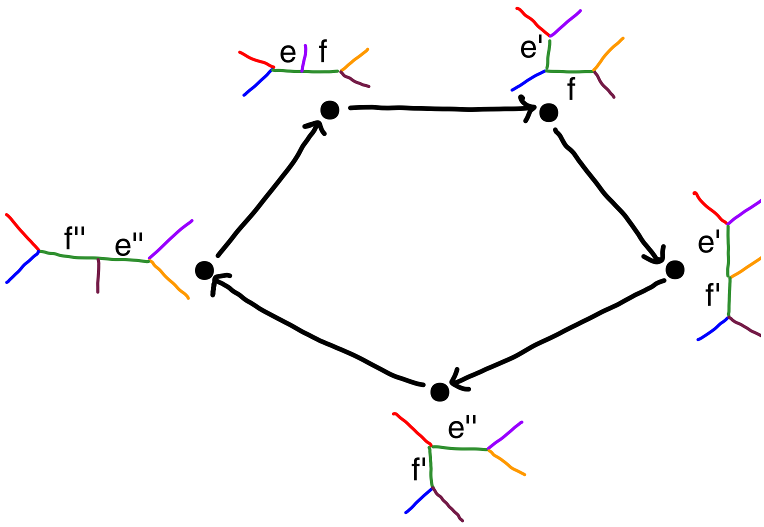
Definition 5.4. The *marked bordered fatgraph complex* is the complex with the following cell structure.

The vertices of this complex are (uncollapsed) marked genus g n-bordered fatgraphs. The edges of this complex are Whitehead moves. The faces are from the following two relations. First, performing Whitehead moves on two non-adjacent edges commutes. Second, if e and f are adjacent edges, performing Whitehead moves on e then f then e' then f' then e'', as labelled in Figure 5.2b, is trivial.

Figure 5.2 depicts the faces associated to these relations.



(a) Moves on non-adjacent edges commute.



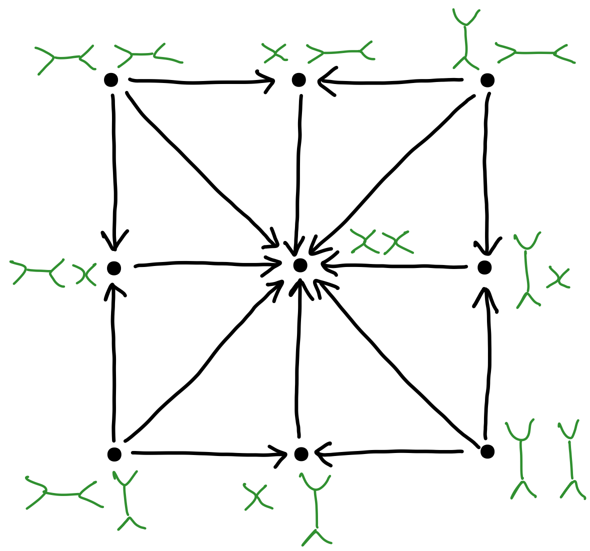
(b) The pentagon relation for adjacent edges.

Figure 5.2: The two types of faces in Theorem 5.5.

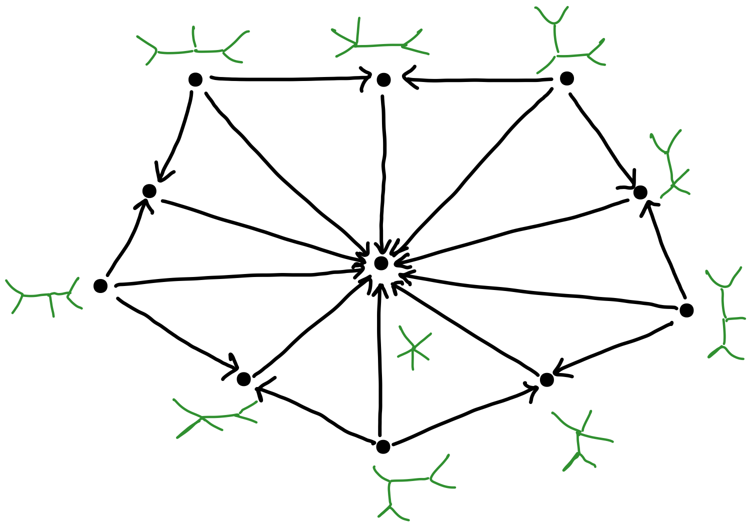
Theorem 5.5 (p.11, [BKP09]). *The 2-skeleton of $|EFat_{g,n}|$ is homotopy equivalent to the marked bordered fatgraph complex. The homeomorphism commutes with the action of $Mod(\Sigma_{g,n})$.*

Remark 5.6. The marked bordered fatgraph complex is constructed by taking the Poincaré dual of $|EFat_{g,n}|$. The two cells of this complex correspond to collapsed bordered fatgraphs formed by collapsing two edges of a bordered fatgraph. The two types of cell correspond to whether or not the two edges are adjacent. Figure 5.3 shows how the faces correspond to these collapsed bordered fatgraphs.

Notation 5.7. From here, $|EFat_{g,n}|$ refers to the marked bordered fatgraph complex.



(a) Non-adjacent edges.



(b) Adjacent edges.

Figure 5.3: The two types of face from taking the Poincaré dual to $|EFat_{g,n}|$. As all of the triangles are 2-cells since they correspond to composition of three morphisms.

Remark 5.8. As shown in Proposition 4.26, the natural action of $Mod(\Sigma_{g,n})$ on $EFat_{g,n}$ is free on marked bordered fatgraphs. Thus by Theorem 5.5, there is an induced $Mod(\Sigma_{g,n})$-action on the marked bordered fatgraph complex that is free on the set of vertices. As the action is free, a mapping class is uniquely determined by the image of any vertex in $|EFat_{g,n}|$ under its action.

Remark 5.9. A Whitehead move on e acts on the dual marking by removing the diagonal of the quadrilateral corresponding to the two vertices on each end of e, then

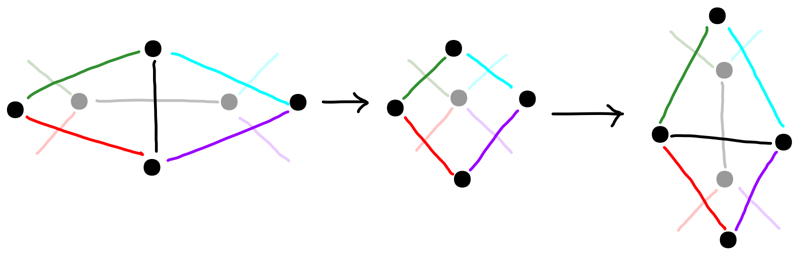


Figure 5.4: The effect of a Whitehead move on the dual.

adding the other diagonal of this quadrilateral. This is shown in Figure 5.4 for the dual of the subgraphs in Figure 5.1.

Consider the fundamental path groupoid of this complex based at the vertices in the cell structure. This will give us a groupoid that induces an infinite presentation of $Mod(\Sigma_{g,n})$.

Definition 5.10. The fundamental path groupoid of $EFat_{g,n}$, $\Pi_1(EFat_{g,n})$, is the full subgroupoid of the fundamental path groupoid of the marked bordered fatgraph complex whose elements start and end at vertices of the complex.

Proposition 5.11. *The following defines an infinite presentation of* $\Pi_1(EFat_{g,n})$. *Let* $\{\Gamma\}$ *be the (finite) set of* n*-bordered genus* g *fatgraphs. The generators of this groupoid are the Whitehead moves on* $\{\Gamma\}$*, starting at a fixed marking. There are three types of relations:*

1. Involution: *Performing a Whitehead move twice on the same edge is the identity.*
2. Commutativity: *The square face. If* e *and* f *are non-adjacent edges,*
$$W(e)W(f) = W(f)W(e).$$
3. Pentagon relation: *The pentagonal face. If* e *and* f *are adjacent edges,*
$$W(e)W(f)W(e')W(f')W(e'') = \text{id}\,.$$

Proof. The fundamental path groupoid is generated by the edges of this complex. As the complex is simply-connected, two paths in the complex are the same in the groupoid if and only if they start and end at the same points. Fix a vertex (that is, a marked bordered fatgraph). Two words in Whitehead moves beginning at this vertex are the same path if and only if they differ by faces in the complex, or by involution. That is, they are the same path if and only if they differ by the relations described above. □

Note that to determine whether two paths are the same, we only need to know the fatgraph they both start at, and not its marking. Thus a sequence of Whitehead moves that starts and ends at the same unmarked bordered fatgraph is trivial on the marking if and only if it is equal to a composition of the involution, commutativity and pentagon relations. Recall that an element of $Mod(\Sigma_{g,n})$ is determined by its action of the marking of a bordered fatgraph.

Corollary 5.12. *There is an infinite presentation of $Mod(\Sigma_{g,n})$, generated by Whitehead moves. Fix some n-bordered (unmarked) fatgraph Γ. The generators are all sequences of Whitehead moves that start and end at Γ. The relations are the involution, commutativity and pentagon relations described in Proposition 5.11.*

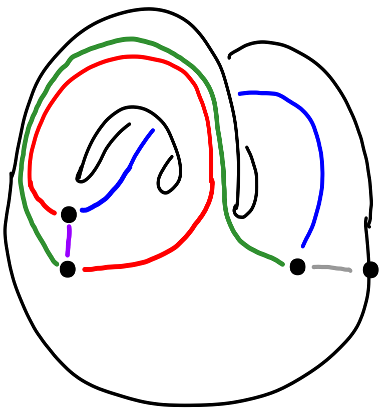
Composition of sequences of Whitehead moves does not correspond to composition of the mapping classes they each represent. We show this as follows. Let (W) be a sequence of Whitehead moves, Γ a fatgraph, and p_1 and p_2 two distinct markings of Γ. In general, (W) considered as starting at (Γ, p_1) corresponds to a different mapping class to (W) starting at (Γ, p_2).

For example, consider Figure 4.4 from Chapter 4. Here we display two different markings of the same fatgraph. If we perform a Whitehead move on the red edge, the result is again the same bordered fatgraph, so we have an associated mapping class. Figure 5.5 shows the result of this Whitehead move. To show the isomorphism between the graphs, the embedded fatgraphs have been recoloured so their edges have the same colours as in Figure 4.4a.

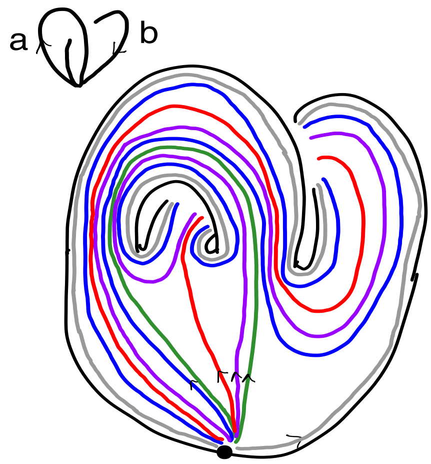
By inspecting the dual of the original markings in Figure 4.7, and comparing it to the duals after the Whitehead move in Figure 5.5, we can see that performing a Whitehead move on the red edge of the first marking gives us the homeomorphism acting on the fundamental group by $a \mapsto a$, $b \mapsto a^{-1}b$. However, performing a Whitehead move on the red edge of the second marking gives us $a \mapsto ab$, $b \mapsto b$, which is a distinct mapping class.

5.2 Whitehead moves and the generators of the fundamental path groupoid

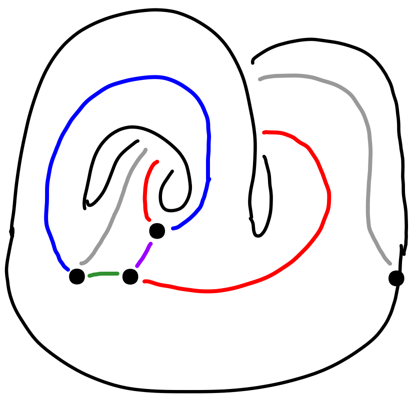
We wish to understand how the Whitehead moves alter the canonical generating set for the fundamental path groupoid of S described in Section 4.3.3. This will allow us to write the Whitehead moves in terms of chord slides in Chapter 6. In all of the figures in this section, we show the subgraph of Γ that is some edge e and the four edges adjacent to it. These adjacent edges may be tails or not. We show both vertices they are incident to, but not any other edges incident to those vertices.



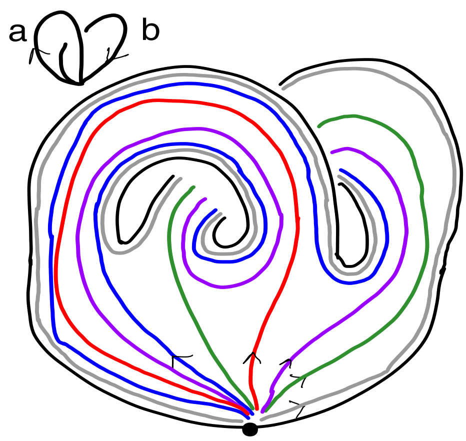
(a) After a Whitehead move on the red edge of Figure 4.4b.



(b) The dual of a.



(c) After a Whitehead move on the red edge of Figure 4.4c.



(d) The dual of c.

Figure 5.5: Performing the same Whitehead move on different markings gives a different mapping class.

Before we give the relevant theorem, we pause for a moment to discuss the cases that determine the effect of $W(e)$.

The effect of $W(e)$ on the generating set is determined by which of the edges adjacent to it are generators before and after the move. Let d through h be the duals of the edges with their preferred orientation (which we identify with the edges themselves) as labelled in Figure 5.6, where $W(e)$ takes the subgraph on the left to the subgraph on the right. We identify the edges f through h with the edges in the image since their duals and orientations do not change under the Whitehead move. The unlabelled edges have no effect on the result. Let u through z be the vertices labelled in Figure 5.6. We identify w, x, y and z with their images under the Whitehead move. Without loss of generality, we assume that the sector below e is the first one traversed. This is the sector that involves edges e and f before the Whitehead move.

The move $W(e)$ fixes the homotopy class of the graph, and fixes the homotopy class of all edges of the graph aside from e itself. Furthermore, as we have assumed that the sector below e is the first one traversed, when we traverse e for the first time we have no edges incident to v in the maximal tree. Thus we will add e to the tree, since this will not create a loop. As e is never a generator, the edges that are generators before the Whitehead move have the same dual before and after the move. Thus the effect of the Whitehead move on the generators is restricted to changing which edges are generators.

We build the maximal tree by traversing the boundary cycles of the graph. Whether or not an edge e is a generator is determined by, when we traverse it for the first time, there is already a path between $v(e)$ and $v(\overline{e})$, which are the two vertices it connects. If there is not, we greedily add it to the tree. If there is, adding it to the tree would create a loop, so it is a generator. The effect of the Whitehead move is to alter when in the boundary cycle traversal the edge e or its image d connects the vertices u and v.

Whether or not an edge is a generator before and after the Whitehead move depends on the order in which the four sectors around this edge are traversed, and whether or not there is already a path in the maximal tree between two vertices when we traverse the depicted edge between them. Recall that the sector below e is the first one traversed. We assign the following numbers to the anticlockwise order in which the four sectors are traversed:

1: 1234

2: 1243

3: 1432

4: 1342

5: 1324

6: 1423.

The sector traversal orders are illustrated in Figure 5.7. Each of the pairs of numbers corresponds to a sector order that differs by swapping 3 and 4. A tick on an edge means it is a generator, and a question mark means it may be a generator depending on other parts of the graph. Edges with neither of these must be in the maximal tree. The preferred orientation of each edge is indicated with an arrow.

Let the first vertex in sector 2 be q. This first vertex is w in sector orders 1 and 2, y in sector orders 3 and 4, and x in sector orders 5 and 6. We define three types of graphs depending on when we form paths in the maximal tree between z and q. Note that the vertex z is added to the maximal tree when we traverse sector 1 both before and after $W(e)$. These types are as follows.

Type a: after $W(e)$, a path is formed between z and q in the maximal tree before we traverse the first edge of sector 2. In this type, before the Whitehead move the first edge of sector 2 will be a generator as adding it to the maximal tree would create a loop.

Type b: after $W(e)$, there is not a path between z and q in the maximal tree before we traverse the first edge of sector 2. However, before we traverse the first edge of sector 3, there is such a path.

Type c: after $W(e)$, there is no path between z and q in the maximal tree before we traverse the first edge of sector 3.

As d joins z and q after we have traversed sector 1, d will be a generator after $W(e)$ in types b and c, but not type a.

In type b, we label an additional edge. Consider the path between u and q in the maximal tree after $W(e)$, which is formed in the maximal tree before we traverse sector 3. There is a unique path between q and z in the maximal tree that does not repeat any edges. Let h be the edge in this path such that, for all other edges p in the path, h (as an unoriented edge) is traversed in the boundary cycles after p (as an unoriented edge). Note that h is the last edge added to this path as we form the maximal tree.

We will discuss the restrictions each type places on the generators more thoroughly in the proof of Theorem 5.13. In the one boundary case, as in [ABP09, §3.1], there is only one boundary cycle, so the point that the maximal tree connects the adjacent edges does not vary. Thus, for a given sector ordering, only one type of the types s a, b and c is possible. This gives only six cases of sector orderings and types.

With n-bordered fatgraphs we must consider eighteen cases of sector orderings and types, up to inverses, though many of these have the same effect. We enumerate these in Figure 5.7, which shows the order in which the four sectors are traversed, and which edges adjacent to e may be generators before and after the Whitehead move. Each case in this figure shows (on the left) e and the four edges adjacent to it, and then (on the right) the result of the Whitehead move. The number of the case is the sector order

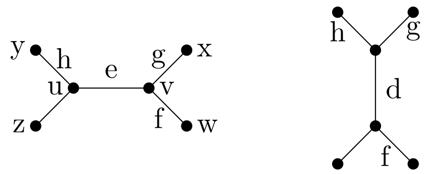


Figure 5.6: Labelling the duals of the edges involved in the Whitehead move.

and the letter is the type. We group them as we will show they have the same effects on the generating set, up to orientation of d.

Theorem 5.13. *Performing a Whitehead move on Γ has the following effect on the (unordered) canonical generating set for the fundamental path groupoid, X_Γ, depending on which of the cases it falls into.*

1. *Cases 1, 2, 3b, 4b, 5c and 6c have no effect.*
2. *Cases 3a and 4a replace c with d.*
3. *Cases 5a and 6a replace b with d.*
4. *Cases 5b and 6b replace h with d.*

Proof. A Whitehead move has no effect on the graph up to homotopy, so it does not change the dual marking of any edges except the one it occurs on. It can only change the generating set by altering which edges are generators. It does this by altering which paths between vertices are formed first, which in turn changes which edges are incorporated into the maximal tree. Let p and r be two paths between u and v in the maximal tree that are disjoint aside from at u and v. Suppose that p is the first path between u and v formed in the maximal tree before the Whitehead move, but after the Whitehead move, r is the first path formed. Then we have the following effect on the generators. Before the move, the last edge to be added to r was a generator. After it, the last edge to be added to p is a generator.

Note that the types a, b and c contain information about the order relative to the sectors in which that we create the first path between q and z that does not use e. As this path does not use e, it is invariant under the Whitehead move.

We can see the cases 1 and 2 are trivial as the move does not alter the order of connections between vertices. After traversing 2, either before or after the Whitehead move, we have a path in the maximal tree between u and all vertices adjacent to e aside from y. Once we traverse sector 3, we create a path from u to y.

For cases 3a and 4a, we have $q = y$. Now, before the Whitehead move, c is a generator. This is as, since this graph is type a, there is a path from y to z in the maximal tree before we traverse the unoriented edge c. Adding c to the maximal tree

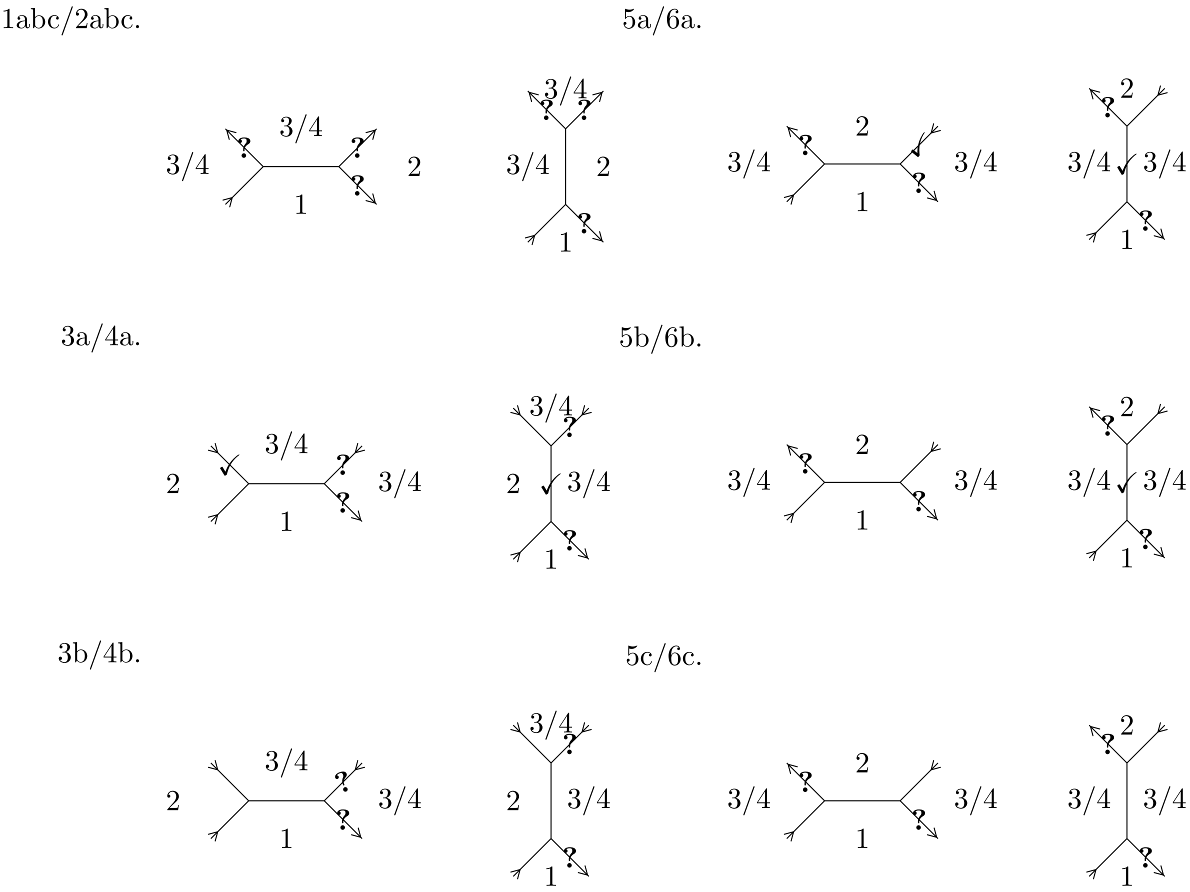


Figure 5.7: The twelve possible cases of Whitehead moves.

would then form a loop, since when we traversed sector 1 we added a path between z and u. In the image, as there is a path from q to z before we traverse sector 2, by the same reasoning d must be a generator. However, c is not a generator, as traversing for the first time (in sector 2) does not connect q and z. Thus the this move changes the path between y and z in the maximal tree such that we form a loop in the maximal tree when traversing d rather than c. Thus this Whitehead move takes c to d. It modifies no other loops in the graph since all other relative orders of connection remain fixed.

For cases 3b and 4b, as the edges adjacent to e are where q and z connect first, neither c nor d creates a loop. Furthermore, the path between q and z that is formed before the Whitehead move by c and d is still created when we traverse sector 2. Thus we have not altered the order of path creation, so the generating set is unchanged.

Cases 3c and 4c are impossible since there is a path from y to z that does not use e as soon as we traverse sector 2 (which is before sector 3).

For cases 5a and 6a, similarly to in cases 3a and 4a, there is a path from x to z before we traverse sector 2. Before the Whitehead move, adding b to the maximal tree would create a loop since it is the last edge in a second path between x and z. After the Whitehead move, the last edge traversed in the path between x and z through b is d. Thus b is replaced by d.

For cases 5b and 6b, we form a path between x and z after we traverse sector 2 but before we traverse sector 3. The first path formed between x and z is through e. Let h be the last edge traversed in this second path. By the reasoning at the beginning of this proof, before the Whitehead move, h is a generator as adding it to the maximal tree would form a loop. After the Whitehead move, however, h is not a generator. This is as we do not connect x and z through the subgraph pictured until we traverse d in sector 3. Thus the path through h occurs before the path in our subgraph. The relative order of traversing paths does not change under the Whitehead move for paths that do not pass through the vertices of the edge we perform the Whitehead move on. Before the Whitehead move, the path through h was the first path between x and z of all the paths that did not go through e. Thus after the Whitehead move, the path through h is the first path between x and z. The effect of the Whitehead move is to remove h as a generator and replace it with d, since the path through e or d is the first path between x and z before the Whitehead move but not the first path after the move.

Finally, for cases 5c and 6c, there are no connections between x and z before sector 3 aside from that at e. Thus no change occurs. As d is traversed before the images of any other paths, we do not alter the order of the connections between x and z. □

Note that one can express d in terms of b and c by the orientability and vertex compatability conditions to explicitly find the action on the generating set for all cases except 5b and 6b.

Chapter 6

Converting the fatgraph presentation to chord slides

In this chapter, we will develop a presentation of $Mod(\Sigma_{g,n})$ whose generators are chord slides. This presentation extends the work of Bene [Ben10] from surfaces with connected boundary to surfaces with disjoint boundary. As noted in Chapter 1, one could potentially use such a presentation to compute invariants in bordered Heegaard Floer homology.

We have a presentation of $\Pi_1(EFat_{g,n})$ whose generating set is Whitehead moves, as discussed in Chapter 5. This presentation gives us an infinite presentation of $Mod(\Sigma_{g,n})$ with the generating set also being sequences of Whitehead moves.

In this chapter, we will define a subclass of marked n-bordered fatgraphs, the chord diagrams. We will define a chord diagram complex $Chord^b_{g,n}$, and show that we have a fully faithful functor $\Pi_1(EFat_{g,n}) \to \Pi_1(Chord^b_{g,n})$ (Proposition 6.11). The image of a marked bordered fatgraph Γ under this map is uniquely determined by the associated canonical generating set for the fundamental path groupoid of $\Sigma_{g,n}$ (Theorem 6.18). We will use this result to show that the images of Whitehead moves in $\Pi_1(EFat^b_{g,n})$ under this functor are sequences of chord slides (a corollary of Theorem 6.20). Thus, we can translate our presentation of $Mod(\Sigma_{g,n})$ with Whitehead move generators to one with chord slide generators (Corollary 6.22). A potential application of this is the ability to compute Heegaard Floer invariants for a 3-manifold presented as an open book with disconnected binding. Computing these invariants does not require the relations of the presentation.

We describe a finite but unwieldy relation set for the presentation. We then give a simple candidate set of generating relations between chord slides, which one could potentially show generates all the relations by checking a finite number of cases that we did not have time to examine. Finally, we will verify that this generating set of relations gives us a correct presentation of $Mod(\Sigma_{0,2})$. Note that the work in [Ben10]

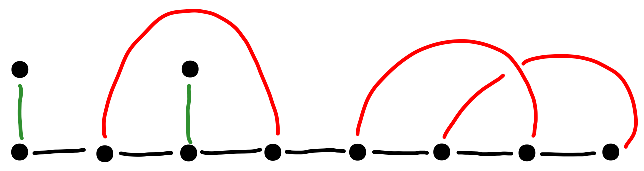


Figure 6.1: An example of a chord diagram. The core is drawn in black, the chords in red and the tails in green.

does not apply to this case as $n > 1$.

Comparing the results here to those in [Ben10], while many of our proofs follow the same ideas, the proof of Theorem 6.20 is substantially more involved. Many other results involve checking substantially more cases than in [Ben10], as our cases in Theorem 5.13 have effects that depend on global properties of the graph whereas all arguments are local in Bene's work.

6.1 A functor from fatgraphs to chord diagram

6.1.1 Chord diagrams

Definition 6.1. A *chord diagram* is an immersed graph in the plane, constructed as follows. First, take the segment $[0, k]$ on the x-axis in the plane, which we call the *core*. Each integer point $(a, 0)$ on the core is a vertex of the graph.

For every vertex $(a, 0)$ with $a > 0$, there is exactly one immersed line segment in the graph with $(a, 0)$ as one of its endpoints. This line segment has its other endpoint either at another integer point on the core, in which case it is called a *chord*, or at $(a, 1)$, in which case it is called a *tail*. All these line segments lie in the upper half of the plane. We further require that the line segment with an endpoint at $(0, 0)$ is a tail.

Notation 6.2. Let $\mathbf{c_1}$ and $\mathbf{c_2}$ be oriented chords. We write $\mathbf{c_1} \preceq \mathbf{c_2}$ if the initial point of $\mathbf{c_1}$ is one less than that of $\mathbf{c_2}$.

As a chord diagram is immersed in the plane, we can view it as a fatgraph. It has one univalent vertex for each tail, and all other vertices are trivalent aside from $(k, 0)$ and $(0, 0)$ which are bivalent.

We say a chord diagram is n-bordered if, viewed as a fatgraph, it has n boundary components and one tail in each boundary component. If we merge the two edges incident each of the bivalent vertices, the chord diagram also satisfies the valency conditions for an n-bordered fatgraph, and we consider it as one.

Figure 6.1 is an example of a chord diagram.

Definition 6.3. A *marked* n-bordered chord diagram is a marked n-bordered fatgraph that satisfies the conditions for a chord diagram, such that the first boundary component starts at the tail at $(0,0)$. The category $Chord^b_{g,n}$ is the full subcategory of $EFat^b_{g,n}$ whose objects are marked n-bordered chord diagrams, and whose morphisms are sequences of Whitehead moves taking a chord diagram to another chord diagram.

Definition 6.4. The groupoid $\Pi_1(Chord^b_{g,n})$ is the full subgroupoid of $\Pi_1(EFat_{g,n})$ that contains all elements of $\Pi_1(EFat_{g,n})$ satisfying the following criterion. Let f be a representative of a path in $\Pi_1(EFat_{g,n})$. Then the homotopy class of f is in $\Pi_1(Chord^b_{g,n})$ if f starts and ends at chord diagrams.

6.1.2 The branch reduction algorithm

We will now translate our presentation of $Mod(\Sigma_{g,n})$ in terms of bordered fatgraphs and Whitehead moves to one in terms of chord diagrams and chord slides. As we have discussed, there is a natural inclusion $Chord^b_{g,n} \hookrightarrow EFat^b_{g,n}$ induced by merging the two edges incident to the bivalent vertices in each chord diagram. From here, we will view each chord diagram as a fatgraph by this inclusion.

We describe a fully faithful functor, the branch reduction algorithm, that is the left inverse of this inclusion map. This extends the branch reduction algorithm for the $n = 1$ case [ABP09, Lemma 5.1].

Definition 6.5. Let the greedy maximal tree T_Γ of Γ be as defined in Definition 4.28. The *trunk* of T_Γ is defined by starting at the first tail vertex of Γ, then following the boundary cycle until we reach an edge not in T_Γ.

Note that the trunk is a line. Figure 6.2 depicts an example of the trunk of the tree.

Given a bordered fatgraph Γ, we define a procedure to take Γ to a chord diagram $C(\Gamma)$ via Whitehead moves only on the greedy maximal tree T_Γ. Denote the trunk of T_Γ by S_Γ.

Definition 6.6. Let e be an edge of $T_\Gamma - S_\Gamma$ such that e is incident to some vertex of S_Γ. We say that such an edge e is a *subtrunk* of T_Γ.

Notation 6.7. For any edge e, $T(e)$ is the subgraph of T_Γ that is disconnected from the first tail vertex in T_Γ if we delete e.

As T_Γ is a tree, $T(e)$ is also a tree and is non-empty as it contains one vertex of e.

Definition 6.8 (Branch reduction algorithm [ABP09])**.** We order the set of subtrunks e of T_Γ by the order in which, walking from the first tail along S_Γ, we pass an endpoint of e. Then the branch reduction algorithm proceeds as follows. Take the first such e that is not a tail. Perform a Whitehead move on this. After this, the set of subtrunks will change. Then take the first subtrunk in this new set, and repeat this process until there are no subtrunks remaining. The resulting fatgraph is called C_Γ.

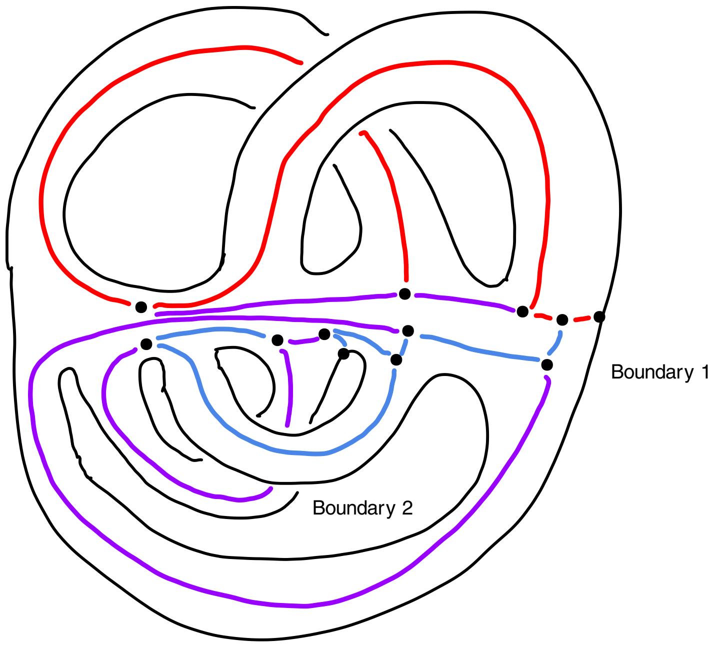


Figure 6.2: Finding the trunk of a fatgraph with two boundary components and genus two. The generators are drawn in purple, the trunk in red, and the rest of the maximal tree in blue.

Figure 6.3 illustrates this process on a twice-bordered genus one fatgraph.

Lemma 6.9. *The branch reduction algorithm halts and the resulting fatgraph is a chord diagram.*

Proof. The branch reduction algorithm increases the length of the trunk by one in each step, so must halt. When the algorithm halts, the trunk has length equal to the number of edges in the whole maximal tree, so we have a chord diagram. □

The branch reduction algorithm induces a map from a marked n-bordered fatgraph Γ to a marked n-bordered chord diagram $C(\Gamma)$. The chord diagram is n-bordered since n-bordered fatgraphs are closed under Whitehead moves. Now, following the description in Chapter 5, a sequence of Whitehead moves induces an action on the marking.

Proposition 6.10. *The generating set for the fundamental path groupoid $\Pi_1(\Sigma_{g,n})$ associated with Γ is the same as that associated with C_Γ.*

Proof. At each Whitehead move when we perform the branch reduction algorithm, we are performing a Whitehead move on an edge e in the tree, incident to a vertex where the other two edges f_1 and f_2 are also in the tree. Since f_1 and f_2 are in the trunk, they must be traversed before any other edge adjacent to e, as well as before e itself.

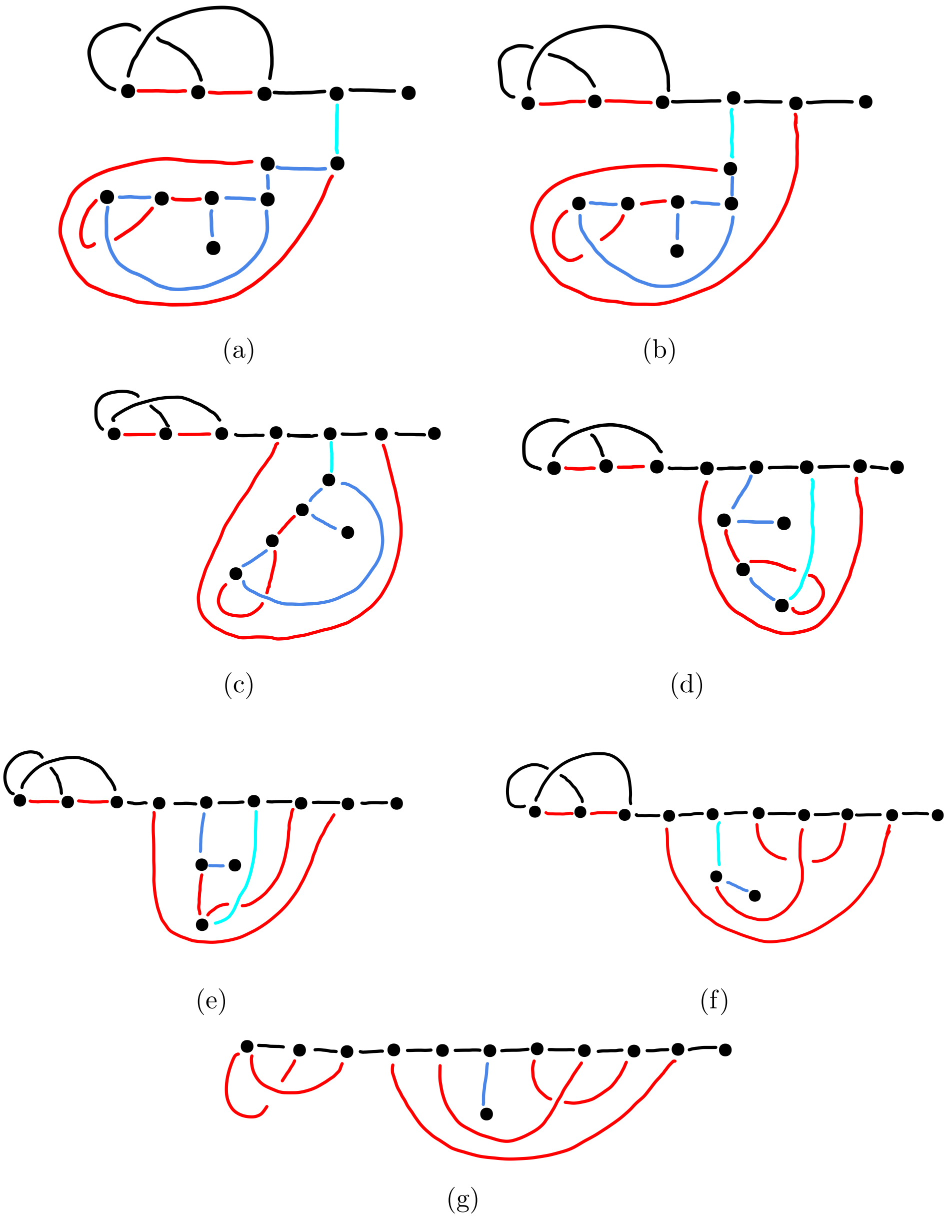


Figure 6.3: Performing branch reduction on a twice-bordered genus one fatgraph. The trunk is marked in red, and the sub-trees in blue. A Whitehead move is performed on the first edge of the first sub-tree, drawn in aqua, at each step.

Also, e must be traversed before the other two edges adjacent to it. Figure 6.4 depicts this situation. This is a case 1 or 2 Whitehead move as defined in Theorem 5.13, which by the same theorem is trivial on the generating set. □

Note that as all the chords and tails are in the upper half of the plane, the maximal tree of a chord diagram is precisely the core. Thus, if we embed this chord diagram in $\Sigma_{g,n}$, the associated generating set of $\Pi_1(\Sigma_{g,n}, \{p_i\})$ is the set of duals of the chords

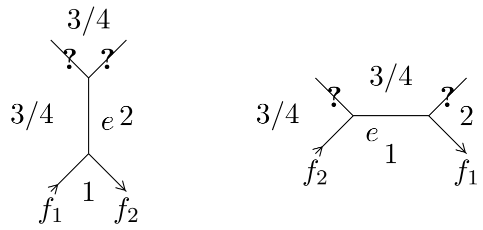


Figure 6.4: The possible orderings of edges in a Whitehead move in branch reduction.

and tails. We identify an edge e of Γ, not in the maximal tree, with the chord or tail of C_Γ that has the same dual.

Proposition 6.11. *The map*

$$C : EFat_{g,n} \to Chord^b_{g,n}$$

given by performing branch reduction on a marked bordered fatgraph Γ *is a left inverse of the inclusion* $Chord^b_{g,n} \hookrightarrow EFat_{g,n}$*, and the induced map between* $\Pi_1(EFat_{g,n})$ *and* $\Pi_1(Chord^b_{g,n})$ *is a fully faithful functor.*

For this proposition we interpret the fundamental path groupoids as categories, with objects the relevant vertices of the complex and morphisms the homotopy classes of paths between them.

Proof. First, C is trivially a left inverse of the inclusion, as the branch reduction algorithm has no effect on bordered fatgraphs that are already chord diagrams.

To show that C is a functor between the fundamental path groupoids, we give a map

$$C_{\Gamma_1,\Gamma_2} : \mathrm{Hom}_{\Pi_1(EFat_{g,n})}(\Gamma_1, \Gamma_2) \to \mathrm{Hom}_{\Pi_1(Chord^b_{g,n})}(C(\Gamma_1), C(\Gamma_2)).$$

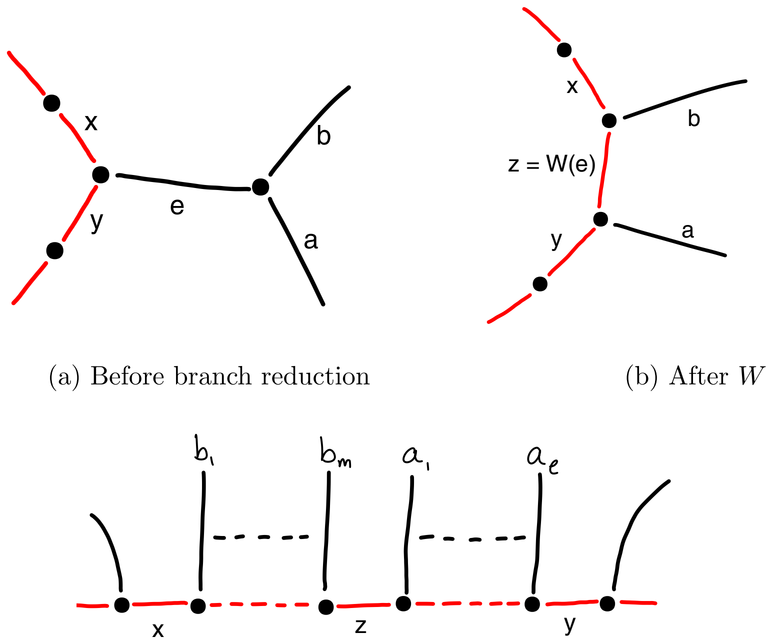
Suppose that $(W) : \Gamma_1 \to \Gamma_2$ is a morphism, that is a sequence of Whitehead moves.

Let $W(e)$ be a Whitehead move and e' be the image of e under $W(e)$. Then the inverse of $W(e)$ is $W(e')$. Thus if (W_{Γ_1}) is the (well-defined) sequence of Whitehead moves taking Γ_1 to $C(\Gamma_1)$, we have a diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{(W)} & \Gamma_2 \\ \downarrow{\scriptstyle (W_{\Gamma_1})} & & \downarrow{\scriptstyle (W_{\Gamma_2})} \\ C(\Gamma_1) & \overset{C(W)}{\dashrightarrow} & C(\Gamma_2) \end{array}$$

where $C_{\Gamma_1,\Gamma_2}(W)$ is defined by the composition

$$C(\Gamma_1) \xrightarrow{(W_{\Gamma_1})^{-1}} \Gamma_1 \xrightarrow{(W)} \Gamma_2 \xrightarrow{(W_{\Gamma_2})} C(\Gamma_2).$$



(a) Before branch reduction

(b) After $W(e)$

(c) After full branch reduction

Figure 6.5: Illustrations for the proof of Lemma 6.12. The edges drawn in red are in the trunk of the fatgraph.

One can then verify that this does indeed give a functor.

Now, to show that C is fully faithful, for fixed Γ_1 and Γ_2, we show that C_{Γ_1,Γ_2} : $Hom_{\Pi_1(EFat_{g,n})}(\Gamma_1,\Gamma_2) \to Hom_{\Pi_1(Chord^b_{g,n})}(C(\Gamma_1),C(\Gamma_2))$ is a bijection. If (W) is a sequence of Whitehead moves taking $C(\Gamma_1)$ to $C(\Gamma_2)$, let $C^{-1}_{\Gamma_1,\Gamma_2}(W)$ be the map

$$\Gamma_1 \xrightarrow{(W_{\Gamma_1})} C(\Gamma_1) \xrightarrow{(W)} C(\Gamma_2) \xrightarrow{(W_{\Gamma_2})^{-1}} \Gamma_2.$$

One can check this is a two-sided inverse for C_{Γ_1,Γ_2}, so the functor is fully faithful. □

Lemma 6.12. *Suppose that e is an edge of a bordered fatgraph Γ that is not in the trunk. Let $\{e_i\}_{i=1}^n$ be the oriented edges such that e_i or $\overline{e_i}$ is in the minimal generating set for the fundamental path groupoid, and the initial vertex $v(\overline{e_i})$ of each edge is in the subtree $T(e)$. Order these edges in clockwise order around $T(e)$. (Note some pairs of these edges may be the two orientations of the same unoriented edge.)*

After branch reduction, the chord diagram C_Γ has $e_1 \preceq \cdots \preceq e_n$, and in the dual marking, we have $e = e_1 \cdots e_n$.

Proof. We show this by induction on the depth of $T(e)$, where e has a fixed orientation away from the core. Note that $T(e)$ as defined cannot contain the first tail unless e is the first tail, and that as the branch reduction algorithm does not affect the generating set, the sequence $e_1, \cdots, e_n$ corresponds to edges in a well-defined way during branch reduction. Each step of the branch reduction algorithm is a Whitehead move on a subtrunk. Note that this does not alter any part of the subtrees of the maximal tree aside from the edges immediately adjacent to the subtrunk. Thus, if we perform branch reduction up to the point where e is adjacent to the core, there is no change to $T(e)$. Without loss of generality, we can thus assume that e is either a subtrunk or a generator adjacent to the core, and if e is a subtrunk, that the fatgraph is such that the next step of branch reduction is $W(e)$. Let x be the edge in the core that is incident to the start point of e that is traversed first by the boundary cycle, and y the edge in the core incident to the start point of e that is traversed second.

First, if e is a generator (that is, if $T(e)$ is of depth 0) then $T(e)$ contains one vertex of e and $\{e_i\} = \{e\}$. As e is a generator, it will be a chord or tail. Also, note that the chord e is still incident to the endpoint of x with its preferred orientation, and that as branch reduction does not involve performing any Whitehead moves on the core and does not change the orientation of edges in the core, this will remain true in the chord diagram. Similarly, the chord e is incident to the start point of y with its preferred orientation, and this will remain true in the chord diagram.

Suppose we have shown the result for edges e' with $T(e')$ having depth d. Let k be the number of generators incident to $T(e')$. Suppose furthermore that for edges with $T(e')$ having depth d, after branch reduction, e'_1 is incident to the endpoint of x and e'_k is incident to the start point of y.

Suppose that $T(e)$ is of depth $d+1$. We wish to show that the inductive result still holds. Note that e is not a generator, as otherwise $T(e)$ would be of depth 0. As e is a subtrunk we have the order of sector traversal shown in Figure 6.5a.

After performing $W(e)$, we have the subgraph shown in Figure 6.5b where the image of e under the Whitehead move is labelled z. Note that $T(b)$ and $T(a)$ are of depth at most d. Define $\{b_i\}_{i=1}^{m}$ and $\{a_j\}_{j=1}^{\ell}$ as with e_i, so they are the generators with a vertex in $T(b)$ and $T(a)$ respectively, with clockwise ordering. By induction, after branch reduction, along the core we have $b_1 \preceq \cdots b_m$ such that b_1 is incident to the endpoint of x and b_m is incident to the start point of z. Also, along the core $a_1 \preceq \cdots a_\ell$ such that a_1 is incident to the endpoint of z and a_ℓ is incident to the start point of y. We illustrate this in Figure 6.5c. Thus, along the core we have $b_1 \preceq \cdots b_m \preceq a_1 \preceq \cdots a_\ell$. Furthermore, b_1 is incident to the endpoint of x and a_ℓ is incident to the start point of y. Thus the inductive result holds.

We now show that $e = e_1 \cdots e_n$. We show this also by induction. Fix the orientation of all edges in $T(e)$ as pointing towards deeper levels of the tree, that is, away from the vertex of e that is still connected to the first tail by a path in the maximal tree after deleting e. We have already fixed the orientations of the e_i as being away from the tree.

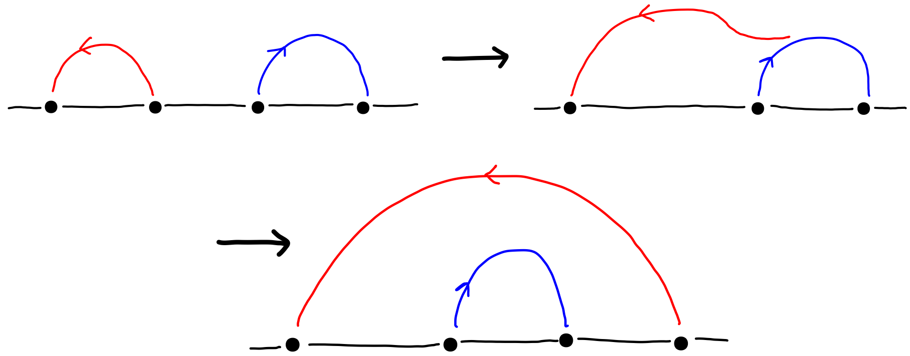


Figure 6.6: Illustrating a chord slide of c, the red chord, over d, the blue chord.

For e a generator, trivially $e = e_1$. For the inductive step, suppose that we have $e' = e'_1 \cdots e'_m$ for all e' such that $T(e')$ is of depth d. Let e be an edge with $T(e)$ of depth $d+1$. Let a and b be the other two edges incident to its endpoint with cyclic ordering as in Figure 6.5a, oriented away from e. By the orientation and vertex compatability conditions of a dual marking, $e\bar{a}\bar{b}$ is the identity at the start point of e. Thus $e = ba$. We have the depths of $T(a)$ and $T(b)$ at most d. By induction,

$$e = b_1 \cdots b_m a_1 \cdots a_\ell = e_1 \cdots e_n$$

since the b_i and a_j are precisely the e_i in clockwise order around $T(e)$. □

6.2 Translating Whitehead moves to chord slides

We now define an operation on chord diagrams, the *chord slide*, under which $Chord^b_{g,n}$ is closed.

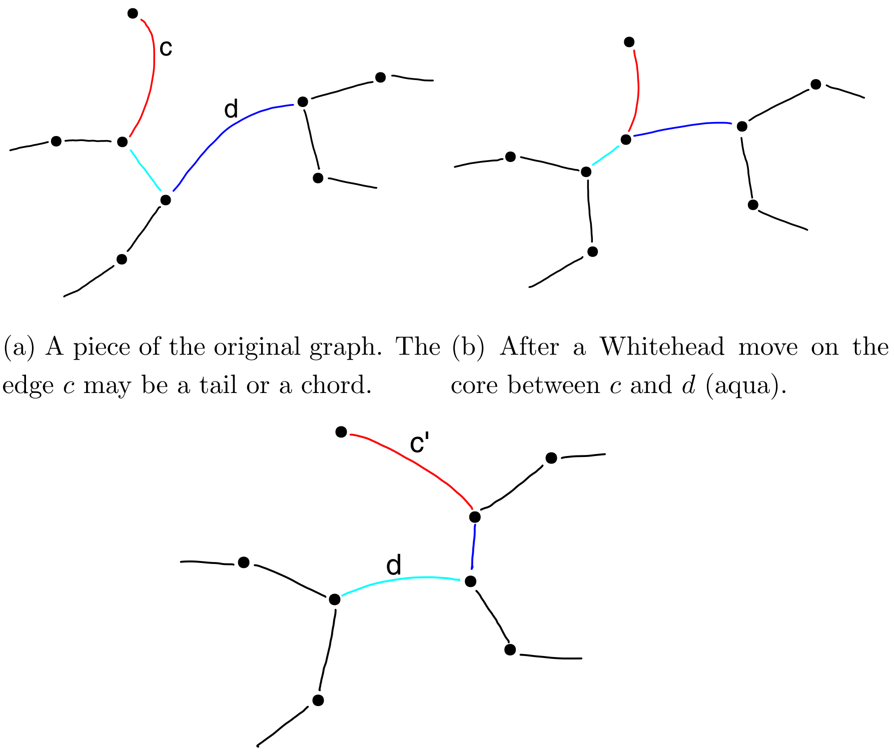
Definition 6.13. Let C be a chord diagram. Let **c** be either a tail or chord of C, and **d** a chord in C, such that an endpoint c of **c** and an endpoint d of **d** are adjacent. A *chord slide* of c over d is performed by sliding the end of **c** from c to d, then over **d**.

Figure 6.6 is an example of a chord slide.

We wish to present $\Pi_1(EFat_{g,n})$ with the generators being chord slides in $Chord^b_{g,n}$ rather than Whitehead moves in $EFat^b_{g,n}$. This will extend the work of Bene [Ben10] from $n = 1$ to $n \geq 1$. As noted in Chapter 1, this has a number of potential applications to low-dimensional topology.

Lemma 6.14. *Any chord slide on a n-bordered chord diagram C can be written as two Whitehead moves on the edges of C.*

When $n = 1$, this is [Ben10, Observation 4.2].



(a) A piece of the original graph. The edge c may be a tail or a chord.

(b) After a Whitehead move on the core between c and d (aqua).

(c) After a Whitehead move on d (blue).

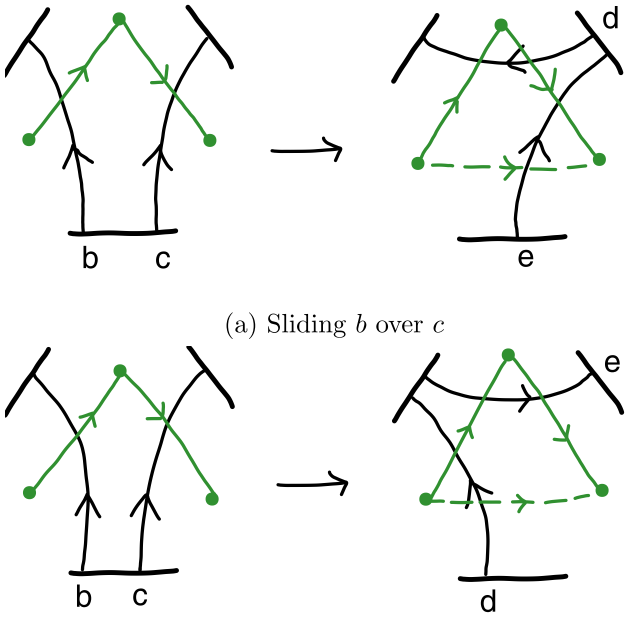
Figure 6.7: A chord slide of c over d written in terms of Whitehead moves. The resulting chord diagram has c replaced with c'.

Proof. For a pictorial proof sketch, we direct the reader to Figure 6.7. This figure depicts a portion of a chord diagram containing two adjacent chords, **c** and **d**. After performing a Whitehead move on the core segment between **c** and **d**, and then a Whitehead move on the image of **d**, we see we have reached a chord diagram with the chord **d** and a new chord, from the endpoint of **c** to a point next to **d**. This is the same as the result of a chord slide of **c** over **d**. Note that the Whitehead moves have the same effect on the marking as the chord slide.

Similarly, one can check that performing a Whitehead move on the core segment between **c** and **d**, and then one on the image of **c**, has the same effect as a chord slide of **d** over **c**. □

Lemma 6.15. *Let b and c be oriented chords (with b possibly a tail), with orientations fixed as in Figure 6.8. Sliding b over c fixes b considered as a dual edge, and takes c to bc. Sliding c over b fixes the dual of c, and takes b to bc.*

Proof. Figure 6.8 depicts the chord slides in this lemma, as well as the duals of the chords with orientations as in Definition 4.6. Note that as the chord diagram is em-



(a) Sliding b over c

(b) Sliding c over b

Figure 6.8: The action of chord slides on the dual.

bedded as a spine, its complement is a series of disjoint contractable components.

Let d be the image of b after the slide, and e the image of c. In the left of the figure, we show the chords before the slide. Their duals are drawn in green. On the right, we show the chords after the slide. We draw the dual of e with a dotted line. We see that the dual edge of d is the same as the dual edge of b, as this is the unique homotopy class of paths between the vertices corresponding to the faces that d separates that passes through d once, with the relative orientation shown in Figure 4.6. The dual edge of e, which is added as a dotted line, is homotopic to the composition of the duals of b and c, which is bc.

By the same reasoning, the dual of b after sliding c over b is bc while the dual of c remains fixed. □

Corollary 6.16 (Figure 4.2 [Ben10]). *There are six cases of chord slides where we take chords with their preferred orientations, shown in Figure 6.9. In each case, we slide c over d. These cases are determined by the relative order in which the three labelled sectors are traversed. For all cases, the generator d is removed, and some word in c and d is added. They have the following effects:*

1. $d \mapsto cd$

2. $d \mapsto \bar{c}d$

3. $d \mapsto dc$

4. $d \mapsto \overline{d}c$

5. $d \mapsto c\overline{d}$

6. $d \mapsto d\overline{c}$.

Remark 6.17. These cases are fully determined by local properties, in contrast to the cases of Whitehead moves in Theorem 5.13. We can calculate the effect on the generating set of the inverses of these six cases. For the inverses of the cases in Figure 6.9, where we slide c over d, we have:

1. $d \mapsto \overline{c}d$
2. $d \mapsto cd$
3. $d \mapsto d\overline{c}$
4. $d \mapsto c\overline{d}$
5. $d \mapsto \overline{d}c$
6. $d \mapsto dc$.

6.2.1 Translating Whitehead moves between chord diagrams into chord slides

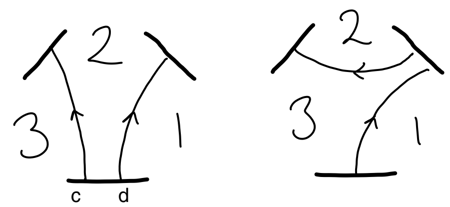
We now show that we can write any sequence of Whitehead moves between two chord diagrams as a sequence of chord slides.

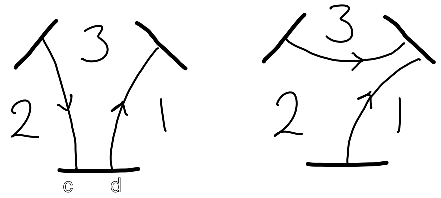
We first prove that a chord diagram is entirely determined by the associated generating set for $\Pi_1(\Sigma_{g,n}, \{p_i\})$.

Theorem 6.18. *Let X be a choice of minimal generating set for $\Pi_1(\Sigma_{g,n}, \{p_i\})$. If there exists a marked chord diagram whose associated set of generators for $\Pi_1(\Sigma_{g,n}, \{p_i\})$, up to reordering and inverses, is X, then this chord diagram is unique.*

To prove this, we first show that in $\Pi_1(\Sigma_{g,n}, \{p_i\})$ there is a unique reduced word in a given set of generators representing any element. Here *reduced* means that for all $x \in X$, the word does not contain a substring xx^{-1} (or $x^{-1}x$).

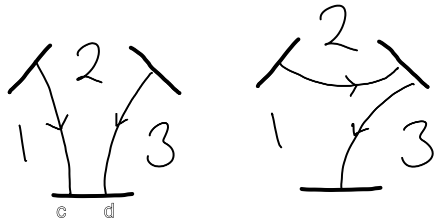
Proposition 6.19. *Let w be an element of $\Pi_1(\Sigma_{g,n}, \{p_i\})$. Let X be a minimal generating set for $\Pi_1(\Sigma_{g,n}, \{p_i\})$. Then there is a unique reduced word in X representing w.*

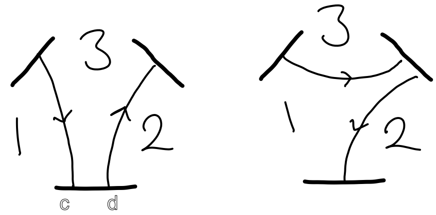




(a) Case 1

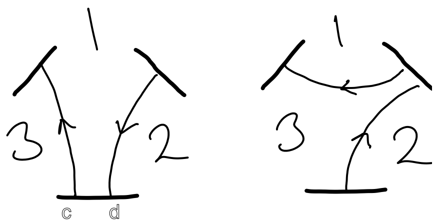
(b) Case 2

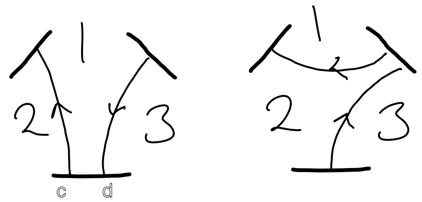




(c) Case 3

(d) Case 4





(e) Case 5

(f) Case 6

Figure 6.9: The six cases of chord slides. Note these are characterised by local properties, unlike Whitehead moves. Each of these slides takes the diagram on the left to the diagram on the right by sliding c over d. From [Ben10, Figure 4.2].

Proof. Note that as $\Sigma_{g,n}$ has boundary, the fundamental group is free. Thus any element of the fundamental group is a unique reduced word in any minimal set of group generators.

We wish to show that if two reduced words in X are equal as elements of the fundamental path groupoid, they are equal as words. Without loss of generality, we will only consider words that start and end at the marked point on the first boundary component. (Otherwise, we can compose with some generators such that this is the case.)

Now, we give a method to derive a minimal generating set for the fundamental group from X, and write these words in terms of the fundamental group generators.

First, we can think of the elements of X as edges in a graph with n vertices on $\Sigma_{g,n}$, where the vertices are the n points in $\{p_i\}$, one on which is on each boundary component. Pick a minimal spanning tree for this graph. Let P be the elements of X

that are in this tree.

Now, for each basepoint k, we have a unique reduced word in elements of P that is a path from the first basepoint to k, corresponding to the unique path from the first basepoint to k in the minimal spanning tree. Let this path be p_k. Note that the reduced form of $(p_j)^{-1}p_k$ is the unique reduced word in elements of P from j to k.

Write $X = \{x_1, \ldots, x_p\}$. Let s_i be the start point of x_i, and e_i the endpoint of x_i. Now, let

$$\widetilde{X} = \{p_{s_i} x_i p_{e_i}^{-1} \mid x_i \in X\}.$$

Each element of $\widetilde{X}$ is a homotopy class of paths starting and ending at the first basepoint.

If x_i is used in constructing the paths p_k then $p_{s_i} x_i p_{e_i}^{-1}$ is the trivial path. Otherwise, this word is non-trivial. If it were trivial, we would have $x_i = p_{s_i}^{-1} p_{e_i}$. But p_{s_i} and p_{e_i} are words in $X - \{x_i\}$. We have then written x_i in terms of the other generators, which is a contradiction since we assumed X was minimal. Thus $\widetilde{X}$ contains $(2g+2n-2)-(n-1) = 2g + n - 1$ non-trivial elements.

Now, we show that the non-trivial elements of $\widetilde{X}$ are a minimal generating set for the fundamental group. As $|\widetilde{X}| = 2g + n - 1$, which is the size of such a set, it suffices to show that $\widetilde{X}$ does indeed generate the fundamental group.

Let ϕ be an element of $\pi_1(\Sigma_{g,n})$. Then we can consider ϕ as an element of the fundamental path groupoid, and write it as a word $w_1 \cdots w_n$. Let t_i and f_i be the start and end-point of w_i respectively. Note that $f_i = t_{i+1}$, and $t_1 = f_n = 1$. Now,

$$\phi = \left(p_{t_1} w_1 p_{f_1}^{-1}\right) \left(p_{t_2} w_2 p_{f_2}^{-1}\right) \cdots \left(p_{t_n} w_n p_{f_n}^{-1}\right)$$

by observing that p_1 is trivial and otherwise the inserted p_i elements cancel. Thus, we can write any element of $\pi_1(\Sigma_{g,n})$ as a word in elements of $\widetilde{X}$, so $\widetilde{X}$ is a minimal generating set of the fundamental group.

Now, let $w_1 \cdots w_n$ and $v_1 \cdots v_m$ be two reduced words in X that start and end at the first basepoint and are equal as elements of the fundamental path groupoid. We wish to show they are equal as words.

Considering $w_1 \cdots w_n$ as an element of the fundamental group, we can write it in terms of $\widetilde{X}$ as

$$w_1 \cdots w_n = \left(p_{t_1} w_1 p_{f_1}^{-1}\right) \left(p_{t_2} w_2 p_{f_2}^{-1}\right) \cdots \left(p_{t_n} w_n p_{f_n}^{-1}\right).$$

The i^{th} character in this will be trivial if and only if w_i is used in constructing the paths p_k. Removing all such entries by letting (n_i) be the subsequence of indices such that w_{n_i} is not used in the paths p_k, we have

$$w_1 \cdots w_n = \left(p_{t_{n_1}} w_{n_1} p_{f_{n_1}}^{-1}\right) \left(p_{t_{n_2}} w_{n_2} p_{f_{n_2}}^{-1}\right) \cdots \left(p_{t_{n_k}} w_{n_k} p_{f_{n_k}}^{-1}\right).$$

We know that this word in $\widetilde{X}$ contains no trivial characters. Furthermore, it is reduced. If it were not, we would have some substring of $w_1 \cdots w_n$ of the form

$$w_j \alpha_1 \cdots \alpha_p w_k$$

where the α_i are used to build the paths p_k, and w_j and w_k are not. This converts to $\left(p_{t_j} w_j p_{f_j}^{-1}\right)\left(p_{t_k} w_k p_{f_k}^{-1}\right)$ upon concatenation, such that

$$\left(p_{t_j} w_j p_{f_j}^{-1}\right) = \left(p_{t_k} w_k p_{f_k}^{-1}\right)^{-1}.$$

Now, if $w_j \neq w_k^{\pm 1}$, this gives us an expression for w_k in terms of the other generators, a contradiction. If $w_j = w_k$, we know $t_j = t_k$ and $f_k = f_j$. Then $(p_{t_j} w_j p_{f_j}^{-1})^2$ is the trivial loop at 1, which is a contradiction as the fundamental group of a surface with boundary is free. Finally, if $w_j = w_k^{-1}$, $f_j = t_k$. But then the the path $\alpha_1 \cdots \alpha_p$ is from f_j to t_k which is trivial, so the original word contained a substring $w_j w_j^{-1}$, which is a contradiction as we assumed it was reduced.

Thus, this word in $\widetilde{X}$ is reduced.

Now, by the same argument, if s_i and e_i are the start and end-points of v_i respectively,

$$v_1 \cdots v_m = \left(p_{t_{m_1}} v_{m_1} p_{f_{m_1}}^{-1}\right)\left(p_{t_{m_2}} v_{m_2} p_{f_{m_2}}^{-1}\right) \cdots \left(p_{t_{m_k}} v_{m_k} p_{f_{m_k}}^{-1}\right).$$

and this is a reduced word in $\widetilde{X}$.

Now, as $w_1 \cdots w_n = v_1 \cdots v_m$ as elements of the fundamental path groupoid, they are equal in the fundamental group. Thus

$$\left(p_{t_{n_1}} w_{n_1} p_{f_{n_1}}^{-1}\right) \cdots \left(p_{t_{n_k}} w_{n_k} p_{f_{n_k}}^{-1}\right) = \left(p_{t_{m_1}} v_{m_1} p_{f_{m_1}}^{-1}\right) \cdots \left(p_{t_{m_k}} v_{m_k} p_{f_{m_k}}^{-1}\right).$$

As the fundamental group of a surface with boundary is free, there is a unique reduced form of any element. Thus, these are equal as words in $\widetilde{X}$. We have

$$p_{t_{n_i}} w_{n_i} p_{f_{n_i}}^{-1} = p_{t_{m_i}} v_{m_i} p_{f_{m_i}}^{-1}$$

with each word having the same number of characters. The same reasoning used in showing the word in $\widetilde{X}$ was reduced implies that, $w_{n_i} = v_{m_i}$ for all i.

Once we have determined the v_{m_i}, the elements between them are determined, as these are the paths from the endpoint of v_{m_i} to the start-point of $v_{m_{i+1}}$ in the elements $\{\alpha\}$. Thus, $w_1 \cdots w_n = v_1 \cdots v_m$ as words in X, as required. □

Using this proposition, we can now show that chord diagrams are determined by their associated generating set (up to inverses and reordering).

Proof of theorem. Note that for a chord diagram Γ, the associated set of generators X is the set of duals of the chords, and duals of all tails but the first.

By the orientation and vertex compatibility conditions on the dual marking, we can read off a word in the generators equal to the dual of the first tail as follows. From the first tail, walk along the core. At each vertex, append the generator of its chord or tail, oriented pointing away from the vertex.

For example, if we label the chords in Figure 6.3 as c_i from right to left, and the second tail as t_2, we find that the dual of the first tail t is given by

$$t = \overline{c_1}\,\overline{c_2}\,\overline{c_3} c_2 \overline{t_2} c_3 c_1 c_4 \overline{c_5}\,\overline{c_4} c_5.$$

Observe that this word is reduced, as if it were not, the start and end of a chord would be adjacent. Then this chord would contain a boundary component with no tail, which is impossible by the definition of an n-bordered fatgraph.

Now, by Proposition 6.19, there is a unique such reduced word. Thus any chord diagram with this generating set, up to reordering and inverses, must have chords corresponding to these generators in this order. But this determines a chord diagram and its marking, as required. □

We now prove that chord slides generate all sequences of Whitehead moves between chord diagrams, generalising Theorem 5.3, [Ben10] from $n = 1$ to $n \geq 1$.

Theorem 6.20. *Let $W(e) : \Gamma_1 \to \Gamma_2$ be the Whitehead move on an edge e of Γ_1. The associated morphism $C_{W(e)} : C_{\Gamma_1} \to C_{\Gamma_2}$ in $\Pi_1(Chord^b_{g,n})$ has a representative which is a sequence of chord slides from $C_{\Gamma_1} \to C_{\Gamma_2}$.*

This representative is a series of chord slides along one chord, which is incident to a leaf of $T(b)$ or $T(c)$ before the branch reduction.

In the proof of this theorem, if $\mathbf{e}$ is an oriented edge, then e is the edge with no assigned orientation.

Proof. We have shown that a chord diagram is determined entirely by the associated generating set of the fundamental path groupoid, and there is a unique morphism in $\Pi_1(Chord^b_{g,n})$ between any two chord diagrams. Thus it suffices to show that for any Whitehead move, there is a sequence of chord slides that has the same effect on the fundamental path groupoid generating set. We break the Whitehead moves into cases as in Theorem 5.13.

In cases 1, 2, 3b, 4b, 5c and 6c the move has no effect, so this is the trivial morphism.

Let $\mathbf{b}$ and $\mathbf{c}$ be oriented edges defined as follows. The oriented edges $\mathbf{b}$ and $\mathbf{c}$ are the unoriented edges b and c as labelled in Figure 5.6 with orientations pointing away from e, the edge we perform the Whitehead move on. Note that $\mathbf{b}$ and $\mathbf{c}$, if they are in the tree, are not in the trunk. This is as the first sector traversed around the edge e does not involve either of them. Let $\{\mathbf{b}_i\}_{i=1}^m$ be the oriented edges whose initial vertex is in the tree $T(\mathbf{b})$, ordered such that $\mathbf{b_i} \preceq \mathbf{b_{i+1}}$, as in Lemma 6.12. Recall from the lemma that

$$\mathbf{b} = \mathbf{b}_1 \cdots \mathbf{b}_m.$$

Similarly, for $\{\mathbf{c}_i\}_{i=1}^n$ the generators whose initial vertices are in $T(\mathbf{c})$, ordered such that $\mathbf{c_i} \preceq \mathbf{c_{i+1}}$, we have

$$\mathbf{c} = \mathbf{c}_1 \cdots \mathbf{c}_n.$$

Consider cases 3a and 4a. In these cases, c is a generator and is replaced by d in the generating set. Let $\mathbf{d}$ be d with its preferred orientation. We can write $\mathbf{d}$ in terms of $\mathbf{b}$ and $\mathbf{c}$ by using the orientability and vertex compatibility conditions on the dual marking. Then 3a replaces $\overline{\mathbf{c}}$ with $\overline{\mathbf{b}}\overline{\mathbf{c}}$ and 4a replaces $\overline{\mathbf{c}}$ with $\mathbf{b}\overline{\mathbf{c}}$. If we slide the $\mathbf{b}_i$ over $\mathbf{c}$, we find that this chord slide has the same effect.

This slide would be impossible only if some $\mathbf{b}_i = \overline{\mathbf{c}}$. However, if this were the case, we would be replacing $\mathbf{c}$ in the generating set by $x\mathbf{c}y\overline{\mathbf{c}}$ for x and y sequences of the form $\overline{\mathbf{b}_j \mathbf{b}_{j-1}} \cdots$. This does not give us a generating set for the fundamental path groupoid. If it did, we would be able to write c as a word in the $\mathbf{b_i}$ and $x\mathbf{c}y\overline{\mathbf{c}}$. As x is a word in the $\mathbf{b_i}$, we would equivalently be able to write c as a word in $\mathbf{c}y\overline{\mathbf{c}}$ and the $\mathbf{b_i}$. But this is impossible. We know there is a unique reduced form of any word in the fundamental path groupoid in a set of generators. Any concatenation of $\mathbf{c}y\overline{\mathbf{c}}$ with some $\mathbf{b_i}$ creates a reduced word with an equal number of copies of $\mathbf{c}$ and $\overline{\mathbf{c}}$. Thus we can never have a reduced word with one copy of $\mathbf{c}$, so in particular $\mathbf{c}$ is not generated by this set. Thus this set does not generate the fundamental path groupoid, which is a contradiction.

Next, consider cases 5a and 6a. In these cases, b is a generator and is replaced by d. As with the previous case, we can write $\mathbf{d}$ in terms of $\mathbf{b}$ and $\mathbf{c}$. If we slide the $\mathbf{c}_j$ over $\mathbf{b}$, one can check that (with preferred orientations) this has the desired effect. By the reasoning in case 3a/4a, this is always possible.

Finally, we consider cases 5b and 6b separately. Let x be defined as in Theorem 5.13, with its preferred orientation. Let $\mathbf{d}$ be d with its preferred orientation. We know that $x \mapsto d$ under the Whitehead move. Furthermore, we can write d as a word in the $\mathbf{b}_i$ and $\mathbf{c}_j$. As the set of generators both generates the fundamental path groupoid of S and is minimal, we can show that $x = \mathbf{b}_i$ or $x = \mathbf{c}_j$ for some i or j. If not, the set of generators after the Whitehead move is not minimal. This is as d is a word in $\mathbf{b}_i$ and $\mathbf{c}_j$, so if we have removed none of these generators, we can write the generator d as a word in the other generators. This is a contradiction.

For case 5b, we traverse the sectors in the order 1324 going anticlockwise. Then $\mathbf{d} = \mathbf{cb}$. After branch reduction, we arrive at a chord diagram where $\mathbf{c}_1, \ldots, \mathbf{c}_n, \mathbf{b}_1, \ldots, \mathbf{b}_m$ are adjacent (by Lemma 6.12). Suppose we slide all the $\mathbf{c}_j$ and $\mathbf{b}_i$ over x, aside from of course x itself. Note that some of these slide from the left and some from the right. However, as all chords are oriented away from the core, by Lemma 6.15, these chord slides take x to the concatenation of all these chords (including x itself) along the core, which is

$$\mathbf{c}_1 \cdots \mathbf{c}_n \mathbf{b}_1 \cdots \mathbf{b}_m = \mathbf{d}$$

as required. By the reasoning in case 3a/4a, this is always possible.

Finally, for case 6b, we traverse the sectors in the order 1423 going anticlockwise. We have $\mathbf{d} = \overline{\mathbf{b}\mathbf{b}}$. After branch reduction, we arrive at a chord diagram where $\mathbf{c}_1, \ldots, \mathbf{c}_n, \mathbf{b}_1, \ldots, \mathbf{b}_m$ are adjacent (by Lemma 6.12). Sliding all of these over x, we take x to the concatenation of all these chords along the core, which is $\overline{\mathbf{d}}$. However, the preferred orientation of x after the chord slide is towards the core, so the dual of x with its preferred orientation is $\mathbf{d}$. As shown previously, this slide is always possible.

In any of the cases of this procedure, we slide a series of chords over the chord associated to b, c or x (which is equal to $\mathbf{b}_i$ or $\mathbf{c}_j$) in C_{Γ_1}. All of these are incident to leaves of either $T(b)$ or $T(c)$ (as for example if b is a generator, $T(b)$ is a vertex of b). □

The following statement is then a straightforward corollary, as sequences of Whitehead moves between chord diagrams can be written as chord slides, and the morphisms in $\Pi_1(Chord^b_{g,n})$ are precisely these sequences of Whitehead moves.

Corollary 6.21. *The morphisms of* $\Pi_1(Chord^b_{g,n})$ *are generated by chord slides.*

6.3 Presenting the mapping class group in chord slides

As a corollary of Proposition 5.11 and Corollary 6.21, we have the following result.

Corollary 6.22. *The group* $Mod(\Sigma_{g,n})$ *has an (infinite) presentation as follows. Fix an* n*-bordered (unmarked) chord diagram. The generators are all sequences of chord slides beginning and ending at this chord diagram. The relations are described as follows. Two such sequences of chord slides are equivalent if, when converted to Whitehead moves by Lemma 6.14, the sequences of Whitehead moves differ by the relations in Proposition 5.11.*

To give an explicit presentation of $\Pi_1(Chord^b_{g,n})$, we wish to characterise a finite generating set of relations for chord slides.

Remark 6.23. We know that the relations in $\Pi_1(Chord^b_{g,n})$ are the images of relations in $\Pi_1(EFat_{g,n})$ under the branch reduction functor. The relations in $\Pi_1(EFat_{g,n})$ are generated by the square, pentagon and involutivity relations. To prove a finite set of relations between chord slides generates all relations, it suffices to show that the finite set generates the image of these classes of relations between Whitehead moves.

Proposition 6.24. *The groupoid* $\Pi_1(Chord^b_{g,n})$ *has a finite relation set.*

Proof. By Lemma 4.27, for fixed n and g, every n-bordered genus g fatgraph has the same number of edges and vertices. There are a finite number of graphs satisfying these two conditions up to the cyclic ordering of edges around vertices. Once we fix such a graph, there are a finite number of possible cyclic orderings around each edge. Thus there are a finite number of n-bordered genus g fatgraphs.

There are a finite number of Whitehead moves possible for each such fatgraph. Thus, for an unmarked n-bordered genus g fatgraph, there are a finite number of involutivity, square and pentagon relations that start with a Whitehead move on this graph. These relations generate all relations between Whitehead moves. Thus the images of this finite set of relations under the branch reduction algorithm generates all relations between chord slides in $\Pi_1(Chord^b_{g,n})$, as required. □

However, we wish to find a finite generating set of chord slide relations that is easier to describe. Consider the set of relations between chord slides in Figure 6.10. We will conjecture that these generate all of the relations and give an partial proof of this result.

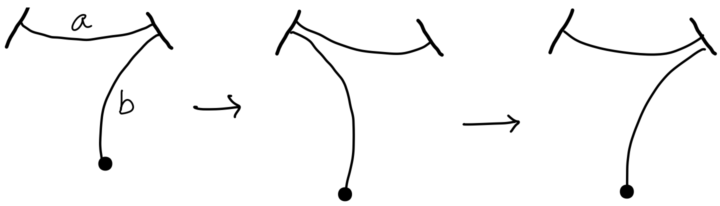
Using Lemma 6.15, one can verify that each of these relations holds between chord slides. The last three can be interpreted as follows. The left and right pentagons are relations showing that two different ways of sliding two chords over one chord are equivalent. Adjacent commutativity shows that sliding chords over the two sides of a single chord commutes. We have another relation that we can derive from this set, Opposite End Commutativity, shown in Figure 6.11, which is that slides of the two ends of a single chord commute.

Lemma 6.25. *Opposite end commutativity follows from the I, T, L and R relations.*

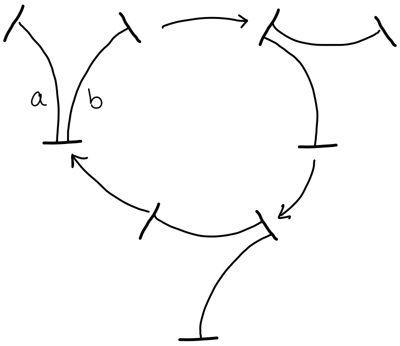
Proof. Figure 6.12 depicts the derivation. We observe that we can write the right pentagon relation in terms of the triangle and left pentagon relations, as well as one copy of opposite end commutativity. □

Remark 6.26. Not all chords in these relations can be tails, since we cannot slide over tails. The list of chords that could be tails is as follows:

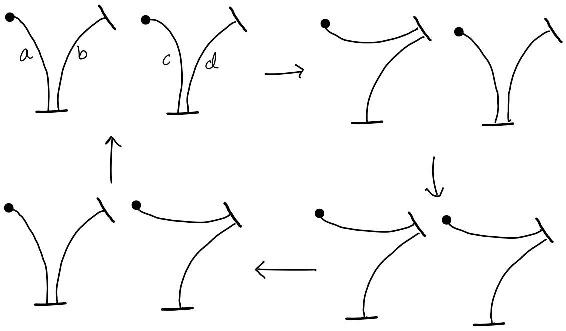
1. Involutivity: b
2. Triangle: no chords
3. Commutativity: a, c
4. Left Pentagon: c
5. Right Pentagon: a
6. Adjacent Commutativity: a, c
7. Opposite End Commutativity: no chords



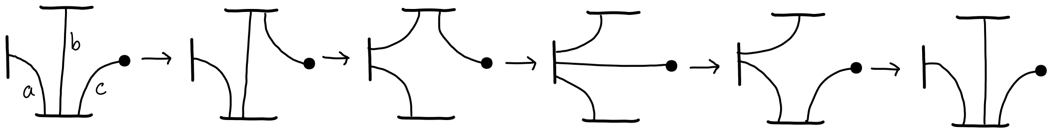
(a) I: Involutivity



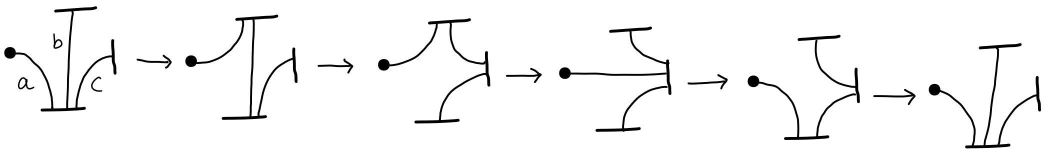
(b) T: Triangle



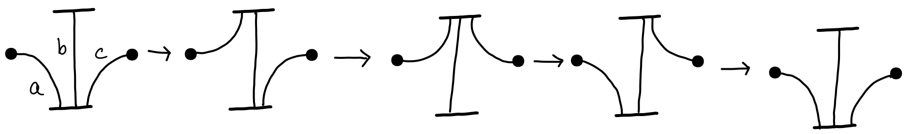
(c) C: Commutativity



(d) L: Left Pentagon



(e) R: Right Pentagon



(f) A: Adjacent Commutativity

Figure 6.10: A set of relations between chord slides.

As we do not slide over any of these chords, we can replace them with multiple chords, as in the following lemma.

Lemma 6.27. *In these relations, we can replace the following chords with multiple chords:*

1. *Involutivity:* b
2. *Commutativity:* a, c
3. *Left Pentagon:* c
4. *Right Pentagon:* a
5. *Adjacent Commutativity:* a *and* c.

Proof. We show each of these by induction. For involutivity, suppose that the relation holds where we replace b with m chords. Then by the image in Figure 6.13a, it holds when we replace b with $m+1$ chords. This is as the involutivity relation for $m+1$ chords factors as the involutivity relation for m chords and the relation for one chord.

For commutativity, we must induct on two variables. As shown in Figure 6.13b, if the relation holds when we replace a with m chords and b with n chords, it holds when we replace a with $m+1$ chords and b with n as it factors as commutativity with m and n chords and commutativity with one and n chords. By involutivity, we have the relation for $m+1$ and n chords. By the symmetry of a and b, it holds when we replace a and b with any number of chords.

For the multiple chord left pentagon, as shown in Figure 6.13c, the relation for $m+1$ strands decomposes into the pentagon on one strand, the pentagon on m strands and the commutativity relation for m strands. As all our relations hold if we flip them horizontally, showing that the multiple chord left pentagon holds implies that the multiple chord right pentagon holds. The figure giving the derivation would be a mirror image of that for the left pentagon, with left pentagons replaced by right pentagons.

Finally, for the multiple chord adjacent end commutativity, we must again induct on two variables. As shown in Figure 6.13d, if the relation holds when we replace a with m chords and c with n chords, it holds when we replace a with $m+1$ chords and c with n chords. By the symmetry of a and c, it holds for all m and n. ☐

For the connected boundary case, Bene shows that this set of relations generates all the relations between chord slides. In fact, the relation A can be derived from T, L and R, using a commutative diagram that does not hold if both a and c in A are tails.

Theorem 6.28 (Theorem 6.2, [Ben10])**.** *For $n=1$, the relations I, C, L, R and T in Figure 6.10 generate all relations between sequences of chord slides in* $\Pi_1(Chord^b_{g,1})$.

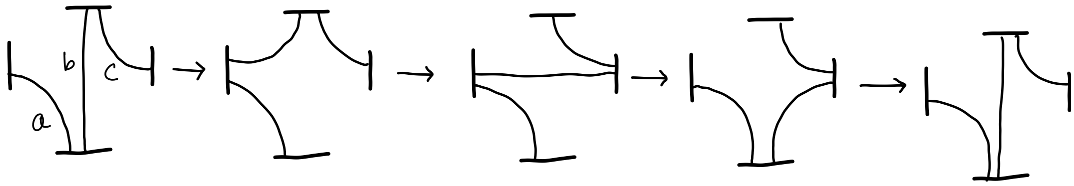


Figure 6.11: Opposite end commutativity

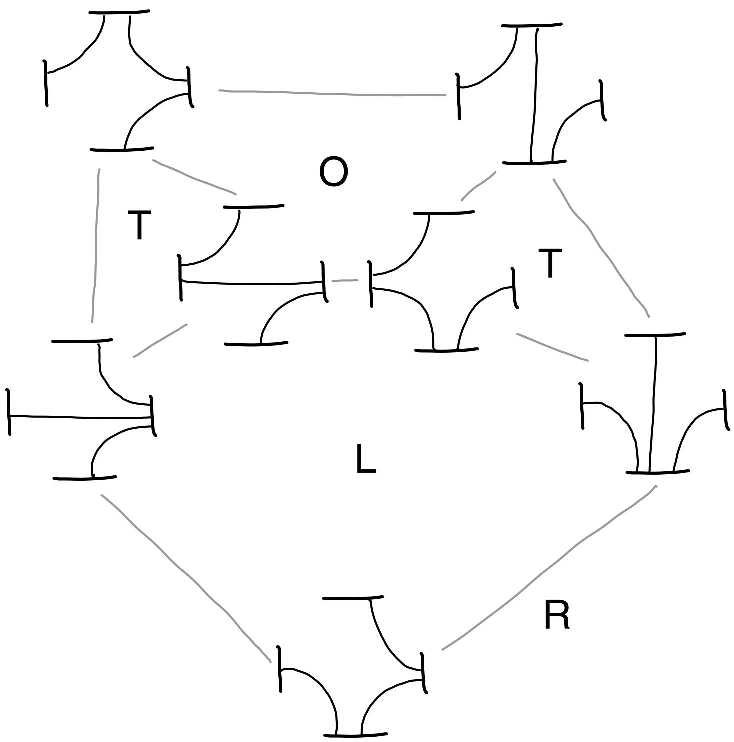


Figure 6.12: Derivation of opposite end commutativity

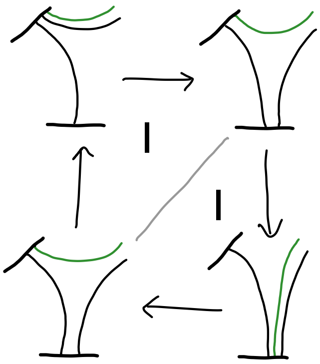
We conjecture that for the n-boundary case, this is still a generating set.

Conjecture A. *The relations in Figure 6.10 generate all relations between sequences of chord slides in* $\Pi_1(Chord^b_{g,n})$.

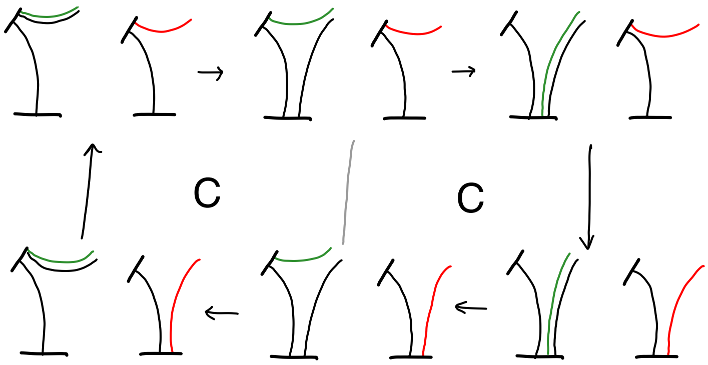
We did not prove this conjecture. However, we reduced the proof to checking a finite number of cases. In the following results, we describe the progress we made so far.

We introduce some notation for the following proofs. We sometimes conflate chord endpoints and oriented chords. When we do this, **c** as an endpoint is the initial point of **c** the oriented chord. If **c** is an oriented chord, c is the same chord with no assigned orientation.

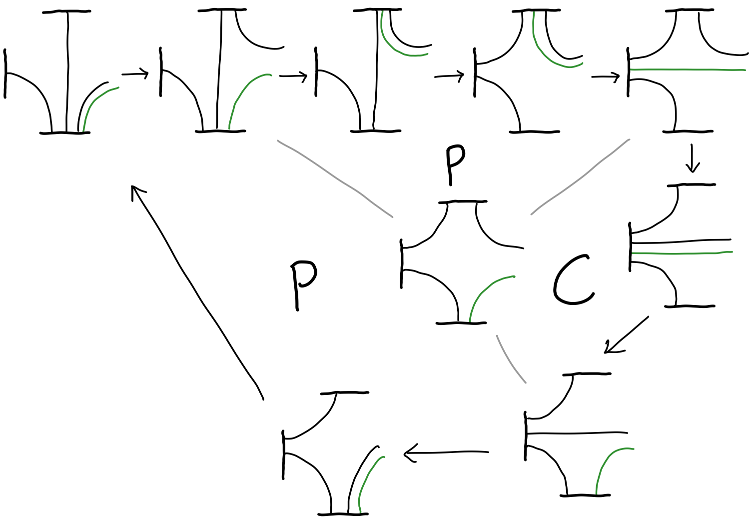
For all of the figures referenced for the inductive steps in these proofs, our notation is as follows. Coloured lines represent some number of chords that have adjacent



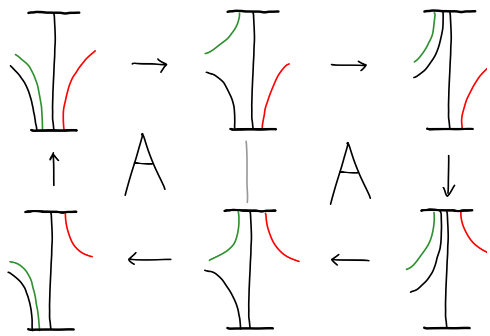
(a) The inductive step for multiple chord involutivity.



(b) The inductive step for multiple chord commutativity.



(c) The inductive step for the multiple chord left pentagon.



(d) The inductive step for the multiple chord adjacent commutativity.

Figure 6.13: The inductive steps for the multiple chord relations. The green line represents m chords with adjacent endpoints at the drawn point where they touch the core. They may have non-adjacent endpoints at their other end. Similarly, the red lines represents n chords with adjacent endpoints at one end.

endpoints. Black lines represent a single chord. We label arrows with the Whitehead move they are the image of. When the image of a Whitehead move is a number of consecutive chord slides, the image is the labelled arrow and the unlabelled arrows that consecutively follow it.

Lemma 6.29. *The images of involutivity relations between Whitehead moves are generated by the involutivity relations between chord slides.*

Proof. One Whitehead move, by Theorem 6.20, corresponds in chords slides to a slide of some number of chords along (possibly both sides of) a single chord. The inverse of this move corresponds to sliding these chords back along the single chord, as this reverses the effect on the generating set. By the multiple chord version of involutivity, this is trivial as a chord slide. □

Next, we consider the commutativity relations.

Proposition 6.30. *Let y and z be non-adjacent edges. Suppose that $W(y)$ and $W(z)$ are both not of type 5b or 6b. Then the image of the commutativity relation between these Whitehead moves is generated by chord slide relations I, C, T, L, R and A.*

Proof of proposition. If $W(y)$ is trivial on the generating set, is the trivial morphism in chord slides, so the commutativity relation reduces to involutivity. We hence assume $W(y)$ and $W(z)$ are not trivial on the generators. Thus without loss of generality, by Theorem 6.20, in chord slides, $W(y)$ corresponds to sliding some set of oriented chords $\{\mathbf{y}_i\}_{i=1}^n$ over a chord $\mathbf{y_0}$ such that either $\mathbf{y_0} \preceq \mathbf{y}_1 \preceq \cdots \preceq \mathbf{y_n}$, or $\mathbf{y}_1 \preceq \cdots \preceq \mathbf{y}_n \preceq \mathbf{y}_0$.

The corresponding result holds for $W(z)$, which in chord slides is the slide of some set of chords $\{\mathbf{z}_j\}_{j=1}^m$ over a chord $\mathbf{z_0}$.

Furthermore, $\mathbf{y_0}$ is adjacent to y in Γ, and $\mathbf{z_0}$ is adjacent to z in Γ. Thus since y and z are not adjacent, as oriented edges, $\mathbf{y_0} \neq \mathbf{z_0}$. By Theorem 5.13, one can check that $\overline{\mathbf{y_0}} \neq \mathbf{z_0}$. If this were the case, we would have a contradiction in the order in which the sectors are traversed, since we know that the core is traversed before all other sectors. Thus, $y_0 \neq z_0$ as unoriented chords.

Note that the $\{\mathbf{y}_i\}$ is the set of generating edges whose start point is in $T(y')$ for some y' adjacent to y, since we are not in case 5b or 6b. Similarly $\{\mathbf{z}_i\}$ is the set of generating edges whose start point is in $T(z')$ for z' adjacent to z. By the definition of $T(e)$, if $T(y')$ and $T(z')$ are not disjoint, either $T(y') \subseteq T(z')$ or $T(z') \subseteq T(y')$.

We consider the possible cases. Note that if $\mathbf{y}_i = \overline{\mathbf{z}_j}$, by opposite end commutativity, the slides of the two ends commute with each other. Thus so long as i and j are not 0, so we never slide over these chords, this does not affect the relation in the chord slides.

First, suppose that $T(y')$ and $T(z')$ are not disjoint, so $T(y') \subseteq T(z')$ or $T(z') \subseteq T(y')$. Without loss of generality, we suppose $T(y') \subseteq T(z')$. As the branch reduction algorithm preserves the clockwise ordering of the generators around a subtree of the maximal tree (Lemma 6.12), if $\mathbf{y}_1 = \mathbf{z}_j$, $\mathbf{y}_i = \mathbf{z}_{j+i-1}$. That is, the $\mathbf{y}_i$ are nested in the $\mathbf{z}_j$. We show the relation follows from the chord slide relations L, R, O and I by induction.

Before we analyse this case, we give a restriction on the nesting. We must have $\mathbf{y}_0 = \mathbf{z}_j$ for some j. (Recall that $j \neq 0$.) We show this as follows. If $m = n$, $T(y') = T(z')$. But then we must have $y' = z'$, since $T(e)$ is the unique maximal subtree of the maximal tree whose root is the vertex of e that is disconnected from the first tail in the maximal tree if we remove e. Note that by the translation of the Whitehead move into chord slides, y is incident to the start point of y' oriented away from the core in the maximal tree. However, so is z. Thus y and z share a common vertex, a contradiction. Otherwise, $m > n$, so $T(y')$ is a proper subset of $T(z')$. As y and y' are adjacent, the smallest subtree of the maximal tree that contains $T(y')$ but is not $T(y')$ also contains the common vertex of y and y'. We know that z' is not equal to y, as z' and z are adjacent but y and z are not adjacent. Thus $T(z')$ contains $T(y)$ as a proper subtree, and so contains both vertices of y, one of which is the start point of $\mathbf{y}_0$. Thus $\mathbf{y}_0$ is one of the $\mathbf{z}_j$.

We have four base cases. Suppose that $m = 2$ and $n = 1$. These possible nested cases are as follows:

1. $\mathbf{z}_0 \preceq \mathbf{z}_1 = \mathbf{y}_0 \preceq \mathbf{z}_2 = \mathbf{y}_1$
2. $\mathbf{z}_1 = \mathbf{y}_0 \preceq \mathbf{z}_2 = \mathbf{y}_1 \preceq \mathbf{z}_0$
3. $\mathbf{z}_0 \preceq \mathbf{z}_1 = \mathbf{y}_1 \preceq \mathbf{z}_2 = \mathbf{y}_0$

4. $\mathbf{z}_1 = \mathbf{y}_1 \preceq \mathbf{z}_2 = \mathbf{y}_0 \preceq \mathbf{z}_0$.

We draw the corresponding relation for each of these in Figure 6.17.

We see that the first case is the left pentagon, the second is the left pentagon inverse, the third is the right pentagon inverse, and the fourth is the right pentagon.

Now, we give the inductive step. We know the $\mathbf{y}_i$ are nested in the $\mathbf{z}_j$. There are four positions of $\mathbf{y}_0$ and $\mathbf{z}_0$ to consider, as each may be before or after the chords that slide along it on the core. As for every relation, the same relation flipped left to right is also a relation, it suffices to consider only the cases where $\mathbf{y}_0$ is to the left of the $\mathbf{y}_i$. For the induction, we have three variables. These are the index of the $\mathbf{z}_j$ such that $\mathbf{y}_0 = \mathbf{z}_j$, the number of chords m, and the number of chords n.

We show these inductive steps in Figure 6.18. We go through the first in detail. This is the case where we have $\mathbf{z}_0$ before the $\mathbf{z}_j$ along the core, and we wish to increase the number of chords of $\mathbf{z}_j$ before $\mathbf{y}_0$ by one. Suppose that we have the commutativity relation for the j such that $\mathbf{z}_j = \mathbf{y}_0$ being at most r, n up to q and $m-j-n+1$ (which is the number of chords used in $W(z)$ not used in $W(y)$ to the right of $\mathbf{y}_0$) up to p. We wish to show it for j at most $r+1$. Suppose we take $m-j-n+1 = p$, $j = r+1$ and $n = q$. Now, as shown in the figure, the commutativity relation in chord slides decomposes into a copy of the multichord disjoint commutativity relation, and a copy of the inductive relation with $m-j-n+1 = p$, $j = r+1$ and $n = q$. The remaining figures give us the induction on the remaining variables. Thus we have the result.

If $T(z')$ and $T(y')$ are not nested, the oriented chords $\{\mathbf{y}_i\}_{i=1}^n$ and $\{\mathbf{z}_j\}_{j=1}^m$ are disjoint.

We split this into subcases. First, if the unoriented chords y_i are disjoint from the unoriented chords z_j and z_0, and also the unoriented chords z_j are disjoint from y_0, by multichord disjoint commutativity we have the result. Otherwise, we must have one of the following situations.

If $\mathbf{y}_i = \mathbf{z}_0$, we are in the nested situation or else e and f are adjacent, so we do not consider this case further.

If $y_i \neq z_0$ and $z_j \neq y_0$, we may have some y_i equal to some z_j as unoriented chords. We show we have this relation by induction. For the base case, suppose $m = n = 1$, and $\mathbf{y}_1 = \overline{\mathbf{z}_1}$. As in the nested case, we have four possibilities here depending on whether $\mathbf{y_0}$ is before or after $\mathbf{y_1}$ along the core, and similarly for the z_i. By symmetry, without loss of generality we assume that $\mathbf{y}_0$ is after $\mathbf{y}_1$ along the core. We show the two remaining base cases in Figure 6.19. These are both the O relation.

For the inductive step, we consider two cases. By the symmetry of y and z, it suffices to add chords to the $\mathbf{z}_j$. The first is when we add a chord to the $\mathbf{z}_j$ such that its other orientation is not in the $\mathbf{y}_i$. As shown in Figure 6.20a, this decomposes into the C relation and the inductive relation with m fixed and n reduced by one. The second is when we add a chord to the $\mathbf{z}_j$ whose other orientation is in the $\mathbf{y}_i$. As shown in

Figure 6.20b, this decomposes into two copies of the inductive relation with n reduced. We depict the cases where $\mathbf{z}_0 \preceq \mathbf{z}_1$ and $\mathbf{y}_n \preceq \mathbf{y}_0$. The case where $\mathbf{y}_0 \preceq \mathbf{y}_1$ is very similar, and follows from the same relations. Thus by induction, this relation holds for all m and n.

Finally, we have the case where $\mathbf{y}_i = \overline{\mathbf{z}_0}$ or $\mathbf{z}_j = \overline{\mathbf{z}_0}$. By Lemma 5.5, [Ben10], we cannot have both be the case simultaneously for chord slides coming from Whitehead moves not of type 5b or 6b. Thus without loss of generality, $\mathbf{y}_i = \overline{\mathbf{z}_j}$. Now, after we perform $W(f)$, we are in the nested situation for $W(e)$. Thus by the previous case, we have the relation. □

For the pentagon classes of relations between Whitehead moves, the image of this relation in chord slides depends on the relative order in which we traverse the sectors around the two edges. Recall that the pentagon relation is illustrated in Figure 5.2b. We label our cases by the anticlockwise relative ordering of traversal of their sectors, up to cyclic permutation.

For the cases beginning with the label 12, at least three of the moves are of type 1 or 2 which are trivial. Thus the only relation we can see is involutivity, and we can check that in all cases this relation is sufficient to give the pentagon relation.

Of the 24 possible cases, we have checked the following nine. We list the cases and the relations that they follow from. In general, they follow from the multiple chord versions of the relations listed here.

Case	Follows from relations
12345	Trivial
12354	Trivial
12435	I
12453	I
12534	I
12543	I
13254	I, A
13452	I
13542	I

We give an example of checking one of these relations to show the scope of the work involved.

Lemma 6.31. *The pentagon relation of type 13254 follows from the candidate set of chord slide relations.*

Proof. We show the pentagon relation for this case in Figure 6.14a.

Note that $W(e)$ is of type 5, $W(e')$ is type 3, $W(f')$ is type 5 and the other two moves are trivial on the generating set so are the trivial morphism in chord slides.

We have three cases. First, if $W(e)$ is of type 5a, we have that $W(e')$ is of type 3a and $W(f')$ is of type 5a. For $W(e')$, this is as before we traverse sector 3, there is already a connection between sector 1 and sector 2. As sector 3 and sector 1 share an edge, this is type a of the case 3 move. For $W(f')$, as $W(e)$ is of type 5a, there is a connection between sector 1 and sector 2 through some of g, a, b and c, but not e or f, before we traverse 2. Thus this move is type 5a also. By these conditions, b must be a generator. Then their effects are as follows: $W(e) : b \mapsto e' = c\bar{b}$, $W(e') : e' \mapsto f' = \bar{d}c\bar{b}$, and $W(f') : f' \mapsto b$. Figure 6.14b shows the translation of these moves on generators to chord slides, following the translation from Theorem 6.20. From this theorem, each of the edges a, b, c, d and g, under branch reduction, becomes a series of chords where the composition of the duals is equal to the dual of the original edge. We represent these multiple adjacent chords with coloured edges. We see this is multiple chord involutivity.

Second, if $W(e)$ is of type 5b, we have $W(e')$ of type 3a, and $W(f')$ of type 5b. On the generators, they have the effect of $W(e) : h \mapsto e' = c\bar{b}$, $W(e') : e' \mapsto f' = \bar{d}c\bar{b}$, and $W(f') : \bar{d}c\bar{b} \mapsto h$. By the definition of h in Theorem 5.13, h is some b_i or c_j and is not a tail. Figure 6.14c shows the translation of these moves on generators to chord slides, following the translation from Theorem 6.20. We depict the case where h is some b_i. The case where it is c_j is very similar, and differs only be a relabelling. We see this relation is multiple chord adjacent end commutativity.

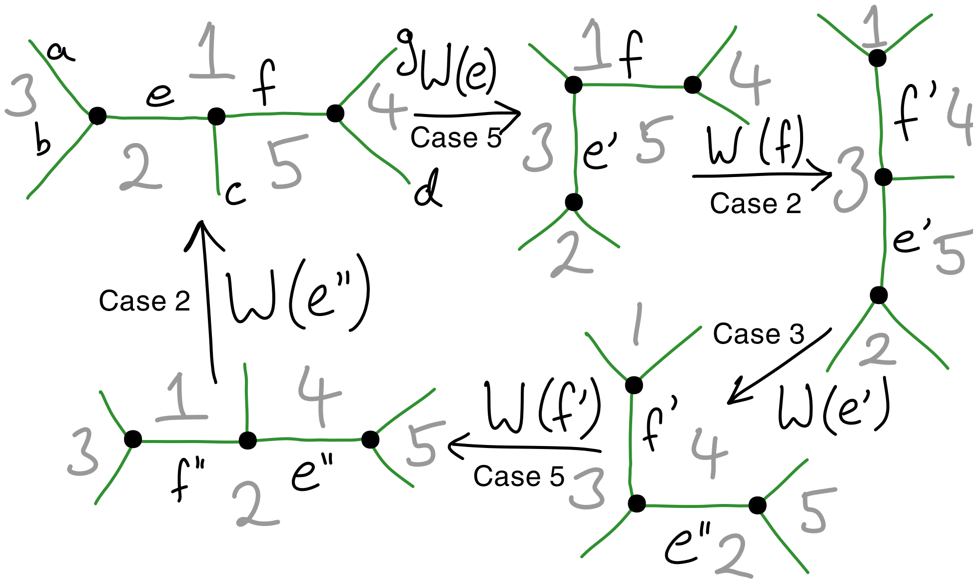
Finally, if $W(e)$ is of type 5c, we have $W(e')$ of type 3b and $W(f')$ of type 5c. Thus all the moves are trivial. Thus, in all cases, we show the relation follows from the chord slide relations A and I, as required. □

To show that this candidate set of relations generates all the relations, the work remaining is the following. First, one must show the commutativity relation follows for type 5b and 6b moves. The author suggests that showing it holds for the inverse of these moves would be simpler. This would be sufficient since if a and b commute, a^{-1} and b commute. If $W(e)$ is an inverse move, in its image in chord slides, the single chord we slide over is the image of e itself. This allows one to restrict the adjacency possibilities between the moves.

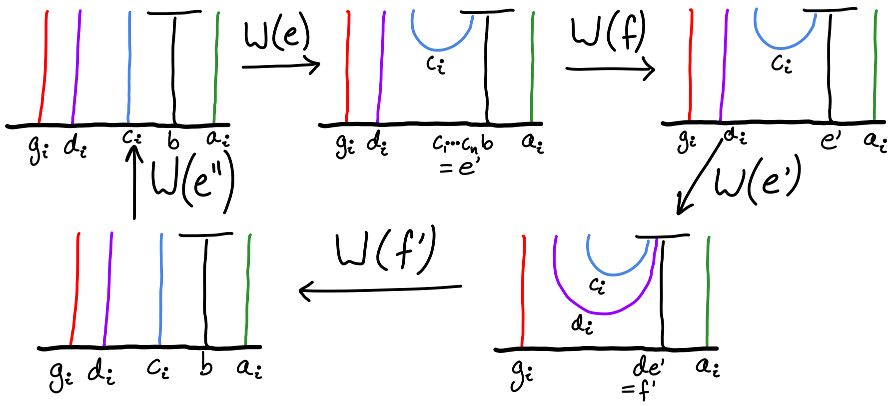
Second, one must check the remaining fifteen cases of pentagon relations. This work is time-consuming but straightforward.

6.3.1 A chord slide presentation of $Mod(\Sigma_{0,2})$

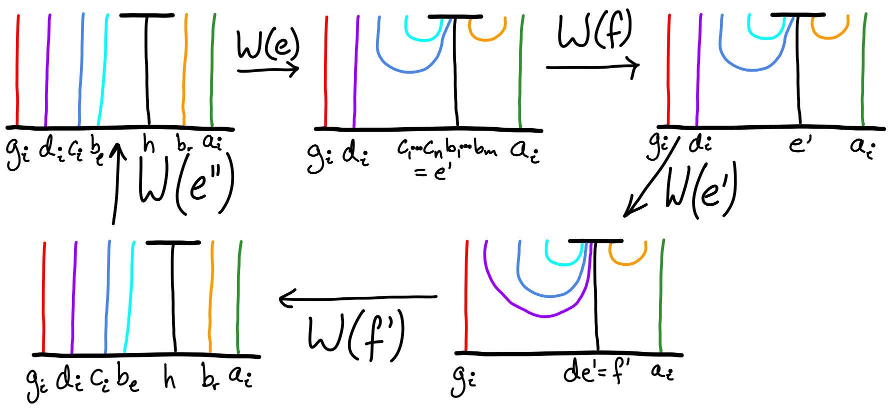
We verify that Conjecture A correctly predicts a presentation for the annulus. The number of n-bordered genus g chord diagrams grows rapidly with both n and g. For example, with $n = 2$ and $g = 1$, there are three chords and one tail. In a chord diagram for this twice-bordered torus, there are five vertices at which we may place the second tail. Then there are five choices as to which vertex we pair the first chord with and



(a) The pentagon relation case 13254.



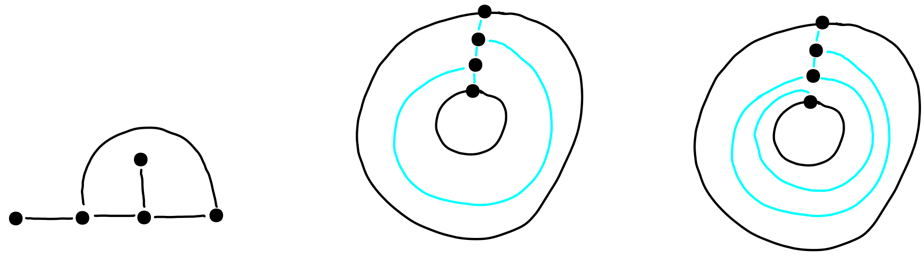
(b) In chord slides if $W(e)$ is case 5a.



(c) In chord slides if $W(e)$ is case 5b.

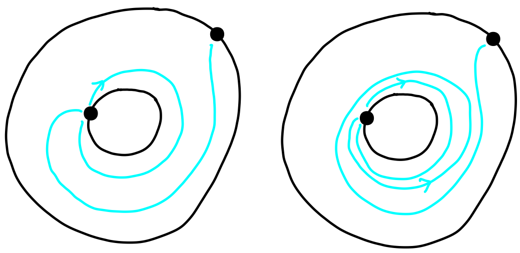
Figure 6.14: The Whitehead move pentagon relation for case 13254.

between one and two choices for which vertex we pair the second chord with. Thus there are at least 25 such chord diagrams, which is already rather large to feasibly use in computations.



(a) A twice-bordered genus zero chord diagram. (b) A marking of the fatgraph on the annulus. (c) The marking after a chord slide.

Figure 6.15: A marked chord diagram for the annulus.



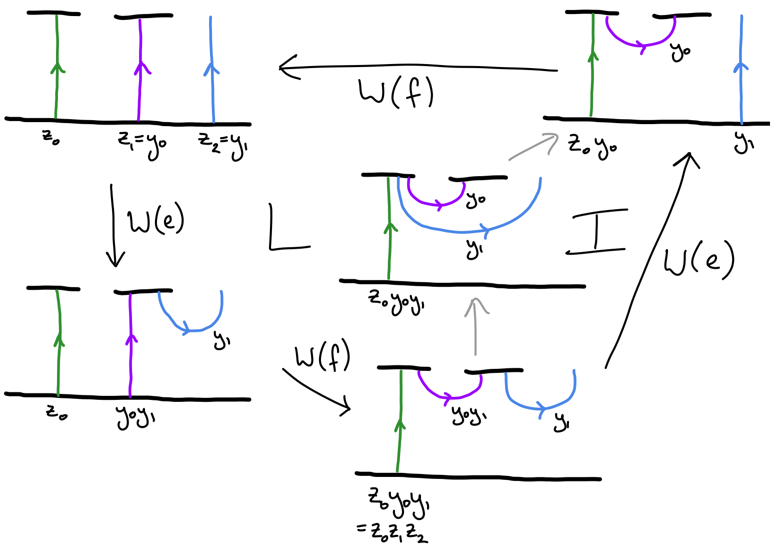
(a) The dual of the generators for Figure 6.15b. (b) The dual of the generators for Figure 6.15c.

Figure 6.16: The generators of the fundamental path groupoid for the annulus before and after the chord slide.

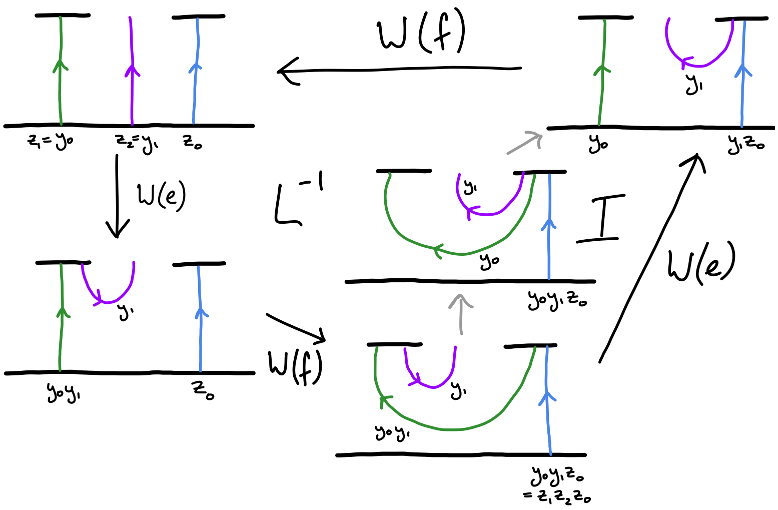
A more feasible example with more than one boundary component is the annulus with the chord diagram shown in Figure 6.15. As there is only one chord, the only possible chord slide on this diagram is to slide the second tail around the chord in one of the two directions (which are inverses of each other). One can check that this does not change the chord diagram, and the action on the marking is shown in Figure 6.15c. In this case, $Mod(\Sigma_{0,2})$ is then generated by these two chord slides, c_1 and c_2. By involutivity, $c_1^{-1} = c_2$. This shows that the mapping class group of the annulus, $Mod(\Sigma_{0,2})$, is the free group on one element, which is isomorphic to $\mathbb{Z}$.

As shown in Figure 6.16, which depicts the action on the marking, performing this chord slide once corresponds to a Dehn twist about a curve parallel to the boundary components. This is the classical generator of the mapping class group of the annulus.

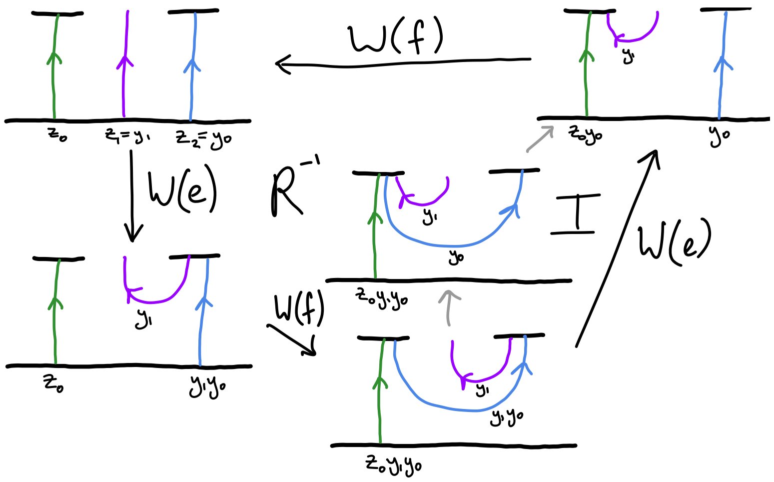
In this case, we can observe that performing this chord slide n times corresponds to n Dehn twists about this curve, so this presentation is the same as the classical one.



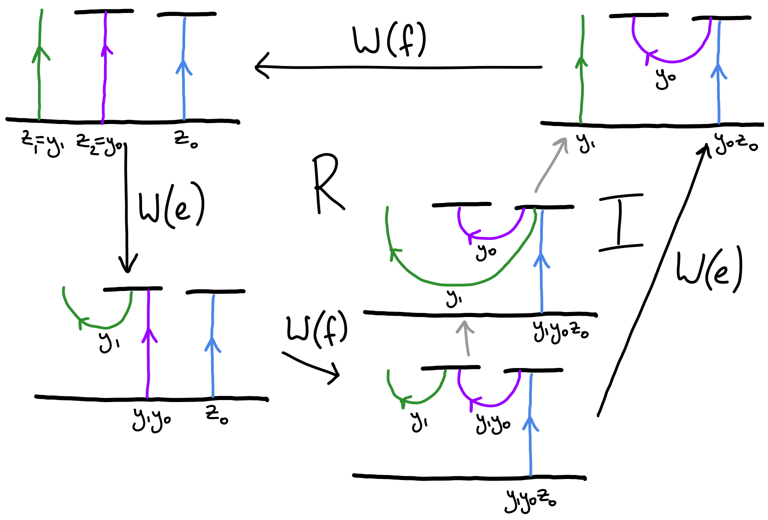
(a) $\mathbf{z}_0 \preceq \mathbf{z}_1 = \mathbf{y}_0 \preceq \mathbf{z}_2 = \mathbf{y}_1$



(b) $\mathbf{z}_1 = \mathbf{y}_0 \preceq \mathbf{z}_2 = \mathbf{y}_1 \preceq \mathbf{z}_0$

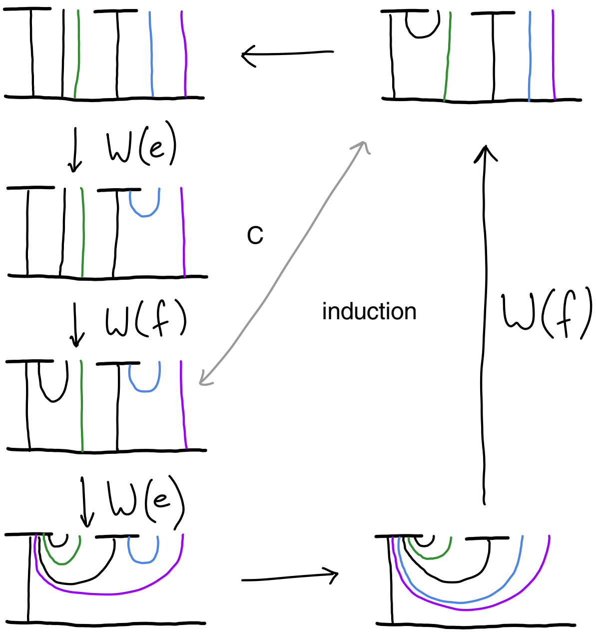


(c) $\mathbf{z}_0 \preceq \mathbf{z}_1 = \mathbf{y}_1 \preceq \mathbf{z}_2 = \mathbf{y}_0$

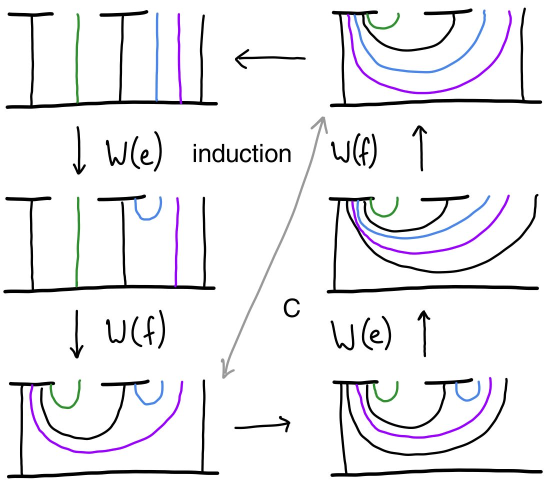


(d) $\mathbf{z}_1 = \mathbf{y}_1 \preceq \mathbf{z}_2 = \mathbf{y}_0 \preceq \mathbf{z}_0$.

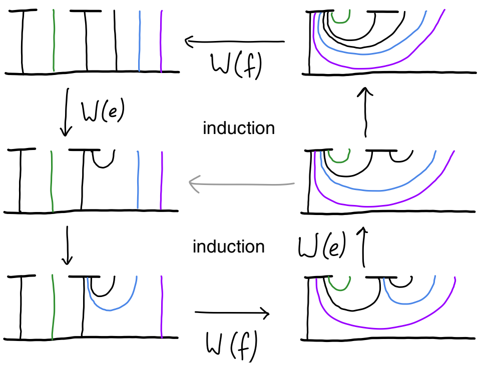
Figure 6.17: The base cases for nested commutativity.



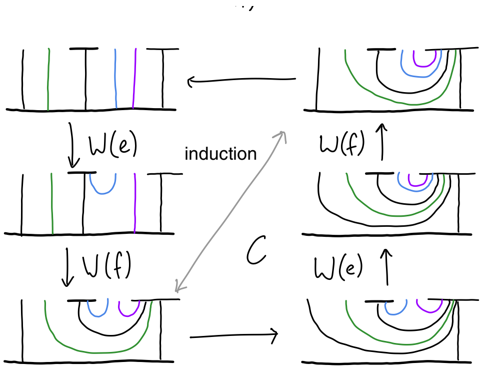
(a) We have $\mathbf{z}_0$ to the left of the $\mathbf{z}_j$. We add one chord to $\mathbf{z}_l$, which increases the index of $\mathbf{y}_0$ in the $\mathbf{z}_j$.



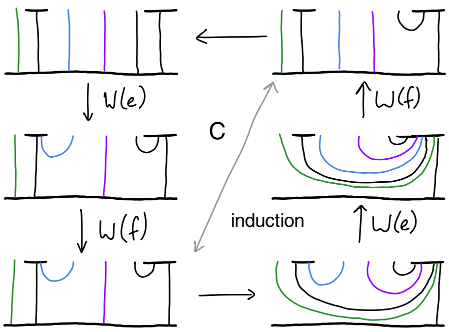
(b) Adding a chord to the $\mathbf{z}_r$.



(c) Adding one to n, which is the chords in $\mathbf{y}_s$.

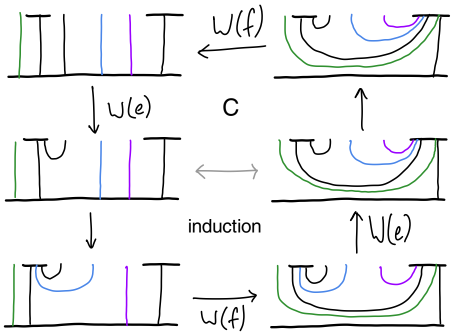


(d) We have $\mathbf{z}_0$ to the right of the $\mathbf{z}_j$. We add one chord to $\mathbf{z}_l$.



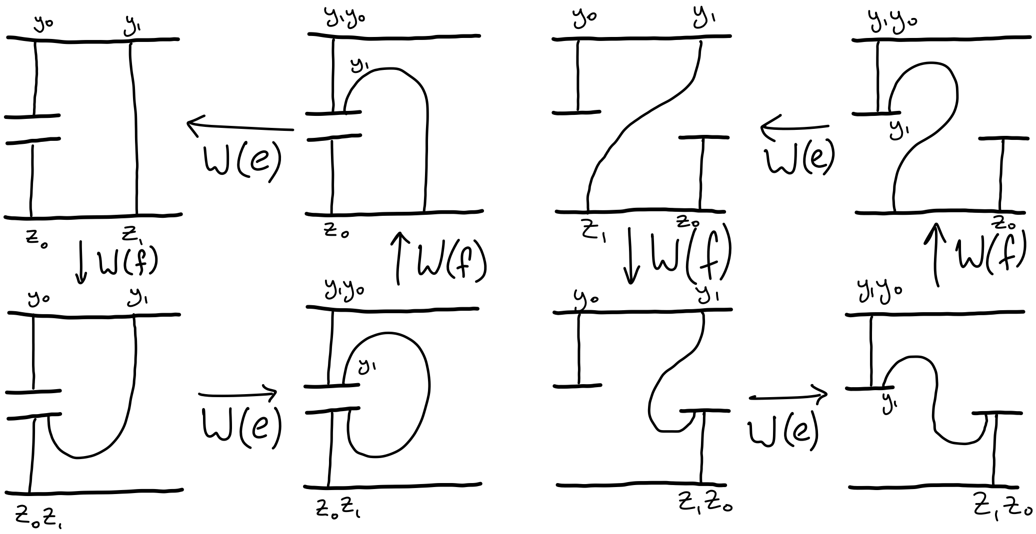
(e) Adding one to n, which is the chords in $\mathbf{y}_s$.

(f) Adding a chord to the $\mathbf{z}_r$.



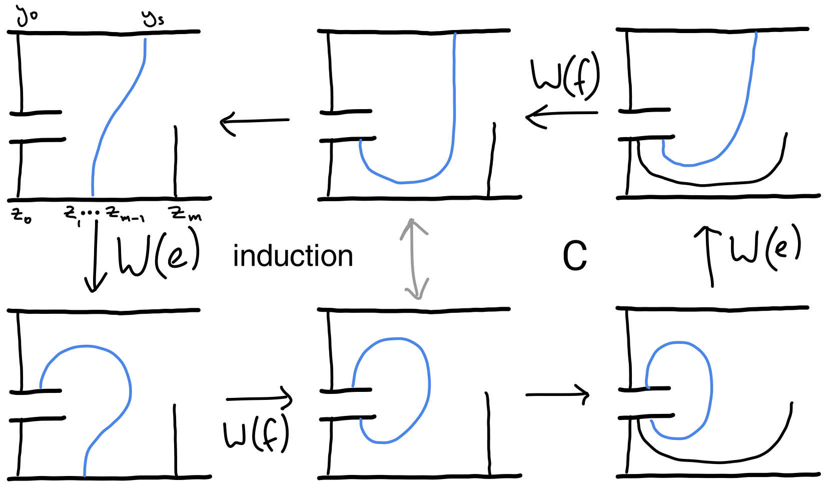
(g) Adding one to n, which is the chords in $\mathbf{y}_s$.

Figure 6.18: The inductive step for the nested case. The coloured lines may be multiple chords or tails. The black lines are a single chord or tail. The edge that we add for the inductive step is the black line whose other end is not shown. In the first three images, $\mathbf{z}_0$ is on the left; in the second three, it is on the right. In order of endpoints from left to right, excluding the single black chord that we do not slide over and $\mathbf{z}_0$, the chords are $\mathbf{z}_l$, $\mathbf{z}_k = \mathbf{y}_0$, $\mathbf{y}_s$ and $\mathbf{z}_r$. All of the $\mathbf{y}_i$ are in the $\mathbf{z}_j$.

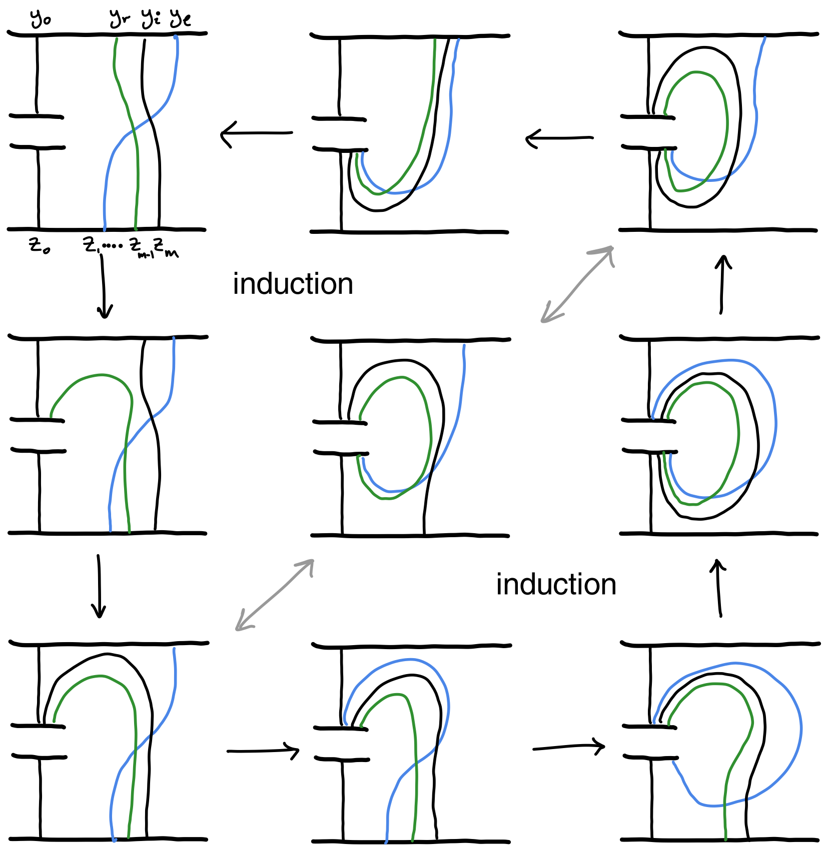


(a) $\mathbf{z}_1 \preceq \mathbf{z}_0$ (b) $\mathbf{z}_0 \preceq \mathbf{z}_1$

Figure 6.19: The base cases for commutativity for $y_i \neq z_0$, $z_j \neq y_0$ and the oriented chords disjoint but $y_i = z_j$.



(a)



(b)

Figure 6.20: The inductive step for commutativity for $y_i \neq z_0$, $z_j \neq y_0$ and the oriented chords disjoint but $y_i = z_j$. In the inductive step, we add z_m. The coloured lines represent multiple chords, all of which are one of the y_i or z_j. Not all of the multiple chords are in both the y_i and z_j.

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