On Generalized Frobenius-Schur Indicators for Spherical Fusion Categories

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Introduction

Classically, for a finite group G and a representation V over $\mathbb{C}$ with character χ, we define the *Frobenius-Schur indicator*, as

$$\nu(V) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

If V is an irreducible representation, then this indicator helps us to determine the flavour of duality of V. Specifically, the Frobenius-Schur theorem states that $\nu(V)$ could only be 1, -1, or 0, and if

- $\nu(V) = 1$: V is symmetrically self-dual

- $\nu(V) = -1$: V is antisymmetrically self-dual

- $\nu(V) = 0$: V is not self-dual

A great deal of work has been done to generalize the Frobenius-Schur(FS) indicators for Hopf algebras (see [LM00], [KSZ06], [MN05], [Sch04], [NS08]) and for categories (see [FS03], [FGSV99]). This culminated in the definition of generalized FS indicators for pivotal categories (see [NS07]) and a formula of generalized FS indicators for spherical fusion categories, given by Ng and Schauenberg in [NS10].

In broad terms, the generalized FS indicators for fusion categories are the traces of generalized rotation operators on homspaces in the category. Generalized rotation operators have been studied extensively and play an important role in the study of subfactor planar algebra. V. Jones used these rotations to show that certain quadratic tangles are linearly independent [Jon12] and to construct annular structures of subfactors [Jon01], which played a crucial part in the classification of subfactors of index at most 5 (see [JMS14] for an overview).

Generalized FS indicators have proven to be a useful tool for analyzing fusion categories. One important application is in the proof of the congruence subgroup conjecture for spherical fusion categories, which states that the kernels of the modular representations of modular categories are congruence subgroups of $SL_2(\mathbb{Z})$ (see [NS10]). The confirmation of this conjecture provides important insight on the relationship between rational conformal field theories and modular categories.

Generalized FS indicators are also useful for classification purposes, as they can be used to create bounds and have nice number theoretic properties. It was used by Bruillard, Ng, Rowell and Wang to show rank finiteness of modular tensor categories, which is that, up to equivalence, there are only finitely many modular categories of any fixed rank [BNRW16]. Furthermore, the indicators have been used to classify fusion categories of small rank (see [Ost14], [Lar15]).

The focus of this thesis is to give a self-contained derivation of the generalized Frobenius-Schur indicator formulas for spherical fusion categories given in [NS10]. This thesis will be presented as follows. The first two chapters give an introduction to the language of monoidal categories, focusing mostly on the theory needed in the remainder of the thesis. Specifically, chapter 1 defines pivotal and semisimple monoidal categories and chapter 2 defines braided monoidal categories, modular data and the Drinfeld center. In chapter 3, we define the induction functor to the Drinfeld center, and give a formula for the generalized Frobenius-Schur indicators in terms of the inductor functor and modular data of the center. Finally, in chapter 4, we follow the work of Barter, C. Jones and Tucker in [BJT16] and use the indicator formula to construct special torus link invariants for modular categories.

Chapter 1

Introduction to monoidal categories

There is a plethora of adjectives used to classify and describe tensor categories. This chapter, along with the next, introduces some of this language. In section 1.1 we give the definitions of monoidal categories and monoidal functors, as well as provide an explanation of string diagrams. Section 1.2 introduces pivotal categories and section 1.3 extends on this and defines spherical categories. Section 1.4 focuses on semisimple categories and makes some important observations about dimensions of simple objects in semisimple categories. Section 1.5 gives an introduction to dual pairings and presents the two basis, dual basis pairs we need for the proof of the indicator formula.

Note that throughout this thesis we assume our categories are essentially small.

1.1 Monoidal categories

We first introduce the notion of a monoidal category, which is the categorification of a monoid.

Definition 1.1. A *monoidal category* is a category $\mathcal{C}$ with the following additional information:

1. *tensor product*: a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

2. *associator*: a family of natural isomorphisms

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

   for $U, V, W \in \mathcal{C}$

3. *unit* : an object $\mathbb{1} \in \mathcal{C}$

4. *left and right unitors*: natural isomorphisms

$$\lambda_V : \mathbb{1} \otimes V \xrightarrow{\sim} V$$

$$\rho_V : V \otimes \mathbb{1} \xrightarrow{\sim} V$$

for all $V \in \mathcal{C}$

such that they satisfy the following conditions:

- For all $W, X, Y, Z \in \mathcal{C}$, we have the commutative diagram

$$((W \otimes X) \otimes Y) \otimes Z$$

with $\alpha_{W,X,Y} \otimes \mathrm{id}_Z$ going to $(W \otimes (X \otimes Y)) \otimes Z$ and $\alpha_{W \otimes X, Y, Z}$ going to $(W \otimes X) \otimes (Y \otimes Z)$,

$\alpha_{W, X \otimes Y, Z}$ from $(W \otimes (X \otimes Y)) \otimes Z$ to $W \otimes ((X \otimes Y) \otimes Z)$,

$\alpha_{W, X, Y \otimes Z}$ from $(W \otimes X) \otimes (Y \otimes Z)$ to $W \otimes (X \otimes (Y \otimes Z))$,

$W \otimes ((X \otimes Y) \otimes Z) \xrightarrow{\mathrm{id}_W \otimes \alpha_{X,Y,Z}} W \otimes (X \otimes (Y \otimes Z))$.

(pentagon axiom)

- For all $V, W \in \mathcal{C}$, we have the commutative diagram

$$(V \otimes \mathbb{1}) \otimes W \xrightarrow{\alpha} V \otimes (\mathbb{1} \otimes W)$$

with $\rho \otimes \mathrm{id}$ and $\mathrm{id} \otimes \lambda$ going to $V \otimes W$.   (triangle axiom)

We now give a brief explanation of why we require the pentagon and triangle axioms. Recall that in a monoid, we write the expression for an element, $m_1 \circ ... \circ m_n$, of the monoid without specifying a parenthesization. This is because multiplication in the monoid is associative. Similarly, we could also add and delete copies of the identity element. As an analogue of the associativity condition of monoids, given two parenthesization of $V_1 \otimes ... \otimes V_n$, X_1 and X_2, in the monoidal category, we require that all isomorphisms composed of α, ρ, λ from X_1 to X_2 to be equal. In other words, we have a canonical isomorphism between X_1 and X_2, By the MacLane Coherence Theorem[1] this is equivalent to showing that the triangle and pentagon axioms are satisfied.

We can also categorify morphisms between monoids.

Definition 1.2. Let $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \lambda, \rho)$ and $(\mathcal{C}', \otimes', \alpha', \mathbb{1}', \lambda', \rho')$ be monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J) where

$$F : \mathcal{C} \to \mathcal{C}'$$

---

[1] For a proof, see [EGNO15].

is a functor and

$$J_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$$

is a family of natural isomorphisms such that:

- $F(\mathbb{1})$ is isomorphic to $\mathbb{1}'$

- For $X, Y, Z \in \mathcal{C}$, we have the commutative diagram

$$
\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\downarrow{\scriptstyle J_{X,Y} \otimes' \mathrm{id}_{F(Z)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(Z)} \otimes' J_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' (F(Y \otimes Z) \\
\downarrow{\scriptstyle J_{X \otimes Y,Z}} & & \downarrow{\scriptstyle J_{X,Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}
$$

$$\text{(monoidal structure axiom)}$$

Definition 1.3. A monoidal functor (F, J) is *an equivalence of monoidal categories* if $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories. In this case, we also say $\mathcal{C}$ and $\mathcal{C}'$ are *monoidally equivalent*.

It can become complicated wrangling with unitors and associators. However, sometimes we are lucky and all unitors and associators are in fact identities. In this case, we have a strict monoidal category.

Definition 1.4. A monoidal category $\mathcal{C}$ is *strict* if

$$V \otimes \mathbb{1} = V = \mathbb{1} \otimes V \ , \ (V \otimes W) \otimes Z = V \otimes (W \otimes Z)$$

for all $V, W, Z \in \mathcal{C}$ and all components of α, ρ, λ are identities.

It is a well-known theorem of MacLane[2] that every monoidal category is monoidally equivalent to a strict monoidal category. Therefore, given any non-strict monoidal category, we can replace it with a strict one as long as we are only concerned with monoidal categories up to monoidal equivalence. Thus in all later definitions[3], we will simplify notation and assume that our monoidal categories are strict.

Now we introduce a diagrammatic calculus for morphisms in a strict monoidal category. We denote a morphism $f : V \to W$ by a box labelled by f with strings labelled by V and W.

---

[2]For a proof, see [EGNO15].

[3]Except when we define braided tensor categories in section 2.1, where the unitors and associators are included for completeness and consistency with the literature.
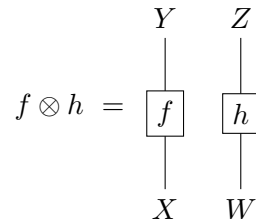
$$
\begin{array}{c}
W \\
| \\
\boxed{f} \\
| \\
V
\end{array}
$$

Note that we read the string diagrams optimistically, that is, upwards. When f is the identity, we just write it as a string without any boxes.

$$
\begin{array}{ccc}
V & & V \\
| & & | \\
\boxed{\text{id}} & := & \\
| & & | \\
V & & V
\end{array}
$$
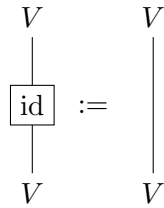
Composition of morphisms is denoted by vertical composition of the string diagrams. For example, given $f : X \to Y$ and $g : Y \to Z$, we write $g \circ f : X \to Z$ as:

$$
g \circ f \;=\;
\begin{array}{c}
Z \\
| \\
\boxed{g} \\
|\,Y \\
\boxed{f} \\
| \\
X
\end{array}
$$
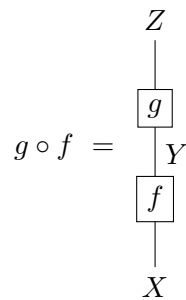
Tensor product of morphisms is denoted by horizontal juxtaposition of the string diagrams, for example, given $f : X \to Y$ and $h : W \to Z$, we write $f \otimes h : X \otimes W \to Y \otimes Z$ as:

$$
f \otimes h \;=\;
\begin{array}{cc}
Y & Z \\
| & | \\
\boxed{f} & \boxed{h} \\
| & | \\
X & W
\end{array}
$$

One advantage of using string diagrams is that most intuitive manipulations of the strings are allowed and correspond to extra structures in the monoidal category[4]. For example, one can imagine morphisms as beads on strings, which can be shifted up or down the strings freely[5].

---

[4]This statement is kept deliberately vague. We will make more sense of it in the later sections, once we have defined more structures on monoidal categories.

[5]For further details on the coherence of the string diagram calculus, the reader should consult [Mue10] for an overview and [JS91], [FY92] for comprehensive explanations.

Lemma 1.5. *Given $f : V \to W$, and $g : X \to Y$, we have*

$$\text{(1.1)}$$

Proof. Since the tensor product is a bifunctor, we know that for morphisms h, k, l, m with the appropriate domains and codomains, we have

$$(h \otimes k) \circ (l \otimes m) = (h \circ l) \otimes (k \circ m).$$

Then substituting h for f, g for m, and k, l for identities give us

$$(f \otimes id_Y) \circ (id_V \otimes g) = f \otimes g = (id_W \otimes g) \circ (f \otimes id_X).$$

which is precisely what the diagram says. $\qquad\square$

1.2 Pivotal categories

Motivated by the concept of duals in the category of vector spaces, pivotal categories gives us a way of defining duals of objects. Note that many varying definitions exist and we follow the approach taken in [Mue03a], that is, we give the definition of a strict pivotal category. It can be shown that theorems for strict pivotal categories can be translated into results for general pivotal categories, up to inserting some isomorphisms [Mue03a].

Definition 1.6. A *strictly pivotal category* is a strict monoidal category $\mathcal{C}$ with:

1. *duals on objects*: A map $Obj(\mathcal{C}) \to Obj(\mathcal{C})$ when sends $V \mapsto \overline{V}$ such that

$$\overline{\overline{V}} = V, \ \overline{V \otimes W} = \overline{W} \otimes \overline{V}, \ \overline{\mathbb{1}} = \mathbb{1}.$$

2. *evaluation and coevaluation*: For all $V \in \mathcal{C}$, we have morphisms

$$\varepsilon_V : V \otimes \overline{V} \to \mathbb{1}$$
$$\iota_V : \mathbb{1} \to V \otimes \overline{V}$$

such that both

$$V = V \otimes \mathbb{1} \xrightarrow{\text{id}_V \otimes \iota_{\overline{V}}} V \otimes \overline{V} \otimes V \xrightarrow{\varepsilon_V \otimes \text{id}_V} \mathbb{1} \otimes V = V$$

$$V = \mathbb{1} \otimes V \xrightarrow{\iota_V \otimes \text{id}_V} V \otimes \overline{V} \otimes V \xrightarrow{\text{id}_V \otimes \varepsilon_{\overline{V}}} V \otimes \mathbb{1} = V$$

$$\text{(1.2)}$$

are equal to id_V.

3. *coherence of objects*: For all $V, W \in \mathcal{C}$ we have the following commutative triangles:

$$
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\;\;\iota_V\;\;} & V \otimes \overline{V} \\
& \searrow^{\iota_{V \otimes W}} \quad \nwarrow_{\mathrm{id}_V \otimes \iota_W \otimes \mathrm{id}_{\overline{V}}} & \\
& V \otimes W \otimes \overline{V \otimes W} = V \otimes W \otimes \overline{W} \otimes \overline{V} &
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{1} & \xleftarrow{\;\;\varepsilon_V\;\;} & V \otimes \overline{V} \\
& \nwarrow^{\varepsilon_{V \otimes W}} \quad \nearrow_{\mathrm{id}_V \otimes \varepsilon_W \otimes \mathrm{id}_{\overline{V}}} & \\
& V \otimes W \otimes \overline{V \otimes W} = V \otimes W \otimes \overline{W} \otimes \overline{V} &
\end{array}
$$

4. *coherence of morphisms*: For all morphisms $s : V \to W$ the following composite morphisms are equal:

$$
\overline{W} = \overline{W} \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{\overline{W}} \otimes \iota_V} \overline{W} \otimes V \otimes \overline{V} \xrightarrow{\mathrm{id}_{\overline{W}} \otimes s \otimes \mathrm{id}_{\overline{V}}} \overline{W} \otimes W \otimes \overline{V} \xrightarrow{\varepsilon_{\overline{W}} \otimes \mathrm{id}_{\overline{V}}} \mathbb{1} \otimes \overline{V} = \overline{V}
$$

$$
\overline{W} = \mathbb{1} \otimes \overline{W} \xrightarrow{\iota_{\overline{V}} \otimes \mathrm{id}_{\overline{W}}} \overline{V} \otimes V \otimes \overline{W} \xrightarrow{\mathrm{id}_{\overline{V}} \otimes s \otimes \mathrm{id}_{\overline{W}}} \overline{V} \otimes W \otimes \overline{W} \xrightarrow{\mathrm{id}_{\overline{V}} \otimes \varepsilon_W} \overline{V} \otimes \mathbb{1} = \overline{V}
$$

$$\tag{1.3}$$

Graphically, we denote ε_V as

Similarly, we denote ι_V as

Sometimes, to avoid clutter in the string diagrams, we may choose to only label the middle of the string instead of the ends. For example, we can also denote ε_V as

Then (1.2) can be graphically denoted as:

And (1.3) can be represented as:

Furthermore, ε and ι give us a way of defining morphisms of $\mathcal{C}(\overline{V}, \overline{W})$ from morphisms in $\mathcal{C}(W, V)$, a technique we will make frequent use of in chapter 3.

Definition 1.7. Given a pivotal category $\mathcal{C}$ and $s \in \mathcal{C}(V, W)$, define $\overline{s} \in \mathcal{C}(\overline{W}, \overline{V})$ as:

$$(1.4)$$

Remark 1.8. By (1.3), we know we could have equivalently defined $\overline{s}$ to be:

$$(1.5)$$

1.3   Spherical structure

Definition 1.9. Given V an object in a strict pivotal category $\mathcal{C}$, and $f \in \mathrm{End}(V)$ we can define the *left categorical trace* of f as follows:

$$\mathbb{1} \xrightarrow{\iota_V} V \otimes \overline{V} \xrightarrow{f \otimes \mathrm{Id}_{\overline{V}}} V \otimes \overline{V} \xrightarrow{\varepsilon_V} \mathbb{1} \tag{1.6}$$

Diagrammatically, the left categorical trace, denoted by $tr_L(f)$, is

$$tr_L(f) \;=\; \boxed{f} \qquad \overline{V} \quad . \tag{1.7}$$

Similarly, we define the right categorical trace, denoted by $tr_R(f)$, to be

$$tr_R(f) \;=\; \overline{V} \qquad \boxed{f} \quad . \tag{1.8}$$

When taking the categorical trace, if the domains are codomains are compatible, the order in which we compose the morphisms does not matter.

Lemma 1.10. *For a pivotal category $\mathcal{C}$, $f \in \mathcal{C}(V, W)$, and $g \in (W, V)$, we have:*

$$tr_L(f \circ g) = tr_L(g \circ f)$$

$$tr_R(f \circ g) = tr_R(g \circ f)$$

Proof. To prove $tr_L(f \circ g) = tr_L(g \circ f)$, we need to show that

$$\tag{1.9}$$

By repeated use of (1.2), we have that

$$\tag{1.10}$$

Observe that by (1.3) and (1.2),

$$(1.11)$$

Combining the two facts above gives us (1.9). Similarly, one can show that $tr_R(g \circ f) = tr_R(g \circ f)$. □

Definition 1.11. A pivotal category $\mathcal{C}$ is *spherical* if for all objects $V \in \mathcal{C}$ and morphisms $s \in \text{End}(V)$, $tr_R(f) = tr_L(f)$. When this is the case, we drop the letters R and L and just use the notation $tr(f)$.

Definition 1.12. For V an object in a spherical category $\mathcal{C}$, the *dimension* of V, denoted by d_V, is defined by

$$d_V = tr(id_V).$$

1.4 Semisimple categories

To define a semisimple category, we have to first define a $\Bbbk$-linear category, which can be thought of as an enrichment over $\Bbbk$-vector spaces[6]. That is, instead of thinking of the homspaces just as sets, we require them to be vector spaces. Note that for the rest of the thesis, we only work with the case where $\Bbbk$ is an algebraically closed field with characteristic 0.

Definition 1.13. Let $\Bbbk$ be a field. A category $\mathcal{C}$ is a $\Bbbk$-linear category if it satisfies the following:

- all homspaces are finitely generated $\Bbbk$-vector spaces

- compositions of homspaces are $\Bbbk$-linear

We now define the concept of direct sums and simple objects in a $\Bbbk$-linear category.

Definition 1.14. For a k-linear category $\mathcal{C}$ and $X_1, \ldots, X_n$ objects in $\mathcal{C}$, the *direct sum* of $X_1, \ldots, X_n$ exists if there are $Y \in \mathcal{C}$, $v_i \in \mathcal{C}(X_i, Y)$, and $v_i' \in \mathcal{C}(Y, X_i)$ such that:

$$\sum_{i \in [1,\ldots,n]} v_i \circ v_i' = \text{id}_Y$$

$$v_i' \circ v_j = \delta_{i,j} \text{id}_{X_i}$$

$$(1.12)$$

Then we say Y is a *direct sum* of $X_1, \ldots, X_n$. The category $\mathcal{C}$ *has direct sums* if the direct sum of W, Z exists for all $W, Z \in \mathcal{C}$.

---

[6]Note that one could let $\Bbbk$ to be a commutative ring, and think about $\Bbbk$-modules instead.

Definition 1.15. In a $\Bbbk$-linear category $\mathcal{C}$, an object $V \in \mathcal{C}$ is *simple* if $\mathrm{End}(V) = \Bbbk \, \mathrm{id}_V$.

At last, we define semisimple categories.

Definition 1.16. A $\Bbbk$-linear category $\mathcal{C}$ is *semisimple* if:

- it has direct sums

- all idempotents split, that is, for all $f = f \cdot f \in \mathrm{End}(X)$, there exists $Y \in \mathcal{C}$ and $u : Y \to X$, $u' : X \to Y$ such that $u' \cdot u = \mathrm{id}_Y$ and $u \cdot u' = f$

- the simple objects are mutually disjoint, that is, let the set of simple objects be $\{X_i\}_{i \in I}$, then $\mathcal{C}(X_i, X_j) \simeq \delta_{i,j} \Bbbk$

- for all $Y, Z \in \mathcal{C}$, the map by composition:

$$\bigoplus_{i \in I} \mathcal{C}(X_i, Z) \otimes_{\Bbbk} \mathcal{C}(Y, X_i) \to \mathcal{C}(Y, Z)$$

  is an isomorphism.

Remark 1.17. Furthermore, if $\mathcal{C}$ is a monoidal category, we also require that $\mathbb{1}$ is a simple object for semisimplicity.

For $\mathcal{C}$ a semisimple category, we denote the set of simple objects in $\mathcal{C}$ by $\mathrm{Irr}(\mathcal{C})$. It can be shown that every object in a semisimple category is isomorphic to a finite direct sum of simple objects[7]. As a consequence of this, for any $X \in \mathcal{C}$, we have that

$$X \simeq \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})} Y^{n_Y} \tag{1.13}$$

where $n_Y = \dim(\mathcal{C}(Y, X))$.

In particular, when $X = A \otimes B$ for $A, B \in \mathrm{Irr}(\mathcal{C})$, we can define the *fusion coefficients*

$$N_{A,B}^Y = \dim(\mathcal{C}(A \otimes B, Y)),$$

which satisfy

$$A \otimes B \simeq \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})} Y^{N_{A,B}^Y}.$$

Now we prove some lemmas for semisimple, pivotal categories.

Lemma 1.18. *Let $\mathcal{C}$ be semisimple, pivotal category and $J \in Irr(\mathcal{C})$. Then $\overline{J} \in Irr(\mathcal{C})$.*

---

[7]We won't prove this here. In Cor. 1.27, however, we give a proof for the case when $\mathcal{C}$ is a semisimple category with finitely many simple objects.

Proof. We can define a linear isomorphism $\Phi : \text{End}(J) \to \text{End}(\overline{J})$ as the map that sends

$$
\begin{array}{c} J \\ | \\ \boxed{f} \\ | \\ J \end{array} \quad \mapsto \quad \left(\begin{array}{c} \overline{J} \\ \boxed{f} \\ \overline{J} \end{array} \right) . \tag{1.14}
$$

This map has an inverse that sends

$$
\begin{array}{c} \overline{J} \\ | \\ \boxed{g} \\ | \\ \overline{J} \end{array} \quad \mapsto \quad \left(\begin{array}{c} J \\ \boxed{g} \\ J \end{array} \right) . \tag{1.15}
$$

It is easy to check that Φ is in fact a vector space isomorphism, so $\text{End}(\overline{J}) \simeq \text{End}(J) = \mathbb{k}\,\text{id}_J$. $\square$

Lemma 1.19. *For a semisimple, pivotal category $\mathcal{C}$, and $J \in \mathcal{C}$, we have*

$$d_J = d_{\overline{J}}$$

$$d_{\mathbb{1}} = 1.$$

Proof. By definition of dimension and sphericality of $\mathcal{C}$, we have

$$
d_J \;=\; tr_L(\text{id}_J) \;=\; \bigcirc\; \overline{J} \;=\; tr_R(\text{id}_{\overline{J}}) \;=\; d_{\overline{J}}. \tag{1.16}
$$

For the second equation, observe that since $\mathcal{C}$ is a $\mathbb{k}$-linear category, for all $X \in \mathcal{C}$, we have

$$d_X = d_{\mathbb{1}\otimes X} = d_{\mathbb{1}} d_X. \tag{1.17}$$

Therefore $d_{\mathbb{1}} = 1$. $\square$

Lemma 1.20. *For a semisimple, pivotal category, and $L, J \in Irr(\mathcal{C})$, $Hom(L \otimes \overline{J}, \mathbb{1})$ is one-dimensional if $L = J$ and zero otherwise.*

Proof. By a similar argument as in Lemma 1.18, we know that $\text{Hom}(L\overline{J}, \mathbb{1}) \cong \text{Hom}(L, \mathbb{1}\otimes J) = \text{Hom}(L, J)$. Since L, J are simple objects, by definition, $\text{Hom}(L, J)$ is one-dimensional if and only if $L = J$.

$\square$

We can also require there to be only finitely many simple objects.

Definition 1.21. A monoidal category $\mathcal{C}$ is *finitely semisimple* if it is semisimple and has a finite number of isomorphism classes of simple objects. A monoidal category $\mathcal{C}$ is *fusion* if it is finitely semisimple and pivotal.

Definition 1.22. The *categorical dimension*, $\mathcal{D}$, of a spherical fusion category $\mathcal{C}$ is

$$\mathcal{D} = \sum_{L \in \mathrm{Irr}(\mathcal{C})} d_L^2.$$

1.5 Dual pairings

Let $\mathcal{C}$ be a semisimple category and $\Bbbk$ be an algebraically closed field with characteristic 0. Since the homspaces are vector spaces, we can give a basis. The goal of this section is to review of the concept of dual pairing, which is a gadget that generates a dual basis from a basis. Moreover, we highlight two dual pairings we use in the latter chapters.

Definition 1.23. Let V and W be vector spaces over a field $\Bbbk$. Then a bilinear map $\langle\,,\,\rangle : V \times W \to \Bbbk$ is *non-degenerate* if it satisfies:

- if $v \in V$ is such that for all $w \in W$, $\langle v, w \rangle = 0$, then $v = 0$

- if $w \in W$ is such that for all $v \in V$, $\langle v, w \rangle = 0$, then $w = 0$

Definition 1.24. For V and W vector spaces over a field $\Bbbk$, a *dual pairing* of V with W is a non-degenerate bilinear map $\langle\,,\,\rangle : V \times W \to \Bbbk$.

Theorem 1.25. *Let V and W be finite dimensional vector spaces over a field $\Bbbk$ and*

$$\langle\,,\,\rangle : V \times W \to \Bbbk$$

is a dual pairing of V with W. Then there is an isomorphism from V to W^*, which sends v to λ_v where $\lambda_v(w) = \langle v, w \rangle$.

For a proof, see [Gar09]. Thus, given $\{v_i\}$ a basis of V, the dual pairing provides a corresponding dual basis $\{w_i\}$ of $W \simeq V^*$, given by the equation $\langle v_i, w_j \rangle = \delta_{ij}$. Now we prove some properties of finitely semisimple categories which will help us to define our first dual pairing.

Theorem 1.26. *For a finitely semisimple category $\mathcal{C}$ and $L \in Irr(\mathcal{C})$, define*

$$\langle\,,\,\rangle : \mathcal{C}(Y, L) \otimes_{\Bbbk} \mathcal{C}(L, Y) \to \Bbbk \tag{1.18}$$

to be the bilinear map that sends $g \otimes_{\Bbbk} f$ to c where $g \circ f = c \cdot id_L$. This map is non-degenerate.

Proof. Suppose there exists $v \in \mathcal{C}(L, Y)$ such that for all $w \in \mathcal{C}(Y, L)$, $\langle w, v \rangle = w \circ v = 0$. By semisimplicity, for $x \in \mathrm{End}(Y)$, there exist $f_J \in \mathcal{C}(Y, J)$ and $g_J \in \mathcal{C}(J, Y)$ such that

$$x \simeq \sum_{J \in \mathrm{Irr}(\mathcal{C})} g_J \circ f_J.$$

Consider the morphism

$$\sum_{J \in \mathrm{Irr}(\mathcal{C})} g_J \circ f_J \circ v,$$

as L, J are simple objects, $\mathcal{C}(L, J) = 0$ if $L \neq J$, so most of the terms in the sum will be zero and we are left with $g_L \circ f_L \circ v$. By assumption, $f_L \circ v = 0$, thus $x \circ v = 0$ for all $x \in \mathrm{End}(Y)$. Let $x = \mathrm{id}_Y$. Then we have that $0 = \mathrm{id}_Y \circ v = v$.

By a similar argument, one can show that if there exists $w \in \mathcal{C}(Y, L)$ such that for all $v \in \mathcal{C}(L, Y)$, $\langle w, v \rangle = 0$, it must be the case that $w = 0$. $\square$

Corollary 1.27. *In a finitely semisimple category $\mathcal{C}$, every object can be written as a finite direct sum of simple objects.*

Proof. Let $\{\beta\}$ be a basis of $\mathcal{C}(Y, L)$, which we will denote as $\beta \in \mathfrak{B}(Y, L)$. Observe that the dual pairing above gives a dual basis $\{\beta^*\} \in \mathcal{C}(L, Y)$. Since $\beta \circ \alpha = \delta_{\alpha, \beta} \mathrm{id}_L$, it suffices to show that

$$\sum_{\substack{L \in \mathrm{Irr}(C) \\ \beta \in \mathfrak{B}(Y, L)}} \beta^* \circ \beta = \mathrm{id}_Y.$$

Let

$$g = \sum_{\substack{L \in \mathrm{Irr}(C) \\ \beta \in \mathfrak{B}(Y, L)}} \beta^* \circ \beta.$$

Then it suffices to show that for all $f \in \mathcal{C}(Z, Y), f' \in \mathcal{C}(Y, Z)$, we have that $g \circ f = f$ and $f' \circ g = f'$.

By semisimplicity, there exist $w_J \in \mathcal{C}(J, Y)$ and $v_J \in \mathcal{C}(Z, J)$ such that

$$f \simeq \sum_{J \in \mathrm{Irr}(\mathcal{C})} w_J \circ v_J.$$

Consider

$$g \circ \sum_{J \in \mathrm{Irr}(\mathcal{C})} w_J \circ v_J,$$

it is easy to see that for, $L \neq J$ the term is zero as L, J are simple objects. Thus we are left with

$$\sum_{\substack{L \in \mathrm{Irr}(C) \\ \beta \in \mathfrak{B}(Y, L)}} \beta^* \circ \beta \circ w_L \circ v_L.$$

By writing w_L in terms of $\beta^* \in \mathfrak{B}(L, Y)$, we see that,

$$\sum_{\substack{L \in \mathrm{Irr}(C) \\ \beta \in \mathfrak{B}(Y, L)}} \beta^* \circ \beta \circ w_L = w_L.$$

Therefore, $g \circ f = f$. By a similarly argument, one can show that $f \circ g = f$. $\square$

Thus, for this dual pairing, we have a nice expression of the identity morphism in terms of a basis and its dual basis.

Corollary 1.28. *Let $\mathcal{C}$ be a finitely semisimple category and $X, Y \in \mathcal{C}$. For each $L \in Irr(\mathcal{C})$, pick a basis of $\mathcal{C}(X \otimes Y, L)$. Let the dual pairing be the map*

$$\langle\,,\,\rangle : \mathcal{C}(L, X \otimes Y) \otimes_{\Bbbk} \mathcal{C}(X \otimes Y, L) \to \Bbbk \tag{1.19}$$

that sends $f \otimes_{\Bbbk} g$ to c where $g \circ f = c \cdot id_L$. Then

$$\tag{1.20}$$

Remark 1.29. The fact that the RHS of (1.20) does not depend on the choice of basis stems from the linearity of hom-spaces in $\mathcal{C}$. It is well-known that if $\{\alpha_i\}$ and $\{\beta_j\}$ are two different bases for $\mathcal{C}(X \otimes Y, L)$, then $\sum_i \alpha_i \otimes_{\Bbbk} \alpha_i^*$ and $\sum_j \beta_j \otimes_{\Bbbk} \beta_j^*$ represent the same element in $\mathcal{C}(X \otimes Y, L) \otimes_{\Bbbk} \mathcal{C}(L, X \otimes Y)$. Consider the map

$$\Gamma : \mathcal{C}(X \otimes Y, L) \times \mathcal{C}(L, X \otimes Y) \to \mathcal{C}(X \otimes Y, X \otimes Y) \tag{1.21}$$

which sends $f \times g$ to the following morphism

Since $\mathcal{C}$ is a $\Bbbk$-linear category, Γ is in fact a $\Bbbk$-linear map. Therefore

$$\Gamma\left(\sum_L \alpha_i \times \alpha_i^*\right) = \Gamma\left(\sum_j \beta_j \times \beta_j^*\right)$$

and the definition does not depend on the basis. In general, in a $\Bbbk$-linear category, if we define a morphism using the sum of some basis and dual basis, then the definition does not depend on the choice of basis. This observation is extremely useful and we frequently use this in the proofs that follow to choose a convenient basis.

Note that the pairing $\langle\,,\,\rangle$ requires L to be a simple object. When this is not the case, we use another dual pairing defined as follows. Let $\mathcal{C}$ be a fusion category. For $X, Y \in \mathcal{C}$, define the bilinear map

$$(\,,\,) : \mathcal{C}(X, Y) \otimes_{\Bbbk} \mathcal{C}(Y, X) \to \Bbbk \tag{1.22}$$

be the map that sends $\beta \otimes \alpha$ to c, where

$$c \;=\; \begin{array}{c} X \\ \alpha \\ Y \\ \beta \end{array} \qquad . \tag{1.23}$$

It is easy to check that $(\,,\,)$ is non-degenerate by decomposing X, Y into direct sums of simple objects.

Remark 1.30. A remark on notation: instead of explicitly stating the dual pairing, we annotate with either an asterisk or a star to differentiate between the two pairings. For instance, let $\{\beta\}$ be a basis for the vector space V, then $\{\beta^*\}$ is the dual with respect to $\langle\,,\,\rangle$ and $\{\beta^\star\}$ is the dual with respect to $(\,,\,)$.

Lastly, we recall the definition of the trace of a linear operator. Note that by remark 1.29, this definition does not depend on the choice of basis.

Definition 1.31. Let $\mathcal{C}$ be a $\Bbbk$-linear category. For a linear map $T : \mathcal{C}(X, Y) \to \mathcal{C}(X, Y)$ and a basis $\{\alpha\}$ of $\mathcal{C}(X, Y)$, the *trace* of T is

$$Tr(T) := \sum_{\alpha \in \mathfrak{B}(X,Y)} (T\alpha_i, \alpha_i^\star)$$

where $\{\alpha_i^\star\}$ is the corresponding dual basis under the dual pairing $(\,,\,)$.

Chapter 2

Braided monoidal categories

This chapter continues the introduction to the language of monoidal categories. In section 2.1 we define braided monoidal categories and braided monoidal functors. In section 2.2 we define twists for pivotal, braided tensor categories and elaborate on twists of simple objects when the category is also semisimple. In section 2.3 we discuss modular categories. In particular, we define the s and t matrices and present the Verlinde formula. Finally, in section 2.4 we define the Drinfeld center.

2.1 Braided monoidal categories

Braided monoidal categories capture in categorical terms what happens when we commute the terms in a tensor product. This is done by specifying a family of natural isomorphisms $\sigma_{V,W} : V \otimes W \to W \otimes V$, which is referred to as the braiding.

Definition 2.1. A *braided* monoidal category is a monoidal category $\mathcal{C}$ equipped with a family of natural isomorphisms

$$\sigma_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$$

satisfying the hexagon axioms, that is, for all $X, Y, Z \in \mathcal{C}$

$$
\begin{array}{ccc}
 & X \otimes (Y \otimes Z) \xrightarrow{\sigma_{X,Y \otimes Z}} (Y \otimes Z) \otimes X & \\
\quad\nearrow{\scriptstyle \alpha_{X,Y,Z}} & & \searrow{\scriptstyle \alpha_{Y,Z,X}} \\
(X \otimes Y) \otimes Z & & Y \otimes (Z \otimes X) \\
\quad\searrow{\scriptstyle \sigma_{X,Y} \otimes \mathrm{id}_Z} & & \nearrow{\scriptstyle \mathrm{id}_Y \otimes \sigma_{X,Z}} \\
 & (Y \otimes X) \otimes Z \xrightarrow[\alpha_{Y,X,Z}]{} Y \otimes (X \otimes Z) &
\end{array}
$$

$$(X \otimes Y) \otimes Z \xrightarrow{\sigma_{X \otimes Y, Z}} Z \otimes (X \otimes Y)$$

$$\alpha_{X,Y,Z}^{-1} \nearrow \qquad\qquad\qquad\qquad \searrow \alpha_{Z,X,Y}^{-1}$$

$$X \otimes (Y \otimes Z) \qquad\qquad\qquad\qquad\qquad (Z \otimes X) \otimes Y$$

$$\mathrm{id}_X \otimes \sigma_{Y,Z} \searrow \qquad\qquad\qquad\qquad \nearrow \sigma_{X,Z} \otimes \mathrm{id}_Y$$

$$X \otimes (Z \otimes Y) \xrightarrow[\alpha_{X,Z,Y}^{-1}]{} (X \otimes Z) \otimes Y$$

commutes.

The hexagon axioms ensure that the braiding is in fact a good one, in the sense that it is compatible with the tensor product. One way to see this is as follows. For strict tensor categories, the hexagon axioms simplify to the following conditions:

$$\sigma_{X, Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes \mathrm{id}_Z)$$
$$\sigma_{X \otimes Y, Z} = (\sigma_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma_{Y,Z})$$
$$\tag{2.1}$$

In our graphical notation, we denote $\sigma_{V,W}$ as a crossing of the strings of the following type

$$\tag{2.2}$$

and we denote $\sigma_{V,W}^{-1}$ as a crossing of the strings of the following type

$$\tag{2.3}$$

Then (2.1) can be expressed as:

$$
\begin{array}{c}
\text{(2.4)}
\end{array}
$$

Since the braiding is natural, we can also move morphisms up and down the string like beads. For example, for $U, V, W \in \mathcal{C}$, $f \in \mathcal{C}(U, V)$ we have

$$
, \qquad \text{(2.5)}
$$

and

$$
. \qquad \text{(2.6)}
$$

We can extend on the definition of monoidal functor when the functor also behaves well with the braiding.

Definition 2.2. Let $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \lambda, \rho)$ and $(\mathcal{C}', \otimes', \alpha', \mathbb{1}', \lambda', \rho')$ be braided monoidal categories with braiding σ and σ' respectively. A monoidal functor (F, J) from $\mathcal{C}$ to $\mathcal{C}'$ is called *braided* if for all $X, Y \in C$, we have the commutative diagram

$$
\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{\sigma'_{(F(X), F(Y))}} & F(Y) \otimes' F(X) \\
{\scriptstyle J_{X,Y}} \downarrow & & \downarrow {\scriptstyle J_{Y,X}} \\
F(X \otimes Y) & \xrightarrow{\sigma_{(X,Y)}} & F(Y \otimes X)
\end{array} \qquad .
$$

If F is also an equivalence of categories, then $\mathcal{C}$ and $\mathcal{C}'$ are *braided monoidally equivalent*.

2.2   Twists

Once we have a braided structure, we can define the notion of twists.

Definition 2.3. For a pivotal, braided tensor category $\mathcal{C}$ and $V \in \mathcal{C}$. We can define the twist of V, denoted by θ_V, as the composition of the following morphisms:

$$V = V \otimes \mathbb{1} \xrightarrow{\ \mathrm{id}_V \otimes \iota_V\ } V \otimes V \otimes \overline{V} \xrightarrow{\ \sigma_{V,V} \otimes \mathrm{id}_{\overline{V}}\ } V \otimes V \otimes \overline{V} \xrightarrow{\ \mathrm{id}_V \otimes \varepsilon_V\ } V \otimes \mathbb{1} = V$$

In our graphical notation, θ_V is interpreted as a twist in the string:

Now we prove that twists allow morphisms to pass through them.

Lemma 2.4. *Let $\mathcal{C}$ be a pivotal braided tensor category and $f \in \mathcal{C}(X,Y)$. Then*

                                                              (2.7)

Proof. This is a simple computation using the functoriality of the braiding and the observation

                                                              (2.8)

$\square$

Remark 2.5. Furthermore, one can show that, for $\mathcal{C}$ a pivotal braided tensor category and $V, W \in \mathcal{C}$, we have

$$\begin{array}{cc}
\begin{matrix} V & W \\ \boxed{\theta_{V \otimes W}} \\ V & W \end{matrix}
& =
\begin{matrix} V & W \\ \boxed{\theta_V}\ \boxed{\theta_W} \\ V & W \end{matrix}
\end{array} . \qquad (2.9)$$

For readers familiar with the notion of ribbon categories, observe that this is the ribbon relation. In fact, for semisimple categories, spherical braided structures uniquely define ribbon structures of $\mathcal{C}$[1].

For a semisimple category $\mathcal{C}$ and $L \in \mathrm{Irr}(C)$, we can identify θ_L with a scalar as $\mathrm{Hom}_{\mathcal{C}}(L, L) \simeq \Bbbk$.

Definition 2.6. Let $\mathcal{C}$ be a semisimple, pivotal, braided category. Given $L \in \mathrm{Irr}(\mathcal{C})$, we define $\theta_L \in \Bbbk$ as follows:

$$\begin{matrix} \\ L \end{matrix} = \theta_L \begin{matrix} \\ L \end{matrix}$$

Now we prove some useful lemmas which tell us how the different twists relate with each other.

Lemma 2.7. *For $\mathcal{C}$ a braided spherical fusion category, and $L \in \mathrm{Irr}(\mathcal{C})$, we have:*

1.

$$\begin{matrix} \\ L \end{matrix} = \begin{matrix} \\ L \end{matrix}$$

2.

$$\begin{matrix} \\ L \end{matrix} = \theta_L^{-1} \begin{matrix} \\ L \end{matrix}$$

Proof. Since $L \in \mathrm{Irr}(\mathcal{C})$, there exists $c_1, c_2 \in \Bbbk$ such that

$$\begin{matrix} \\ L \end{matrix} = c_1 \begin{matrix} \\ L \end{matrix} \qquad (2.10)$$

---

[1]See [EGNO15] Prop. 8.10.12.

$$\text{(diagram)} \; = \; c_2 \; \text{(diagram)} \; . \tag{2.11}$$

Now take the trace of (2.10) and (2.11), since $\mathcal{C}$ is spherical, we know that the two traces must be equal, so $c_1 = c_2$.

To prove the second statement, observe that by the first statement, we have

$$\text{(diagram)} \; = \; \text{(diagram)} \; . \tag{2.12}$$

By functoriality of the braiding, we have

$$\text{(diagram)} \; = \; \text{(diagram)} \; . \tag{2.13}$$

By the properties of the pivotal structure, we know that

$$\text{(diagram)} \; = \; \text{(diagram)} \; . \tag{2.14}$$

Since L is simple, then it must be that case that

$$\text{(diagram)} \; = \; \theta_L^{-1} \; \text{(diagram)} \; . $$

$\square$

2.3   Modular categories

In this section, we define modular data for premodular categories and give a definition of modular categories. Then we present some of the key theorems for modular categories.

Note that this section only contains the theorems needed for later chapters, for a more nuanced introduction to modular categories and complete proof of theorems, consult [BJ00].

First we define premodular categories.

Definition 2.8. A monoidal category $\mathcal{C}$ is a *premodular* category if it is semisimple, spherical and braided.

For a premodular category $\mathcal{C}$ and simple objects $L, J \in \mathrm{Irr}(\mathcal{C})$, we can define $\tilde{s}_{LJ} \in \Bbbk = \mathrm{End}(\mathbb{1}, \mathbb{1})$ by

$$\tilde{s}_{LJ} := \quad \underset{L \qquad\qquad \overline{J}}{\bigcirc\!\!\!\bigcirc} \quad . \tag{2.15}$$

Observe that

$$\tilde{s}_{L\mathbb{1}} = d_L. \tag{2.16}$$

Also, by remark 2.5 and decomposing $L \otimes J$ into simples, we can alternatively express $\tilde{s}_{LJ}$ as,

$$\tilde{s}_{LJ} = \theta_L^{-1}\theta_J^{-1} \sum_{K \in \mathrm{Irr}(\mathcal{C})} N_{\overline{L}J}^K \theta_K d_K. \tag{2.17}$$

Thus by (2.17), one can show that

$$\tilde{s}_{LJ} = \tilde{s}_{JL} = \tilde{s}_{\overline{L}\,\overline{J}} = \tilde{s}_{\overline{J}\,\overline{L}} \tag{2.18}$$

by observing $\mathcal{C}(\overline{L} \otimes J, K)$, $\mathcal{C}(\overline{J} \otimes K, K)$, $\mathcal{C}(L \otimes \overline{J}, \overline{K})$ and $\mathcal{C}(J \otimes \overline{L}, K)$ are isomorphic as vector spaces, where the isomorphisms are given by some suitable compositions of braiding and morphisms from the pivotal structure.

We can collect this information in the form of a matrix, giving us the $\tilde{s}$ matrix. In a similar fashion, we can define the t and c matrices.

Definition 2.9. For a premodular category $\mathcal{C}$ we define the following matrices

$$\tilde{s} := (\tilde{s}_{LJ})$$

$$t := (t_{LJ})$$

$$c := (c_{LJ}),$$

with entries indexed by $L, J \in \mathrm{Irr}(\mathcal{C})$ and

$$t_{L,J} := \delta_{L,J}\theta_L$$

$$c_{L,J} := \delta_{L,\overline{J}}.$$

Definition 2.10. A premodular category $\mathcal{C}$ is *modular* if is finitely semisimple and $\tilde{s}$ is invertible.

When $\mathcal{C}$ is modular, we can normalize $\tilde{s}$ by

$$s := \frac{\tilde{s}}{\sqrt{\mathcal{D}}}$$

where $\mathcal{D}$ is the categorical dimension.

The s, t, and c matrices of a modular category satisfy nice relations.

Theorem 2.11. *For a modular category $\mathcal{C}$, we have:*

$$(st)^3 = \left(\frac{p^+}{p^-}\right)^{\frac{1}{2}} s^2$$

$$s^2 = c$$

$$ct = tc$$

$$c^2 = 1$$

where

$$p^+ := \sum_{L \in Irr(\mathcal{C})} \theta_L d_L^2$$

$$p^- := \sum_{L \in Irr(\mathcal{C})} \theta_L^- d_L^2$$

and are non-zero.

From the relations above, one can show that the s and t matrices give a projective representation of the modular group, $SL_2(\mathbb{Z})$, hence the name modular categories. The s and t matrices provide a surprisingly large amount of information about the category. In particular, we can use them to calculate the fusion coefficients. This is done using the Verlinde formula.

Theorem 2.12 (Verlinde formula)**.** *For a modular category $\mathcal{C}$, we have*

$$N_{LJ}^K = \sum_{R \in Irr(\mathcal{C})} \frac{s_{LR} s_{JR} s_{\overline{K}R}}{s_{\mathbb{1}R}}. \tag{2.19}$$

2.4   Drinfeld center

For a strict tensor category $\mathcal{C}$, one can construct the Drinfeld center, denoted by $\mathcal{Z}(\mathcal{C})$, out of half-braidings.

Definition 2.13. A *half-braiding* on $X \in \mathcal{C}$ is a family of isomorphisms $\{e_X(Y) : XY \to YX\}_{Y \in \mathcal{C}}$ satisfying:

1. *naturality*: For all $Y, Z \in \mathcal{C}$ and all morphisms $f \in \mathcal{C}(Y, Z)$,

$$e_X(Z) \circ (\mathrm{id}_X \otimes f) = (f \otimes \mathrm{id}_X) \circ e_X(Y). \tag{2.20}$$

2. *braid relation*: For all $Y, Z \in \mathcal{C}$,

$$e_X(YZ) = (\mathrm{id}_Y \otimes e_X(Z)) \circ (e_X(Y) \otimes \mathrm{id}_Z). \tag{2.21}$$

As with braidings, we denote half-braiding by crossings of the strings. To avoid any confusion between braidings and half-braiding, we follow a similar convention to that in [JB10][2], and denote strings coming from objects in the center by double green lines. Therefore, any strings that cross under the double green line should be interpreted as a half-braiding. Note that, for convenience, we sometimes label the double green line with the underlying object in the category, rather than the object in the center. For example, We can express the naturality and braid relations of half-braiding using the following diagrammatic equations:

$$\tag{2.22}$$

$$\tag{2.23}$$

Now we define the Drinfeld center of a tensor category.

Definition 2.14. The *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ is a category with objects (X, e_X) where $X \in C$ and e_X is a half braiding of X. Given objects $(X, e_X), (Y, e_Y)$, we define $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((X, e_X), (Y, e_Y))$ as the set of morphisms $f \in \mathcal{C}(X, Y)$ satisfying for all $Z \in C$,

$$\tag{2.24}$$

---

[2]We differ from [JB10] by having the green strings on top in the braiding rather than under.

Remark 2.15. A point on notation: sometimes we may write a morphism $f \in \mathcal{C}(V, W)$, as follows

$$
\begin{array}{c}
W \\
| \\
\boxed{f} \quad . \\
\| \\
(V, e_V)
\end{array}
$$

This allows us to encode the half-braiding of $(V, e_V) \in \mathcal{Z}(\mathcal{C})$ diagrammatically even though f is a morphism in the category.

We can give $\mathcal{Z}(\mathcal{C})$ a strict monoidal structure. Let the tensor product be

$$(X, e_X) \otimes (Y, e_Y) = (XY, e_{XY}) \tag{2.25}$$

where

$$e_{XY}(Z) = (e_X(Z) \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes e_Y(Z)), \tag{2.26}$$

and the unit be $(\mathbb{1}, e_{\mathbb{1}})$ where $e_{\mathbb{1}}(X) = \mathrm{id}_X$. In fact, one can do more, [Mue03b] showed that $\mathcal{Z}(\mathcal{C})$ inherits more structures from $\mathcal{C}$ and is in fact modular.

Theorem 2.16 ([Mue03b] Thm. 1.2)**.** *Let $\Bbbk$ be an algebraically closed field, and $\mathcal{C}$ a spherical fusion category with categorical dimension $\mathcal{D} \neq 0$. Then $\mathcal{Z}(\mathcal{C})$ is a modular category.*

Chapter 3

Generalized Frobenius-Schur indicators

As every object/morphism in $\mathcal{Z}(\mathcal{C})$ corresponds to an object/morphism in C, we can define the *forgetful functor*

$$F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$$

which sends (X, e_X) to X and morphisms in $\mathcal{Z}(\mathcal{C})$ to the underlying morphisms in $\mathcal{C}$. We now have the language to give the formal definition of the generalized Frobenius-Schur indicators.

Definition 3.1. For a spherical fusion category $\mathcal{C}$, $n \in \mathbb{N}$, $X \in \mathcal{C}$ and $W \in \mathcal{Z}(\mathcal{C})$, we define the *rotation operator*

$$\rho_{n,X}^W : \mathcal{C}(F(W), X^{\otimes n}) \to \mathcal{C}(F(W), X^{\otimes n})$$

as the following map
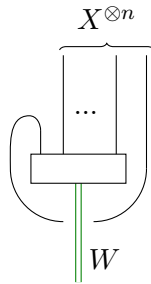
Definition 3.2. The *generalized Frobenius-Schur indicator*, denoted by $\nu_n^W(X)$, is defined as the trace of the rotation operator, that is,

$$\nu_n^W(X) := Tr(\rho_{n,X}^W).$$

The aim of this chapter is to derive a formula for the generalized Frobenius-Schur indicators in terms of modular data of the center. This formula is given in [NS10] as

Cor. 5.6, here we give a direct and self-contained proof. Roughly speaking, we do this by extending the adjunction between the forgetful and induction functor to construct a suitable algebra isomorphism between morphisms in the category and morphisms in the center. Then, using this isomorphism, we find a way of expressing the trace of the rotation operator in terms of the categorical trace of some morphism in the Drinfeld center.

The outline of this chapter is as follows. Section 3.1 is dedicated to defining the induction functor to the Drinfeld center. Section 3.2 gives a proof of the fact that the induction functor is left adjoint to the forgetful functor. Then in section 3.3, we extend the bijection of hom-sets from the adjunction into an algebra isomorphism. In section 3.4 we prove some useful equalities between traces of morphisms. Finally, in section 3.5 we present a closed formula for the generalized Frobenius-Schur indicators.

Note that for simplicity, we abbreviate tensor product as concatenation, for example, $XY := X \otimes Y$.

3.1   Induction to the Drinfeld center

The goal of this section is to define the induction functor to the Drinfeld center, following the approach taken in [JB10].

First we define a special half-braiding for objects $i(X) \in \mathcal{C}$. For $X \in \mathcal{C}$, define $i(X)$ as

$$i(X) = \bigoplus_{J \in \mathrm{Irr}(\mathcal{C})} \overline{J} X J.$$

Then we define a half-braiding for objects of the form $i(X) \in \mathcal{C}$. For

$$e_{i(X)}(Z) : \left(\bigoplus_{J \in \mathrm{Irr}(\mathcal{C})} \overline{J} X J \right) Z \to Z \left(\bigoplus_{J \in \mathrm{Irr}(\mathcal{C})} \overline{J} X J \right),$$

we let the L, J component be

$$\tag{3.1}$$

where

$$
\begin{array}{c}
\vcenter{\hbox{\includegraphics{fig}}}
\end{array}
\qquad (3.2)
$$

Remark 3.3. Once again this does not depend on the choice of basis by remark 1.29, and we remind the reader that the dual basis is with respect to the dual pairing $\langle\,,\,\rangle$ defined in Theorem 1.26.

Remark 3.4. For simplicity, we shorten $e_{i(X)}(Z)_{L,J}$ to just $e_{L,J}$ when the half-braiding can be easily deduced from the string labels.

Now we prove that $e_{i(X)}$ is indeed a half-braiding.

Theorem 3.5. *For a spherical fusion category $\mathcal{C}$ and $X \in \mathcal{C}$, we have that* (3.1) *defines a half braiding of $i(X)$.*

Proof. To show that $e_{i(X)}$ is an isomorphism, we give an inverse, which has components:

$$
\begin{array}{c}
\vcenter{\hbox{\includegraphics{fig}}}
\end{array}
\; = \; \sum_{\varepsilon \in \mathfrak{B}(JZ,L)} \frac{\sqrt{d_L}}{\sqrt{d_J}}
\begin{array}{c}
\vcenter{\hbox{\includegraphics{fig}}}
\end{array}
\qquad (3.3)
$$

where

$$
\begin{array}{c}
\vcenter{\hbox{\includegraphics{fig}}}
\end{array}
\; := \;
\begin{array}{c}
\vcenter{\hbox{\includegraphics{fig}}}
\end{array}
\qquad (3.4)
$$

We claim that for L, K simple objects in $\mathcal{C}$,

$$
\sum_{J \in \mathrm{Irr}(\mathcal{C})} e_{JK} \circ e_{LJ}^{-1} = \delta_{KL} \cdot \mathrm{id}_{Z\overline{L}XL}. \qquad (3.5)
$$

Observe that

$$e_{JK} \circ e_{LJ}^{-1} = \sum_{\substack{\beta \in \mathfrak{B}(JZ,K) \\ \varepsilon \in \mathfrak{B}(JZ,L)}} \frac{\sqrt{d_L}}{\sqrt{d_K}} \quad \text{(3.6)}$$

Since K, L are simple, $e_{JK} \circ e_{LJ}^{-1} = 0$ unless $K = L$. Furthermore, by definition of dual basis, for $\varepsilon_i, \varepsilon_j \in \mathfrak{B}(JZ,L)$, $\varepsilon_i \circ \varepsilon_j^* = \delta_{i,j} \mathrm{id}_L$, therefore

$$e_{JL} \circ e_{LJ}^{-1} = \sum_{\varepsilon \in \mathfrak{B}(JZ,L)} \overline{J} \qquad . \quad \text{(3.7)}$$

By remark 1.29, if we can find some choice of basis of $\beta \in \mathcal{C}(JZ,L)$ such that β' is a basis of $\mathcal{C}(Z\overline{L}, \overline{J})$ and $\hat{\beta}$ is the dual basis of β' with respect to the pairing $\langle \, , \, \rangle$, then (3.5) follows from Cor. 1.28.

First observe that we can construct a linear isomorphism between $\mathcal{C}(JZ,L)$ and $\mathcal{C}(Z\overline{J}, \overline{L})$ using ε, ι and scalar multiplication, thus, for $\beta \in \mathfrak{B}(JZ,L)$,

is a basis of $\mathcal{C}(Z\overline{J}, \overline{L})$.

We claim that

is the corresponding dual basis. To show this, we observe that since $\overline{J}$ is simple, for β_i, β_j in the basis of $\mathcal{C}(JZ, L)$ there exists some $c \in \Bbbk$ such that,

Taking the categorical trace of LHS gives

$$LHS = \frac{d_J}{d_L} \quad\raisebox{-1em}{\includegraphics{}} = \frac{d_J}{d_L} \quad\raisebox{-1em}{\includegraphics{}} . \tag{3.8}$$

By lemma 1.10, we know that this is just the same as $tr_L(\beta_i * \circ \beta_j)$, so the LHS is 0 when $\beta_i \neq \beta_j$ and d_J when $i = j$. Since the categorical trace of the LSH is just $c\, d_{\overline{L}}$. Therefore $c = 0$ when $i \neq j$ and $c = 1$ when $i = j$.

Therefore, by making a good choice of basis, we have deduced

$$\sum_{L \in \mathrm{Irr}(\mathcal{C})} e_{JK} \circ e_{LJ}^{-1} = \delta_{KL} \cdot \mathrm{id}_{Z\overline{L}XL}.$$

A similar argument can be applied to show

$$\sum_{L \in \mathrm{Irr}(\mathcal{C})} e_{JK}^{-1} \circ e_{LJ} = \delta_{KL} \cdot \mathrm{id}_{\overline{L}XLZ}.$$

To show naturality, we need that for all $W, Z \in \mathcal{C}$, $f \in \mathcal{C}(Z, W)$, and for all $L, J \in \mathrm{Irr}(\mathcal{C})$, we have

$$\sum_{\beta \in \mathfrak{B}(LW, J)} \raisebox{-1em}{\includegraphics{}} = \sum_{\varepsilon \in \mathfrak{B}(LZ, J)} \raisebox{-1em}{\includegraphics{}} . \tag{3.9}$$

Since $\{\varepsilon\}$ forms a basis of $\mathcal{C}(LZ, J)$, then for each $\beta \in \mathfrak{B}(LW, J)$, there exists $c_{\varepsilon,\beta,J} \in \Bbbk$ such that

$$
\begin{array}{ccc}
\begin{array}{c} J \\ \beta \\ W \\ f \\ L \quad Z \end{array}
&=&
\displaystyle\sum_{\varepsilon \in \mathfrak{B}(LZ,J)} c_{\varepsilon,\beta,J}
\begin{array}{c} J \\ \varepsilon \\ L \quad Z \end{array}
\end{array}
\tag{3.10}
$$

Similarly, since $\{\beta^*\}$ forms a basis of $\mathcal{C}(J, LW)$, there exists $d_{\varepsilon,\beta,J} \in \Bbbk$ such that

$$
\begin{array}{ccc}
\begin{array}{c} L \quad W \\ f \\ Z \\ \varepsilon^* \\ J \end{array}
&=&
\displaystyle\sum_{\beta \in (LW,J)} d_{\varepsilon,\beta,J}
\begin{array}{c} L \quad W \\ \beta^* \\ J \end{array}
\end{array}
\tag{3.11}
$$

Thus, we have

$$
\begin{array}{ccc}
\begin{array}{c} W \quad \overline{J} \\ f \\ Z \\ \hat{\varepsilon^*} \\ \overline{L} \end{array}
&=&
\displaystyle\sum_{\beta \in (LW,J)} d_{\varepsilon,\beta,J}
\begin{array}{c} W \quad \overline{J} \\ \hat{\beta^*} \\ \overline{L} \end{array}
\end{array}
\tag{3.12}
$$

Since $\{\beta\}$ and $\{\varepsilon\}$ are bases of $\mathcal{C}(LW, J)$ and $\mathcal{C}(LZ, J)$ respectively, by Cor. 1.28

$$
\begin{array}{ccccc}
\begin{array}{c} L \quad W \\ | \quad f \\ L \quad Z \end{array}
&=&
\displaystyle\sum_{\substack{J \in \mathrm{Irr}(\mathcal{C}) \\ \beta \in \mathfrak{B}(LW,J)}}
\begin{array}{c} L \quad W \\ \beta^* \\ J \\ \beta \\ W \\ f \\ L \quad Z \end{array}
&=&
\displaystyle\sum_{\substack{J \in \mathrm{Irr}(\mathcal{C}), \\ \varepsilon \in \mathfrak{B}(LZ,J)}}
\begin{array}{c} L \quad Z \\ f \\ W \\ \varepsilon^* \\ J \\ \varepsilon \\ L \quad Z \end{array}
\end{array}
\tag{3.13}
$$

Substituting (3.10) and (3.12) into (3.13) then pre-composing and post-composing with ε^* and β give us

$$c_{\varepsilon,\beta,J} = d_{\varepsilon,\beta,J}.$$

Thus we have proven (3.9).

To prove the braid relations, let $\alpha \in \mathfrak{B}(LY, K)$ and $\beta \in \mathfrak{B}(KZ, J)$, then the L, J component of $(\mathrm{id}_Y \otimes e_X(Z)) \circ (e_X(Y) \otimes \mathrm{id}_Z)$ is

$$\sum_{\substack{K \in \mathrm{Irr}(\mathcal{C}), \\ \alpha \in \mathfrak{B}(LY,K), \\ \beta \in \mathfrak{B}(KZ,J)}} \frac{\sqrt{d_L}}{\sqrt{d_J}} \quad \vcenter{\hbox{}} \quad . \tag{3.14}$$

By semisimplicity, the set consisting of
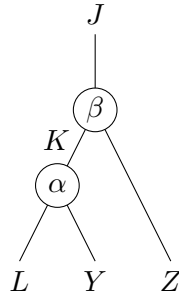
where K ranges over all simple objects in $\mathcal{C}$ and $\alpha \in \mathfrak{B}(LY, K), \beta \in \mathfrak{B}(KZ, J)$ form a basis of $\mathcal{C}(L \otimes (YZ), J)$, with the corresponding dual basis given by
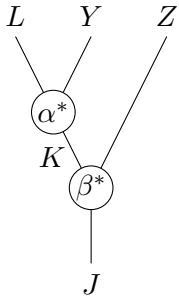
Since $\mathcal{C}$ is pivotal, we have
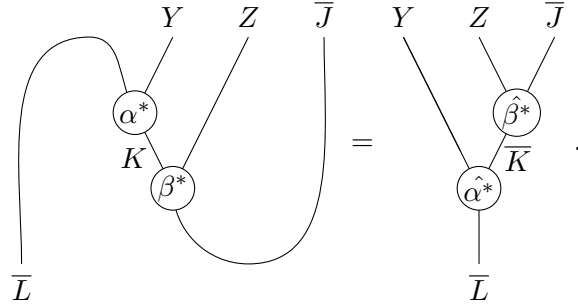
Thus the L, J component of $(\mathrm{id}_Y \otimes e_X(Z)) \circ (e_X(Y) \otimes \mathrm{id}_Z)$ is equal to the L, J component of $e_X(YZ)$ by remark 1.29. Therefore the braid relations are satisfied. $\square$

Now we have all the ingredients needed to define the induction functor $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. For $X \in \mathcal{C}$, let

$$I(X) = (i(X), e_{i(X)})$$

where

$$i(X) = \bigoplus_{J \in \mathrm{Irr}(\mathcal{C})} \overline{J} X J$$

and $e_{i(X)}$ is as defined in Section (3.1).

We define I for morphisms. Let $f : X \to Y$, then $I(f) : I(X) \to I(Y)$ has L, J component 0 if $L \neq J$, and has component

$$(3.15)$$

if $L = J$.

It is easy to see that $I(f)$ is indeed a morphism in $\mathcal{Z}(\mathcal{C})$. Since the only non-zero components are $I(f)_{LL}$ for all $L \in \mathrm{Irr}(\mathcal{C})$, this is equivalent to showing that for all $L, J \in \mathrm{Irr}(\mathcal{C})$ and $Z \in (\mathcal{C})$ we have

$$
\begin{array}{c}
Z \quad \overline{J} \quad Y \quad J \\
\boxed{e_{LJ}} \\
\boxed{\overline{L} \mid Y \mid L} \\
\boxed{I(f)_{LL}} \\
\overline{L} \quad X \quad L \quad Z
\end{array}
\;=\;
\begin{array}{c}
Z \quad \overline{J} \quad Y \quad J \\
\boxed{I(f)_{LL}} \\
\boxed{\overline{J} \mid X \mid J} \\
\boxed{e_{LJ}} \\
\overline{L} \quad X \quad L \quad Z
\end{array}
\; . \qquad (3.16)
$$

An easy substitution of the half-braiding definition shows that the above equality must hold. It is also immediately obvious that $I(f \circ g) = I(f) \circ I(g)$. This confirms that I is in fact a functor.

3.2 Adjoint to the forgetful functor

This section shows that the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is in fact biadjoint to I. We first prove a useful lemma which helps us understand the relationship between the different components $g_K : V \to \overline{K}XK$ of a morphism $g \in \mathcal{Z}(\mathcal{C})(V, I(X))$.

Lemma 3.6. *Let $g : V \to I(X)$ be a morphism in $\mathcal{Z}(\mathcal{C})$, and $g_K : V \to \overline{K}XK$ be the K-th component of g, then*

$$
\begin{array}{c}
\overline{K}XK \\
\boxed{g_K} \\
V
\end{array}
\;=\; \sqrt{d_K}\;
\begin{array}{c}
\overline{K} \quad X \quad K \\
\boxed{g_{\mathbb{1}}} \\
V
\end{array}
\; . \qquad (3.17)
$$

Proof. Since g is a morphism in the center, we know g must satisfy the following

$$
\begin{array}{c}
\overline{K} \quad I(X) \\
\boxed{g} \\
V \quad \overline{K}
\end{array}
\;=\;
\begin{array}{c}
\overline{K} \quad I(X) \\
\boxed{g} \\
V \quad \overline{K}
\end{array}
\; . \qquad (3.18)
$$

Picking the $\mathbb{1}$-th component of (3.18), we have

$$(3.19)$$

Since K and L are both simple, $\mathcal{C}(L\overline{K}, \mathbb{1})$ is one-dimensional if $L = K$ and zero otherwise. Therefore the evaluation map, ε_I, and the coevaluation map, ι_I can be used to give a basis and dual basis of $\mathcal{C}(L\overline{K}, \mathbb{1}) = \mathcal{C}(L\overline{L}, \mathbb{1})$:

$$\beta := \frac{1}{\sqrt{d_K}} \varepsilon_K$$

$$\beta^* := \frac{1}{\sqrt{d_K}} \iota_K$$

Substituting for the basis and dual basis simplifies (3.19) to

$$(3.20)$$

Composing with $\iota_{\overline{K}}$ gives us the desired equality. $\square$

This lemma is crucial in proving that the induction and forgetful functor are biadjoint as it gives us a way of constructing a canonical bijection of hom-sets.

Theorem 3.7. *The induction functor $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is right adjoint to the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$.*

Proof. We need to show that for all $V \in \mathcal{Z}(\mathcal{C})$ and $X \in \mathcal{C}$, there exists a natural isomorphism

$$\Phi : \mathcal{C}(F(V), Y) \to \mathcal{Z}(\mathcal{C})(V, I(Y)).$$

For $f \in \mathcal{C}(F(V), Y)$, we define $\Phi(f) : V \to I(Y)$ component-wise as follows

$$\Phi(f)_{\mathbb{1}} = f \qquad (3.21)$$

$$
\boxed{\Phi(f)_K} = \sqrt{d_K} \quad \begin{array}{c} \overline{K} \quad Y \quad K \\ \boxed{f} \end{array} \tag{3.22}
$$

We first show that for all $f \in \mathcal{C}(F(V), Y)$, $\Phi(f)$ is a morphism in the center. This is equivalent to checking that for all $W \in \mathcal{C}$,

$$
\boxed{\Phi(f)} \quad = \quad \boxed{\Phi(f)} \quad . \tag{3.23}
$$

Observe that the K-th component of the RHS of (3.23) can also be written as:

$$
\sum_{L \in \mathrm{Irr}(\mathcal{C})} \sqrt{d_L} \quad \begin{array}{c} W \quad \overline{K} \quad Y \quad K \\ \boxed{e_{L,K}} \\ \overline{L} \quad \boxed{f} \quad L \end{array} \tag{3.24}
$$

Using $\alpha \in \mathfrak{B}(K\overline{W}, L)$, we can produce the following basis and dual basis of $\mathcal{C}(LW, K)$:

$$
\begin{array}{c} K \\ \beta \\ L \quad W \end{array} := \frac{\sqrt{d_K}}{\sqrt{d_L}} \quad \begin{array}{c} K \\ \alpha^* \\ L \quad W \end{array} \tag{3.25}
$$

$$
\begin{array}{c} L \quad W \\ \beta^* \\ K \end{array} := \frac{\sqrt{d_K}}{\sqrt{d_L}} \quad \begin{array}{c} L \quad W \\ \alpha \\ K \end{array} \tag{3.26}
$$

Substituting the basis into (3.24) gives us

$$\sqrt{d_K} \sum_{\substack{L \in \mathrm{Irr}(\mathcal{C}) \\ \alpha \in \mathfrak{B}(K\overline{W}, L)}} \qquad\qquad\qquad . \tag{3.27}$$

By the pivotal structure of the category, we have

$$\qquad\qquad = \qquad\qquad . \tag{3.28}$$

By the above fact, and that we can pull α under the half-braiding, (3.27) becomes

$$\sqrt{d_K} \sum_{\substack{L \in \mathrm{Irr}(\mathcal{C}) \\ \alpha \in \mathfrak{B}(K\overline{W}, L)}} \qquad\qquad\qquad . \tag{3.29}$$

Since α form a basis of $(K\overline{W}, L)$, by Cor. 1.28, we know (3.27) is equal to

$$\sqrt{d_K} \qquad\qquad\qquad , \tag{3.30}$$

which is the same as the K-th component of the LHS of (3.23), so $\Phi(f)$ is a morphism in the center.

By Lemma 3.6, we know that every morphism in $Z(\mathcal{C})(V, I(Y))$ is uniquely determined by its $\mathbb{1}$-th component, thus Φ is a bijection.

To show that the map is natural, we show that for all $Y, Y' \in \mathcal{C}$, $V, V' \in \mathcal{Z}(\mathcal{C})$, $f \in \mathcal{C}(Y, Y')$, $g \in \mathcal{Z}(\mathcal{C})(V', V)$

$$
\begin{array}{ccc}
\mathcal{C}(F(V), Y) & \xrightarrow{\Phi} & \mathcal{Z}(\mathcal{C})(V, I(Y)) \\
\downarrow{\scriptstyle (F(g), f)} & & \downarrow{\scriptstyle (g, I(f))} \\
\mathcal{C}(F(V'), Y') & \xrightarrow{\Phi} & \mathcal{Z}(\mathcal{C})(V', I(Y'))
\end{array}
$$

commutes. That is, for all $h \in \mathcal{C}(F(V), Y)$, we want

$$\Phi(f \circ h \circ F(g)) = I(f) \circ \Phi(h) \circ g. \tag{3.31}$$

Since the morphisms on both sides are in $\mathcal{Z}(C)(V', I(Y)) \subseteq \mathcal{C}(F(V'), I(Y))$, we can compare the K-th component, which is in $\mathcal{C}(F(V'), \overline{K}YK)$. By the definition of I, we can derive that

$$(I(f) \circ \Phi(h))_K = (I(f)_{K,K} \circ \Phi(h))_K = \Phi(f \circ h)_K. \tag{3.32}$$

So the K-th component of the RHS is

$$\sqrt{d_K} \quad \begin{array}{c} \overline{K} \quad Y \quad K \\ \boxed{f} \\ \boxed{h} \\ \boxed{g} \\ V \end{array} \quad . \tag{3.33}$$

The K-th component of the LHS, by the definition of Φ, is

$$\sqrt{d_K} \quad \begin{array}{c} \overline{K} \quad Y \quad K \\ \boxed{f} \\ \boxed{h} \\ \boxed{F(g)} \\ V \end{array} \quad . \tag{3.34}$$

Since g is a morphism in the centre we can move $F(g)$ down the half-braiding, so we get that it is equal to the RHS.

$\square$

Remark 3.8. In fact, I is biadjoint with F. By flipping the diagrams in proof upside down, we can also show that I is left adjoint to F. That is, we can construct a natural isomorphism

$$\Psi : \mathcal{C}(Y, F(V)) \to \mathcal{Z}(\mathcal{C})(I(Y), V)$$

for $V \in \mathcal{Z}(\mathcal{C})$ and $X \in \mathcal{C}$ as follows:

Let $f \in (Y, F(V))$. Define

$$\Psi(f)_{\mathbb{1}} = f \tag{3.35}$$

$$\tag{3.36}$$

By a similar argument as before, we can show that our definition of Ψ is indeed good.

3.3   The tube algebra on $\mathcal{C}(i(X), X)$

Pick $X \in \mathcal{C}$. Recall that the adjunction in section 3.2 gives us a bijection of sets by Φ^{-1}:

$$\mathcal{C}(i(X), X) = \mathcal{C}(FI(X), X) \simeq \mathcal{Z}(\mathcal{C})(I(X), I(X))$$

Since we are working in $\Bbbk$-linear categories, Φ is also compatible with addition and multiplication by scalars. Thus to produce an algebra isomorphism, it suffices to specify a multiplicative structure on $\mathcal{C}(i(X), X)$. This gives rise to the tube algebra on $\mathcal{C}(i(X), X)$. The construction of the tube algebra is due to [Ocn94].

Definition 3.9. Given $X \in \mathcal{C}$, we define the tube algebra of X as

$$Tube(X) = \mathcal{C}(i(X), X) = \bigoplus_{J \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(\overline{J}XJ, X),$$

which inherits the additive structures of homspaces from $\mathcal{C}$. Given $f, g \in \mathcal{C}(i(X), X)$, define multiplication $g \cdot f$ as

$$g \cdot f := \bigoplus_{L \in \mathrm{Irr}(\mathcal{C})} \sum_{\substack{K,J \in \mathrm{Irr}(\mathcal{C}) \\ \alpha \in \mathfrak{B}(JK,L)}} \frac{\sqrt{d_K}\sqrt{d_J}}{\sqrt{d_L}}$$

(3.37)

where the notation $\overline{\alpha}$ defined in 1.7.

Remark 3.10. It can also be shown that multiplication in the tube algebra is associative by applying remark 1.29.

Theorem 3.11. *Let $\mathcal{C}$ be a spherical fusion category and $X \in \mathcal{C}$. Then Φ defines an algebra isomorphism between $Tube(X)$ and $\mathcal{Z}(\mathcal{C})(I(X), I(X))$.*

Proof. Let $f, g \in \mathcal{C}(I(X), I(X))$. It suffices to show

$$\Phi^{-1}(g \circ f) = \Phi^{-1}(g) \cdot \Phi^{-1}(f).$$

(3.38)

Observe that

$$\Phi^{-1}(g) \cdot \Phi^{-1}(f) = \bigoplus_{L \in \mathrm{Irr}(\mathcal{C})} \sum_{\substack{K,J \in \mathrm{Irr}(\mathcal{C}) \\ \alpha \in \mathfrak{B}(JK,L)}} \frac{\sqrt{d_K}\sqrt{d_J}}{\sqrt{d_L}}$$

(3.39)

On the other hand, we know

$$\Phi^{-1}(g \circ f) = \sum_{K \in \mathrm{Irr}(\mathcal{C})}$$

$$= \sum_{K \in \mathrm{Irr}(\mathcal{C})} \sqrt{d_K}$$

(3.40)

Substituting the definition of the half-braiding into (3.40) gives

$$
\Phi^{-1}(g \circ f) = \bigoplus_{L \in \mathrm{Irr}(\mathcal{C})} \sum_{\substack{K,J \in \mathrm{Irr}(\mathcal{C}) \\ \beta \in \mathfrak{B}(L\overline{K},J)}} \sqrt{d_K}\frac{\sqrt{d_L}}{\sqrt{d_J}} \quad \vcenter{\hbox{[string diagram]}} \quad . \tag{3.41}
$$

For $\{\alpha\}$ a basis for $\mathcal{C}(JK,L)$, we can give a basis $\{\beta\}$ for $\mathcal{C}(L\overline{K},J)$ by letting

$$
\vcenter{\hbox{[diagram]}} \; := \; \frac{\sqrt{d_J}}{\sqrt{d_L}} \; \vcenter{\hbox{[diagram]}} \quad . \tag{3.42}
$$

It is easy to check that the corresponding dual basis $\{\beta^*\}$ is

$$
\vcenter{\hbox{[diagram]}} \; := \; \frac{\sqrt{d_J}}{\sqrt{d_L}} \; \vcenter{\hbox{[diagram]}} \quad . \tag{3.43}
$$

Substituting (3.42) and (3.43) into (3.41) gives us the expression in (3.39). So we are done. $\qquad\square$

3.4 Tying the strings together

Now we prove some useful facts about the relationship between the categorical traces of morphisms in $\mathcal{C}(i(X),X)$ and $\mathcal{Z}(\mathcal{C})(I(X),I(X))$. This gives us the key ingredients we need for the indicator formula.

Theorem 3.12. *Let $\mathcal{C}$ be a spherical fusion category, $X \in \mathcal{C}$ and $f \in \mathcal{C}(i(X),X)$. Then*

$$
\text{} \qquad\qquad (3.44)
$$

where $f_{\mathbb{1}} \in \mathcal{C}(X,X)$ denotes the $\mathbb{1}$-th component of f and $\mathcal{D}$ is the categorical dimension of $\mathcal{C}$.

Proof. First rewrite $i(X)$ as $\sum_L \overline{L}XL$, and observe that since L, J are both simple objects, $\mathcal{C}(L,J)$ is one-dimensional if $L = J$ and is 0 otherwise, therefore we only need to consider the terms when $L = J$. Then using sphericality of $\mathcal{C}$, we get

$$
\text{} \qquad\qquad (3.45)
$$

Applying Lemma 3.6 and the definition of the half-braiding, we get

$$
\sum_{\substack{L,K \in \mathrm{Irr}(\mathcal{C}) \\ \beta \in \mathfrak{B}(L\overline{L},K)}} \sqrt{d_L} \frac{\sqrt{d_L}}{\sqrt{d_K}} \quad \text{} \qquad\qquad (3.46)
$$

Since $\mathcal{C}$ is pivotal,

As β is a morphism from $\mathbb{1}$ to K, $\mathcal{C}(L\overline{L},K)$ must be one-dimensional when $K = \mathbb{1}$ and 0 otherwise. So it is only necessary to consider the term when $K = \mathbb{1}$. Picking the basis of $\mathcal{C}(L\overline{L},\mathbb{1})$ to be $\frac{1}{\sqrt{d_L}}\varepsilon_L$ and the corresponding dual-basis to be $\frac{1}{\sqrt{d_L}}\iota_L$ and

substituting this into (3.46) gives us

$$
\left(\boxed{\Phi(f)} \; I(X) \right) = \sum_{L \in \mathrm{Irr}(\mathcal{C})} \left(X \atop \boxed{f_{\mathbb{1}}} \; L \; \overline{L} \right) = \sum_{L \in \mathrm{Irr}(\mathcal{C})} d_L^2 \left(\boxed{f_{\mathbb{1}}} \atop X \right) . \tag{3.47}
$$

$\square$

We want to find a way of expressing trace of morphisms involving half-braidings in terms of $\mathcal{C}(i(X), X)$. To do this, we first have to prove a lemma on how the half-braidings split under direct sums.

Lemma 3.13. *Let $\mathcal{C}$ be a spherical fusion category, $W \in \mathcal{Z}(\mathcal{C})$ and $X \in C$. We have*

$$
e_W(i(X)) = \bigoplus_{L \in Irr(\mathcal{C})} e_W(\overline{L}XL). \tag{3.48}
$$

Proof. Since $i(X)$ is a direct sum, there exist projections $\pi_L : i(X) \to \overline{L}XL$ and coprojections $\alpha_L : \overline{L}XL \to i(X)$ such that for $L, J \in \mathrm{Irr}(\mathcal{C})$, the following holds:

$$
\pi_L \cdot \alpha_J = \delta_{L,J} \mathrm{id}_L \tag{3.49}
$$

$$
\sum_{L \in \mathrm{Irr}(\mathcal{C})} \alpha_L \cdot \pi_L = \mathrm{id}_{i(X)} \tag{3.50}
$$

By (3.50) and the naturality of the half-braiding e_W, we have

$$
e_W(i(X)) = \sum_{L \in \mathrm{Irr}(\mathcal{C})} \left(\begin{array}{c} i(X) \quad W \\ \boxed{\alpha_L} \\ \overline{L}XL \\ \boxed{\pi_L} \\ W \quad i(X) \end{array} \right) = \sum_{L \in \mathrm{Irr}(\mathcal{C})} \left(\begin{array}{c} i(X) \quad W \\ \boxed{\alpha_L} \\ \overline{L}XL \\ \overline{L}XL \\ \boxed{\pi_L} \\ F(W) \; i(X) \end{array} \right) = \bigoplus_{L \in \mathrm{Irr}(\mathcal{C})} e_W(\overline{L}XL).
$$

$$\tag{3.51}$$

$\square$

Now we give an explicit computation of Φ^{-1} on a specific morphism in $\mathcal{Z}(\mathcal{C})(I(X), I(X))$. As we will find out in the next section, this is a key ingredient for constructing the morphism in the center whose categorical trace is the Frobenius-Schur indicator.

Theorem 3.14. *Let $\mathcal{C}$ be a spherical fusion category. For $X \in \mathcal{C}$ and $W \in Irr(\mathcal{Z}(\mathcal{C}))$,*

$$\Phi \left(\bigoplus_{L \in Irr(\mathcal{C})} \sum_{\alpha \in \mathfrak{B}(F(W),L)} \frac{1}{\sqrt{d_L}} \ \overline{\alpha} \quad \alpha^* \right) = W \quad \text{(3.52)}$$

where $\overline{\alpha}$ is defined in definition 1.7.

Proof. By the braid relation and lemma 3.13,

$$W \left(\ I(X) \ \right) = \bigoplus_{L \in Irr(\mathcal{C})} \sum_{\substack{J \in Irr(\mathcal{C}) \\ \beta \in \mathfrak{B}(I \otimes F(\overline{W}),J)}} \frac{\sqrt{d_L}}{\sqrt{d_J}} \ \hat{\beta}^* \quad \beta \ \overline{W} \quad . \quad \text{(3.53)}$$

Since Φ^{-1} sends $f \in Z(C)(I(X), I(X))$ to the morphism $f_{\mathbb{1}} \in C(i(X), X)$, we can let $J = \mathbb{1}$. Therefore, for $\{\alpha\}$ a basis of $(F(W), L)$ then we can define $\{\beta\}$ a basis of $\mathcal{C}(L \otimes F(\overline{W}), \mathbb{1})$ and a corresponding dual basis as follows:

$$\beta \ := \frac{1}{\sqrt{d_L}} \ \alpha^* \quad \text{(3.54)}$$

$$\beta^* \ := \frac{1}{\sqrt{d_L}} \ \alpha \quad . \quad \text{(3.55)}$$

Using the basis and dual basis defined above gives us

$$
\Phi^{-1}\left(\ \begin{array}{c} I(X) \\ W \\ I(X) \end{array}\ \right) = \bigoplus_{L\in\mathrm{Irr}(\mathcal{C})}\ \sum_{\alpha\in\mathfrak{B}(F(W),L)} \frac{1}{\sqrt{d_L}}\ \ \overline{W} \quad .
$$
(3.56)

By naturality of the half-braiding, we know we can slide morphisms under the braiding, giving us

$$
\Phi^{-1}\left(\ \begin{array}{c} I(X) \\ W \\ I(X) \end{array}\ \right) = \bigoplus_{L\in\mathrm{Irr}(\mathcal{C})}\ \sum_{\alpha\in\mathfrak{B}(F(W),L)} \frac{1}{\sqrt{d_L}}\ \ W \quad . \qquad (3.57)
$$

Since $W \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$, by a similar reasoning as in Lemma 2.7, we know that the two twists will cancel each other, giving us the statement we want.

$\square$

Corollary 3.15. *Let $\mathcal{C}$ be a spherical fusion category, $X \in \mathcal{C}$, $W \in Irr(\mathcal{Z}(\mathcal{C}))$, and $f \in C(i(X), X)$. Suppose also that we denote the Kth component of f by $f_K \in \mathcal{C}(\overline{K}XK, X)$, then*

$$
\frac{1}{\mathcal{D}}\ \ \Phi(f) \ = \ \sum_{\substack{L\in Irr(\mathcal{C}) \\ \alpha\in\mathfrak{B}(F(W),L)}} \frac{1}{\sqrt{d_L}}\ \ f_L
$$
(3.58)

where $\mathcal{D}$ is the categorical dimension of $\mathcal{C}$.

Proof. By Theorem 3.11 and 3.14, we know

$$
\Phi^{-1}\left(\begin{array}{c} I(X) \\ W \\ \Phi(f) \\ I(X) \end{array} \right) = \bigoplus_{K\in\mathrm{Irr}(\mathcal{C})} \sum_{\substack{L,J\in\mathrm{Irr}(\mathcal{C}) \\ \alpha\in\mathfrak{B}(F(W),L) \\ \beta\in\mathfrak{B}(\overline{J}K,L)}} \frac{1}{\sqrt{d_L}}\frac{\sqrt{d_L}\sqrt{d_J}}{\sqrt{d_K}} \quad .
$$

(3.59)

Now consider the $\mathbb{1}$-th component of the morphism above, $\mathcal{C}(JL,K)$ is non-zero if and only if $L = \overline{J}$. In particular, $\mathcal{C}(\overline{L}L,\mathbb{1})$ is one-dimensional, so we can pick $\frac{1}{\sqrt{d_L}}\,\varepsilon_{\overline{L}}$ to be a basis and $\frac{1}{\sqrt{d_L}}\,\iota_{\overline{L}}$ to be the corresponding dual basis. Then, an easy calculation using Theorem 3.12 gives us the statement we are after. $\qquad\square$

3.5 A formula for the Frobenius-Schur indicators

In this section, we show that by finding a morphism f such that $\Phi(f) = \theta^n_{I(X)}$, Cor. 3.15 gives us an expression of the generalized Frobenius-Schur indicators in terms of morphisms in the center.

First, we define $q_X \in \mathcal{C}(i(X),X)$ and show that $q_X = \Phi^{-1}(\theta_{I(X)})$.

Definition 3.16. Let $\mathcal{C}$ be a spherical fusion category. For $X \in \mathcal{C}$, define $q_X \in \mathcal{C}(i(X),X)$ by specifying the components $(q_X)_L \in \mathcal{C}(\overline{L}XL,X)$ to be

$$
(q_X)_L = \sum_{\beta\in\mathfrak{B}(X,\overline{L})} \frac{1}{\sqrt{d_L}} \quad .
$$

Theorem 3.17. *Let $\mathcal{C}$ be a spherical fusion category and $X \in \mathcal{C}$. Then*

$$
\Phi^{-1}(\theta_{I(X)}) = q_X.
$$

Proof. Recalling the definition of the twist, we have

$$
\begin{array}{cc}
I(X) & i(X) \\
\end{array}
$$

$$\tag{3.60}$$

By Lemma 3.13, we know that the half-braiding only has diagonal components, therefore

$$
\theta_{I(X)} = \bigoplus_{L \in \mathrm{Irr}(\mathcal{C})} \sum_{J \in \mathrm{Irr}(\mathcal{C})} \quad
$$

$$\tag{3.61}$$

Taking the component $J = \mathbb{1}$ and writing out the half-braiding, we get

$$
\Phi^{-1}(\theta_{I(X)}) = \bigoplus_{L \in \mathrm{Irr}(\mathcal{C})} \sum_{\gamma \in \mathfrak{B}(LX,\mathbb{1})} \frac{\sqrt{d_L}}{\sqrt{d_{\mathbb{1}}}}
$$

$$\tag{3.62}$$

For $\{\beta\}$ a basis of $\mathcal{C}(X, \overline{L})$, we can define a basis $\{\gamma\}$ of $\mathcal{C}(LX, \mathbb{1})$ and a corresponding dual basis as follows:

$$\tag{3.63}$$

$$\tag{3.64}$$

Substituting for the basis and dual basis gives us the statement we have set out to prove. □

Lemma 3.18. *Let $\mathcal{C}$ be a spherical fusion category and $X \in \mathcal{C}$. Then*

$$\Phi^{-1}(\theta_{I(X)}^n) = \bigoplus_{L \in Irr(\mathcal{C})} \sum_{\beta \in \mathfrak{B}(\overline{X}^n, \overline{L})} \frac{1}{\sqrt{d_L}} \quad \text{(3.65)}$$

Proof. By Theorem 3.11, we know

$$\Phi^{-1}(\theta_{I(X)}^n) = \Phi^{-1}(\theta_{I(X)})^n \quad \text{(3.66)}$$

where the LHS is multiplication in the tube algebra. Further observe that by Theorem 3.17, we know

$$\Phi^{-1}(\theta_{I(X)})^n = (q_X)^n. \quad \text{(3.67)}$$

Then the rest follows from a straight forward computation. $\qquad\square$

Finally, we give a formula for the generalized Frobenius-Schur indicators!

Theorem 3.19 (Generalized FS indicators formula, [NS10])**.** *Let $\mathcal{C}$ be a spherical fusion category, $X \in \mathcal{C}$, $W \in Irr(\mathcal{Z}(\mathcal{C}))$. Then*

$$\nu_n^W(X) = \frac{1}{\mathcal{D}_\mathcal{C}} \sum_{Y \in Irr(\mathcal{Z}(\mathcal{C}))} \tilde{s}_{(W,Y)} \, \theta_Y^n \, dim(\mathcal{C}(F(Y), X))$$

where $\mathcal{D}_\mathcal{C}$ is the categorical dimension of $\mathcal{C}$.

Proof. By the definition of the trace of a linear operator,

$$\nu_n^W(X) = Tr(\rho_{n,X}^W) = \sum_{\gamma \in \mathfrak{B}(F(W), X^n)} \quad \text{(3.68)}$$

Now observe that by Cor. 3.15 and Lemma 3.18,

$$\frac{1}{\mathcal{D}} \quad = \sum_{\substack{L\in\mathrm{Irr}(\mathcal{C}) \\ \alpha\in\mathfrak{B}(F(W),L) \\ \beta\in\mathfrak{B}(\overline{X}^n,\overline{L})}} \frac{1}{d_L} \quad . \tag{3.69}$$

For all $L \in \mathrm{Irr}(\mathcal{C})$ and $\alpha \in \mathfrak{B}(F(W),L)$, $\beta \in \mathfrak{B}(\overline{X}^n, \overline{L})$, we define γ a basis of $\mathcal{C}(F(W), X^n)$ and a dual basis (with respect to the second dual pairing $(\,,\,)$ in chapter 1) by letting:

$$\boxed{\gamma} \quad =: \frac{1}{\sqrt{d_L}} \tag{3.70}$$

$$\boxed{\gamma^\star} \quad =: \frac{1}{\sqrt{d_L}} \tag{3.71}$$

Substituting (3.70) and (3.71) into (3.69) gives us

$$\frac{1}{\mathcal{D}} \quad = \sum_{\gamma\in\mathfrak{B}(W,X^n)} \quad . \tag{3.72}$$

By sphericality and a similar argument to the one in Theorem 3.14, we know that we can move the $\overline{X}$ strand down to match the diagram in (3.68). Therefore,

$$Tr(\rho_{n,X}^W) = \frac{1}{\mathcal{D}} \quad \text{[diagram]} \quad . \tag{3.73}$$

By Theorem 2.16, we know that $\mathcal{Z}(\mathcal{C})$ is semisimple, so we can express $I(X)$ as a direct sum of simple objects in $\mathcal{Z}(\mathcal{C})$, that is

$$I(X) = \bigoplus_{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))} (Y)^{n_Y} \tag{3.74}$$

where n_Y is the number of times Y appears in the direct sum of $I(X)$. By (1.13), we know

$$n_Y = \dim(\mathcal{Z}(\mathcal{C})(Y, I(X))).$$

Since I is right adjoint to F, we know

$$n_Y = \dim(\mathcal{C}(F(Y), X)).$$

For each $Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ and j of n_Y copies of Y in $I(X)$, we have a projection $\pi_{Y,j} \in \mathcal{Z}(\mathcal{C})(I(X), Y)$ and coprojection $\alpha_{Y,j} \in \mathcal{Z}(\mathcal{C})(Y, I(X))$, such that

$$\mathrm{id}_{I(X)} = \sum_{\substack{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C})) \\ j \in [1,2,\ldots,n_Y]}} \alpha_{Y,j} \circ \pi_{Y,j}. \tag{3.75}$$

Therefore,

$$\text{[diagram]} = \sum_{\substack{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C})) \\ j \in [1,2,\ldots,n_Y]}} \text{[diagram]} \quad . \tag{3.76}$$

Since $\pi_{Y,j}$ and $\alpha_{Y,j}$ are morphisms in $\mathcal{Z}(\mathcal{C})$, we can slide them up and down any half-braiding. Furthermore, by Theorem 2.4, $\pi_{Y,j} \circ \theta_{I(X)}^n = \theta_Y^n \circ \pi_{Y,j}$, so we have

$$Tr(\rho_{n,X}^W) = \frac{1}{\mathcal{D}} \sum_{\substack{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C})) \\ j \in [1,2,...,n_Y]}} \quad \text{(diagram)} \qquad (3.77)$$

By, Lemma 1.10, we can move $\pi_{Y,j}$ up to the top, giving us

$$Tr(\rho_{n,X}^W) = \frac{1}{\mathcal{D}} \sum_{\substack{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C})) \\ j \in [1,2,...,n_Y]}} \quad \text{(diagram)} \qquad (3.78)$$

From the definition of finite sum, we know $\pi_{Y,j} \circ \alpha_{Y,j} = \mathrm{id}_Y$. Since Y is a simple object of $\mathcal{Z}(\mathcal{C})$, we can rewrite the twists in terms of θ_Y by definition 2.3, we have

$$Tr(\rho_{n,X}^W) = \frac{1}{\mathcal{D}} \sum_{\substack{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C})) \\ j \in [1,2,...,n_Y]}} \theta_Y^n \quad \text{(diagram)} = \frac{1}{\mathcal{D}} \sum_{\substack{Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C})) \\ j \in [1,2,...,n_Y]}} \tilde{s}_{W,Y} \, \theta_Y^n .$$

$$(3.79)$$

Since $n_Y = \dim(\mathcal{C}(F(Y), X))$, we have the statement we want. $\qquad \square$

Remark 3.20. Though we assumed that W is a simple object in $\mathcal{Z}(\mathcal{C})$. One can easily generalize this formula for any object in $\mathcal{Z}(\mathcal{C})$ by decomposing it into a direct sum of simples.

Remark 3.21. One can also define the *higher* generalized Frobenius-Schur indicators, $\nu_{n,k}^{W}(X)$ as

$$\nu_{n,k}^{W}(X) = Tr((\rho_{n,X}^{W})^{k}) \tag{3.80}$$

The formula we gave in Thm. 3.19 calculates the higher generalized Frobenius-Schur indicators for $k = 1$. In the following chapter, we develop a method of calculating the higher generalized Frobenius-Schur indicators when n is prime. For a complete formula for the higher generalized Frobenius-Schur indicators, see [NS10].

Chapter 4

Link invariants for torus knots

It is well-known that we can obtain oriented ribbon link invariants from braided spherical fusion category. For a braided spherical fusion category $\mathcal{C}$ and X an object in the category, we can interpret the oriented link diagram by thinking of it as a string diagram with the strings labeled by X. Then we obtain a morphism in $\mathcal{C}(\mathbb{1}, \mathbb{1}) \in \Bbbk$ which can be identified with a number in $\Bbbk$ and is invariant under the Reidemeister moves. In practice, calculating the link invariants is a computationally hard process and requires us to know a lot about the category $\mathcal{C}$. This chapter presents a way of generating link invariants of special torus knots using the higher Frobenius-Schur indicators for modular categories. In section 4.1 we give a way of generating some of the higher Frobenius-Schur indicators from the indicator formula using Galois actions. In section 4.2 we discuss a simplification of the indicator formula when the category is modular. In section 4.3 we introduce torus knots and derive a method for generating torus knot invariants from modular data of modular categories. In section 4.4 we present some calculations for link invariants for Drinfeld centers of pointed fusion categories. Finally, section 4.5 gives some concluding remarks.

4.1   Higher Frobenius-Schur indicators

Recall in chapter 3 we defined the higher generalized Frobenius-Schur indicators $\nu_{n,k}^W(X)$ as

$$\nu_{n,k}^W(X) = Tr((\rho_{n,X}^W)^k). \tag{4.1}$$

Now we give a method of computing some of the higher FS indicators $\nu_{n,k}^W(X)$ from $\nu_n^W(X)$. First we observe that the generalized rotation operator is diagonalizable.

Lemma 4.1. *For spherical fusion category $\mathcal{C}$, $W \in Irr(\mathcal{Z}(\mathcal{C}))$ and*

$$\rho_{n,X}^W : \mathcal{C}(F(W), X^n) \to \mathcal{C}(F(W), X^n)$$

the rotation operator, we have

$$(\rho_{n,X}^W)^n = \theta_W^{-1} id_{F(W),X^n} . \tag{4.2}$$

Proof. Let $f \in \mathcal{C}(X^n, F(W))$. Since $\mathcal{C}$ is pivotal, we have

$$(\rho_{n,X}^W)^n(f) = \qquad\qquad = \qquad\qquad . \tag{4.3}$$

We can pass f under the half-braiding, giving us

$$(\rho_{n,X}^W)^n(f) = \qquad\qquad = \theta_W^{-1} \boxed{f} . \tag{4.4}$$

$\square$

Since $\Bbbk$ is an algebraically-closed field with characteristic 0, we know that $\rho_{n,X}^W$ is diagonalizable[1]. Furthermore, let $\{\alpha\}$ be an eigenbasis of $\rho_{n,X}^W$ with eigenvalues $\{\lambda_{W,\alpha}\}$. Picking an nth root of θ_W, then the set

$$\Lambda = \{\theta_W^{\frac{1}{n}} \lambda_{W,\alpha}\}_{\alpha \in \mathfrak{B}(F(W),X^n)}$$

consists only of nth roots of unity.

As the trace of $\rho_{n,X}^W$ is the sum of the eigenvalues, we have

$$\nu_n^W(X) = \sum_{\alpha \in \mathfrak{B}(F(W),X^n)} \lambda_{W,\alpha},$$

and furthermore

$$\nu_{n,k}^W(X) = tr((\rho_{n,X}^W)^k) = \sum_{\alpha \in \mathfrak{B}(F(W),X^n)} \lambda_{W,\alpha}^k.$$

Let $\xi = e^{\frac{2\pi i}{n}}$. Then $\theta_W^{\frac{1}{n}} \nu_n^W(X)$ is an element in the n-cyclotomic field $\mathbb{Q}[\xi]$. Since $\mathbb{Q}[\xi]$ is a Galois extension for $\mathbb{Q}$, for $\gcd(n,k) = 1$, we have an element φ_k of the Galois group, also known as the Frobenius map, which raises ξ to the k-th power. Thus we have

---

[1]This can be shown using representation theory: $\rho_{n,X}^W$ induces a representation of $\mathbb{Z}/n\mathbb{Z}$, since the group is finite, the representation is unitarizable and therefore $\rho_{n,X}^W$ is diagonalizable.

$$\varphi_k(\theta_W^{\frac{1}{n}}\nu_n^W(X)) = \sum_{\alpha \in \mathfrak{B}(F(W),X^n)} (\theta_W^{\frac{1}{n}}\lambda_{L,\alpha})^k = \theta_W^{\frac{k}{n}}\nu_{n,k}^W(X).$$

In particular, when n is prime, then any choice of k would be coprime to n, so in this case, we can compute all higher Frobenius indicator using Galois actions.

Remark 4.2. Whilst we focus on the case of n being prime, a similar process can be done when n is not prime, for details, see Prop. 1.2. in [BJT16].

If we know all higher Frobenius-Schur indicators, by applying the Galois actions or otherwise, then we can explicitly compute the eigenvalues. Let x_l be the number of times ξ^l appears in Λ, where

$$\Lambda = \{\theta_W^{\frac{1}{n}}\lambda_{W,\alpha}\}_{\alpha \in \mathfrak{B}(F(W),X^n)}.$$

Then we can use the discrete Fourier transform[2] to find the multiplicities x_l, given by

$$x_l = \frac{1}{n}\sum_{k=1}^{n} \theta_{(W,e_W)}^{\frac{k}{n}}\nu_{n,k}^W(X)\xi^{-lk}. \tag{4.5}$$

Then the eigenvalues of $\rho_{n,X}^W$ are $\xi^l\theta_W^{\frac{-1}{n}}$ with multiplicity x_l.

4.2 Drinfeld centers of modular categories

The formula given in Theorem 3.19 requires us to know the s and t matrices for the Drinfeld center, as well as the forgetful functor multiplicities. In general, it is difficult to write down the s and t matrices for the center, or even know what the simple objects are! When $\mathcal{C}$ is modular, however, we can derive the the modular data for $\mathcal{Z}(\mathcal{C})$ from the modular data for $\mathcal{C}$. In this section, we explore the implications of this, and give a formula for the generalized FS indicators in purely in terms of the modular data of the category. First we introduce some notation.

Definition 4.3. Let $\mathcal{C}$ be a braided monoidal category, with the braiding given by the family of natural isomorphisms $\sigma_{VW} : VW \to WV$. Define $\tilde{\mathcal{C}}$ to be the braided monoidal category with the inverse braiding $\sigma_{WV}^{-1} : VW \to WV$.

Remark 4.4. To highlight the difference between $\mathcal{C}$ and $\tilde{\mathcal{C}}$, we decorate objects and morphisms in $\mathcal{C}$ with tildes.

Definition 4.5. Let $\mathcal{A}$ and $\mathcal{B}$ be $\Bbbk$-linear categories. Define the *tensor product*, $\mathcal{A} \boxtimes \mathcal{B}$, to be the category consisting of the following:

- objects are finite direct sums of the form $\bigoplus_i A_i \boxtimes B_i$ with $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$

---

[2]For an explanation of the discrete Fourier transform, see the appendices.

- morphisms between objects are defined by:

$$\text{Hom}_{\mathcal{A}\boxtimes\mathcal{B}}(\bigoplus_i A_i \boxtimes B_i \ , \ \bigoplus_j A'_j \boxtimes B'_j) = \bigoplus_{i,j} \mathcal{A}(X_i, X'_j) \otimes_{\Bbbk} \mathcal{B}(Y_i, Y'_j)$$

Remark 4.6. If $\mathcal{A}$ and $\mathcal{B}$ are semisimple categories, it is easy to see that the only objects in $\mathcal{A}\boxtimes\mathcal{B}$ satisfying $\text{End}(\bigoplus_i A_i \boxtimes B_i) = \bigoplus_{i,j} \mathcal{A}(X_i, X_j) \otimes_{\Bbbk} \mathcal{B}(Y_i, Y_j) \simeq \Bbbk$ are of the form $L \boxtimes K$ where L, K are simple objects in $\mathcal{C}$.

Theorem 4.7 ([Mue03b] Theorem 7.10). *Let $\mathcal{C}$ be a braided monoidal category. Define the functor $\mathcal{G} : \mathcal{C}\boxtimes\tilde{\mathcal{C}} \to \mathcal{Z}(\mathcal{C})$ as follows:*

- *on objects: send $X \boxtimes \tilde{Y}$ to $(X \otimes Y, e_{X\otimes Y})$, where for $W \in \mathcal{C}$, $e_{X\otimes Y}(W)$ is defined to be*

$$(\sigma_{XW} \otimes id_Y) \circ (id_X \otimes \sigma_{WY}^{-1}),$$

which can be represented diagrammatically as

- *on morphisms: sends $f \boxtimes \tilde{g}$ to $f \otimes g$*

If $\mathcal{C}$ is modular, then $\mathcal{G}$ yields a braided monoidal equivalence between $\mathcal{C}\boxtimes\tilde{\mathcal{C}}$ and $\mathcal{Z}(\mathcal{C})$.

Under this equivalence, we can view the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ as

$$F : \mathcal{C}\boxtimes\tilde{\mathcal{C}} \to \mathcal{C}$$

which sends $X \boxtimes \tilde{Y}$ to $X \otimes Y$ and $f \boxtimes \tilde{g}$ to $f \otimes g$.

Now that we have an equivalence between a category we understand better and the Drinfeld center, we can write down information about the center in terms of information about the category.

Theorem 4.8. *For a modular category $\mathcal{C}$, we have*

$$\mathcal{D}_{\mathcal{Z}(\mathcal{C})} = \mathcal{D}_{\mathcal{C}}^2$$

where $\mathcal{D}_{\mathcal{Z}(\mathcal{C})}$ is the categorical dimension for the Drinfeld center, $\mathcal{Z}(\mathcal{C})$, and $\mathcal{D}_{\mathcal{C}}$ is the categorical dimension of the category, $\mathcal{C}$.

Remark 4.9. Though we asked for $\mathcal{C}$ to be a modular category, this statement also holds for spherical fusion categories in general.

Proof. This is a straight forward computation. Since isomorphism classes of simple objects in $\mathcal{Z}(\mathcal{C})$ just correspond to $J \boxtimes \tilde{K}$ for $J, K \in \mathrm{Irr}(\mathcal{C})$, then

$$\mathcal{D}_{\mathcal{Z}(\mathcal{C})} = \sum_{J,K \in \mathrm{Irr}(\mathcal{C})} d^2_{J \boxtimes \tilde{K}} = \sum_{J,K \in \mathrm{Irr}(\mathcal{C})} d^2_J d^2_K = \mathcal{D}^2_{\mathcal{C}}. \tag{4.6}$$

$\square$

Theorem 4.10. *For a modular category $\mathcal{C}$, we have*

$$s_{A \boxtimes \tilde{B}, C \boxtimes \tilde{D}} = s_{A,C} s_{\overline{B},D} \tag{4.7}$$

where $s_{A \boxtimes \tilde{B}, C \boxtimes \tilde{D}}$ is an entry of the s matrix for $\mathcal{C} \boxtimes \tilde{\mathcal{C}}$, and $s_{A,C} s_{\overline{B},D}$ is an entry for the s matrix for $\mathcal{C}$.

Proof. Writing out the definition for the s entry gives us that

$$\tag{4.8}$$

Since the braiding is functorial, we can move inner diagram involving B, D outside of the A, C diagram. Then we can use sphericality to show that the RHS is equal to

$$\tilde{s}_{A,C} \tilde{s}_{\overline{B},D}. \tag{4.9}$$

Now we normalize the s matrices. By theorem 4.8, we have

$$s_{A \boxtimes \tilde{B}, C \boxtimes \tilde{D}} = \frac{\tilde{s}_{A \boxtimes \tilde{B}, C \boxtimes \tilde{D}}}{\sqrt{\mathcal{D}_{\mathcal{Z}(\mathcal{C})}}} = \frac{\tilde{s}_{A \boxtimes \tilde{B}, C \boxtimes \tilde{D}}}{\mathcal{D}_{\mathcal{C}}} = \frac{\tilde{s}_{A,C}}{\sqrt{\mathcal{D}_{\mathcal{C}}}} \frac{\tilde{s}_{\overline{B},D}}{\sqrt{\mathcal{D}_{\mathcal{C}}}} = s_{A,C} s_{\overline{B},D}. \tag{4.10}$$

$\square$

Theorem 4.11. *For a modular category $\mathcal{C}$, we have*

$$\theta_{A \boxtimes \tilde{B}} = \frac{\theta_A}{\theta_B} . \tag{4.11}$$

Proof. Similar to the proof Theorem 4.10. We first write out the definition of the twist then use functoriality of the braiding to manipulate the strings to produce the desired diagram. $\square$

Now we derive a simplification for the FS indicators for modular categories. For $X \in \mathcal{C}$ and $L \boxtimes \tilde{M} \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$, by Theorem 3.19, we know

$$\nu_n^{L \otimes \tilde{M}}(X) = \frac{1}{\mathcal{D}_\mathcal{C}} \sum_{J,K \in \mathrm{Irr}(\mathcal{C})} \tilde{s}_{L \boxtimes \tilde{M}, J \boxtimes \tilde{K}} \theta_{J \boxtimes \tilde{K}}^n \dim(\mathcal{C}(F(J \boxtimes \tilde{K}), X)). \qquad (4.12)$$

Then using Theorems 4.10, 4.11 and the fact $F(J \boxtimes \tilde{K}) = J \otimes K$, we have

$$\nu_n^{L \otimes \tilde{M}}(X) = \sum_{J,K \in \mathrm{Irr}(\mathcal{C})} s_{LJ} s_{\overline{M}K} \frac{\theta_J^n}{\theta_K^n} \dim(\mathcal{C}(J \otimes K, X)). \qquad (4.13)$$

By the Verlinde formula (Theorem 2.12), we have

$$\nu_n^{L \otimes \tilde{M}}(X) = \sum_{J,K,R \in \mathrm{Irr}(\mathcal{C})} s_{LJ} s_{\overline{M}K} \cdot \frac{\theta_J^n}{\theta_K^n} \cdot \frac{s_{JR} s_{KR} s_{\overline{X}R}}{s_{\mathbb{1}R}}. \qquad (4.14)$$

To evaluate the formula, we still need a way of finding the duals of simple objects, Theorem 2.11 tells us that this information can be found in the square of the s matrix. Thus, for modular categories, we have a way of writing the Frobenius-Schur indicators solely in terms of the s and t matrices of the category.

4.3   Torus knots
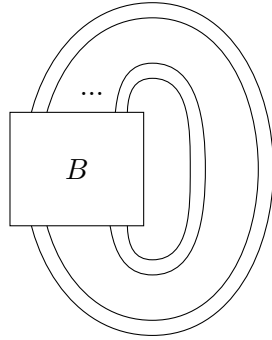
This section gives a brief introduction to torus knots and assumes knowledge of link invariants. Readers without a background in knot theory should consult [Ada04] for a more comprehensive introduction to torus knots and link invariants.

In short, a torus knot is a knot that lies on an unknotted torus. They can be completely characterized by how many times the knot crosses the meridian and the longitude of the torus. We call a torus knot an (n, m)-torus knot if it crosses a meridian curve n times and a longitude curve m times[3]. We can also view the (n, m)-torus knot as the closure of the following braid

$$(4.15)$$

---
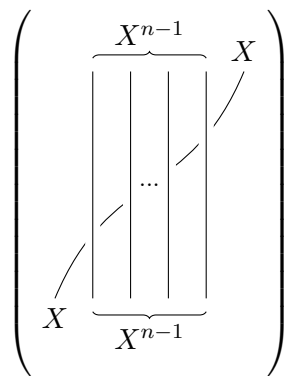
[3]Note that this characterization is not faithful, it can be shown that the (m, n) and (n, m)-torus knots are in fact the same knot. Also. not all choices of m and n produce torus knots, we require $\gcd(n, m) = 1$.

where by closure of a braid B we mean
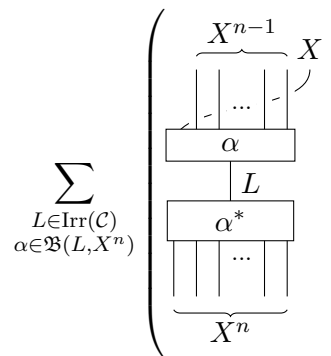
$$(4.16)$$

the loop obtained when we connect the leftmost top string with the leftmost bottom string, the second leftmost string with the second bottom leftmost string and so on.

Let $\mathcal{C}$ be braided spherical fusion category and pick an object $X \in \mathcal{C}$. If we label an oriented link with the object $X \in \mathcal{C}$, then we have a string diagram representing a morphism in $\mathcal{C}(X^n, X^n)$. It can be shown that this morphism in invariant under the Reidemeister moves, giving us an oriented ribbon link invariant. In our case, we have that the torus link invariant, denoted by $T_{n,X}^m$, is the categorical trace of the following morphism in $\mathcal{C}(X^n, X^n)$:

$$(4.17)$$

Since $\mathcal{C}$ is finitely semisimple, then for $L \in \mathrm{Irr}(\mathcal{C})$, and α a basis of $\mathcal{C}(L, X^n)$, we have that $T_{n,X}^m$ is the categorical trace of

$$(4.18)$$

If the linear operator $\mu_{n,X}^L : \mathcal{C}(L, X^n) \to \mathcal{C}(L, X^n)$

$$\mu_{n,X}^L := \qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.19)$$

is diagonalizable, then for $\{\beta\}$ is an eigenbasis of $\mu_{n,X}^L$ with eigenvalues $\{\lambda'_{L,\beta}\}$, we have

$$T_{n,X}^m = \sum_{\substack{L\in\mathrm{Irr}(\mathcal{C}) \\ \beta\in\mathfrak{B}(L,X^n)}} (\lambda'_{L,\beta})^m \, tr \left(\left(\begin{array}{c} \\ \beta \\ L \\ \beta^* \\ \end{array}\right)^m\right) = \sum_{\substack{L\in\mathrm{Irr}(\mathcal{C}) \\ \beta\in\mathfrak{B}(L,X^n)}} (\lambda'_{L,\beta})^m d_L \ . \qquad (4.20)$$

Remark 4.12. Since there are $(n-1)m$ undercrossings and no overcrossings in the the (n,m) torus knot, the writhe of the knot is $-(n-1)m$. Then we can also normalize the invariant by $\theta_X^{(n-1)m}$ to obtain a oriented knot invariant, denoted by $\tilde{T}_{n,X}^m$, where

$$\tilde{T}_{n,X}^m = \theta_X^{(n-1)m} \sum_{\substack{L\in\mathrm{Irr}(\mathcal{C}) \\ \beta\in\mathfrak{B}(L,X^n)}} (\lambda'_{L,\beta})^m d_L. \qquad (4.21)$$

To show $\mu_{n,X}^L$ is diagonalizable, we first show a relationship between $\mu_{n,X}^L$ and $\rho_{n,X}^{(L,\sigma)}$ where

$$\rho_{n,X}^{(L,\sigma)} : \mathcal{C}(L, X^n) \to \mathcal{C}(L, X^n)$$

is the rotation operator

$$\rho_{n,X}^{(L,\sigma)} = \qquad\qquad\qquad\qquad\qquad (4.22)$$

and σ is the braiding coming from the braided structure of $\mathcal{C}$.

Lemma 4.13. *We have the following relationship between $\rho_{n,X}^{(L,\sigma)}$ and $\mu_{n,X}^L$:*

$$\rho_{n,X}^{(L,\sigma)} = \theta_X^{-1}\mu_{n,X}^L \qquad\qquad\qquad (4.23)$$

Proof. Since the half braiding of (L,σ) is also a braiding in $\mathcal{C}$,

$$\rho_{n,X}^{(L,\sigma)} = \qquad = \qquad = \theta_X^{-1}\mu_{n,X}^L. \qquad (4.24)$$

$\square$

In Lemma 4.1, we showed that generalized rotation operators are diagonalizable, thus $\mu_{n,X}^L$ is also diagonalizable. Also, for $\{\alpha\}$ an eigenbasis of $\rho_{n,X}^{(L,\sigma)}$ with eigenvalues $\{\lambda_{L,\alpha}\}$, then $\{\alpha\}$ is an eigenbasis $\mu_{n,X}^L$ with eigenvalues $\{\theta_X\lambda_{L,\alpha}\}$. In particular, we know that the eigenvalues for $\mu_{n,X}^L$ are $\{\theta_X\theta_L^{\frac{-1}{n}}\xi^l\}$ with multiplicity $x_l \in \mathbb{N}$ where

$$x_l = \frac{1}{n}\sum_{k=1}^{n}\theta_L^{\frac{k}{n}}\nu_{n,k}^{L\boxtimes\tilde{\mathbb{1}}}(X)\xi^{-lk}. \qquad\qquad (4.25)$$

Since (L,σ) can be viewed as $L \boxtimes \tilde{\mathbb{1}}$ in $\mathcal{C} \boxtimes \tilde{\mathcal{C}}$, by our work in section 4.2, we have

$$\nu_n^{L\boxtimes\tilde{\mathbb{1}}}(X) = \sum_{J,K,R\in\mathrm{Irr}(\mathcal{C})} s_{LJ}s_{\overline{1}K} \cdot \frac{\theta_J^n}{\theta_K^n} \cdot \frac{s_{JR}s_{KR}s_{\overline{X}R}}{s_{\mathbb{1}R}}. \qquad (4.26)$$

When n is prime, this is all we need to use the Frobenius map to find the higher FS indicators.

Furthermore, observe that

$$d_L = \tilde{s}_{L,\mathbb{1}} = \sqrt{\mathcal{D}_\mathcal{C}}s_{L,\mathbb{1}},$$

Therefore, by (4.20), we can write the torus link invariant to be,

$$T_{n,X}^m = \sum_{\substack{l \in [1,....,n] \\ L \in \mathrm{Irr}(\mathcal{C})}} x_l (\theta_X \theta_L^{-\frac{1}{n}} \xi^l)^m \frac{s_{L,\mathbb{1}}}{\sqrt{\mathcal{D}_\mathcal{C}}}. \tag{4.27}$$

Substituting (4.25), we also have

$$T_{n,X}^m = \frac{1}{n\sqrt{\mathcal{D}_\mathcal{C}}} \sum_{\substack{l \in [1,....,n] \\ k \in [1,....,n] \\ L \in \mathrm{Irr}(\mathcal{C})}} \theta_W^{\frac{k}{n}} \theta_X^m \theta_L^{-\frac{m}{n}} \nu_{n,k}^{L \boxtimes \tilde{\mathbb{1}}}(X) s_{L,\mathbb{1}} \xi^{l(m-k)}. \tag{4.28}$$

4.4   Torus link invariants for Drinfeld centers of pointed fusion categories

Using the computer algebra system GAP[GAP17], we implemented algorithms that calculate both the torus ribbon link invariants and the normalized version. The computer code is included in the appendices. We computed some invariants using modular data for Drinfeld centers of pointed fusion categories given in [Gru17]. This led to some interesting observations.

For example, consider the twisted quantum double of the alternating group A_4, $Rep(D^{\omega_2} A_4)$.

There are 18 equivalence classes of simple objects in the category[4]. We observed that for a given choice of simple object X, $\tilde{T}_{11,X}^m$ is always the same for $m \in [1,...,10]$. Specifically, it is given by

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \end{pmatrix}$$

where the J-th entry gives the invariant for the J-th simple object in the modular data.

When $n = 3$, however, we notice that not all invariants are the same, specifically, we have

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 3 & 16 & 16 & 16 & 3 & 3 & 3 & 3 & 16 & 16 & 16 \end{pmatrix}$$

where the LJ-th entry gives $\tilde{T}_{3,J}^L$. Thus for most choices of simple objects, $\tilde{T}_{3,X}^1 \neq \tilde{T}_{3,X}^2$. We believe this is related to the conductor of the category, that is, the order of the t matrix.

4.5   Concluding remarks

There are several directions for further research. First is to calculate the FS indicators and link invariants for well-known modular categories. For example, it would be inter-

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[4] For the s and t matrices for this category, see the appendices.

esting to calculate the torus link invariants for the extended Haagerup category, which is not currently known.

Another direction is to generalize the algorithm to allow composite values for n, and to improve overall efficiency in order to calculate invariants for categories with bigger modular data. Then we can consider questions such as:

1. Is the normalized link invariant always integral for Drinfeld centers of pointed fusion categories?

2. What is the relationship between the conductor and the variations in the normalized torus invariants?

Appendix A

Discrete Fourier transform

When we have a integral weighted sum of roots of unity, and the sums of the powers of the roots of unity with the same weighting, we can use the discrete Fourier transform to calculate the weights.

Theorem A.1. *Let $\xi = e^{\frac{2\pi i}{n}}$ be the primitive n-th root of unity, and for each ξ^k an integral weight x_k, suppose we know X_j where*

$$X_j = \sum_{k=1}^{n} x_k \xi^{kj} \tag{A.1}$$

Then

$$x_l = \frac{1}{n} \sum_{k=1}^{n} X_k \xi^{-lk}. \tag{A.2}$$

Proof. Substituting the definition of X_k into the RHS gives us

$$\frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} x_j \xi^{(j-l)k}.$$

Now consider $\sum_{j=1}^{n} x_j \xi^{(j-l)k}$ with respect to a fixed k. The $l = j$ term contributes

$$\sum_{k=1}^{n} x_l = n x_l$$

to the sum.

When $l \neq j$,

$$\sum_{k=1}^{n} x_j \xi^{k(j-l)} = x_j \xi^{k(j-l)} \cdot \frac{1 - \xi^{n(j-l)}}{1 - \xi^{j-l}} = x_j \xi^{k(j-l)} \cdot \frac{1-1}{1-\xi^{j-l}} = 0.$$

Thus the RHS is indeed equal to x_l. $\qquad \square$

Appendix B

Modular data for $Rep(D^{\omega_2}A_4)$

$$
s := \begin{pmatrix}
\tfrac{1}{12} & \tfrac{1}{12} & \tfrac{1}{12} & \tfrac{1}{4} & \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{1}{4} & \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} \\[4pt]
-\tfrac{1}{12} & -\tfrac{1}{12} & -\tfrac{1}{12} & \tfrac{1}{4} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} \\[4pt]
\tfrac{1}{12} & \tfrac{1}{12} & \tfrac{1}{12} & \tfrac{1}{4} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} \\[4pt]
-\tfrac{1}{4} & -\tfrac{1}{4} & -\tfrac{1}{4} & \tfrac{3}{4} & 0 & 0 & 0 & -\tfrac{1}{4} & 0 & 0 & 0 & 0 \\[4pt]
\tfrac{1}{3} & -\tfrac{1}{3}e^{\frac{8\pi i}{9}} & -\tfrac{1}{3}e^{\frac{8\pi i}{9}} & 0 & -\tfrac{1}{3}e^{\frac{8\pi i}{9}} & \tfrac{1}{3}e^{\frac{14\pi i}{9}} & 0 & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{4\pi i}{9}} & -\tfrac{1}{3}e^{\frac{10\pi i}{9}} & 0 & 0 \\[4pt]
-\tfrac{1}{3}e^{\frac{2\pi i}{3}} & -\tfrac{1}{3}e^{\frac{2\pi i}{3}} & -\tfrac{1}{3}e^{\frac{2\pi i}{3}} & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{14\pi i}{9}} & \tfrac{1}{3}e^{\frac{8\pi i}{9}} & 0 & -\tfrac{1}{4} & -\tfrac{1}{3}e^{\frac{10\pi i}{9}} & \tfrac{1}{3}e^{\frac{4\pi i}{9}} & 0 & 0 \\[4pt]
-\tfrac{1}{3}e^{\frac{4\pi i}{3}} & -\tfrac{1}{3}e^{\frac{4\pi i}{3}} & -\tfrac{1}{3}e^{\frac{4\pi i}{3}} & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{10\pi i}{9}} & -\tfrac{1}{3}e^{\frac{4\pi i}{9}} & 0 & \tfrac{3}{4} & -\tfrac{1}{3}e^{\frac{8\pi i}{9}} & \tfrac{1}{3}e^{\frac{14\pi i}{9}} & 0 & 0 \\[4pt]
-\tfrac{1}{4} & -\tfrac{1}{4} & -\tfrac{1}{4} & -\tfrac{1}{4} & 0 & 0 & 0 & -\tfrac{1}{4} & 0 & 0 & 0 & 0 \\[4pt]
\tfrac{1}{3} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & 0 & \tfrac{1}{3}e^{\frac{8\pi i}{9}} & -\tfrac{1}{3}e^{\frac{14\pi i}{9}} & 0 & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{10\pi i}{9}} & -\tfrac{1}{3}e^{\frac{4\pi i}{9}} & 0 & 0 \\[4pt]
\tfrac{1}{3} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & 0 & -\tfrac{1}{3}e^{\frac{14\pi i}{9}} & \tfrac{1}{3}e^{\frac{8\pi i}{9}} & 0 & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{4\pi i}{9}} & \tfrac{1}{3}e^{\frac{10\pi i}{9}} & 0 & 0 \\[4pt]
\tfrac{1}{3} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & 0 & \tfrac{1}{3}e^{\frac{14\pi i}{9}} & -\tfrac{1}{3}e^{\frac{8\pi i}{9}} & 0 & -\tfrac{1}{4} & \tfrac{1}{3}e^{\frac{8\pi i}{9}} & -\tfrac{1}{3}e^{\frac{14\pi i}{9}} & 0 & 0 \\[4pt]
\tfrac{1}{3} & \tfrac{1}{3}e^{\frac{4\pi i}{3}} & \tfrac{1}{3}e^{\frac{2\pi i}{3}} & 0 & \tfrac{1}{3}e^{\frac{4\pi i}{9}} & \tfrac{1}{3}e^{\frac{10\pi i}{9}} & 0 & 0 & \tfrac{1}{3}e^{\frac{14\pi i}{9}} & -\tfrac{1}{3}e^{\frac{8\pi i}{9}} & 0 & 0
\end{pmatrix}
$$

$$
t := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{14\pi i}{9}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{\frac{8\pi i}{9}} - e^{\frac{14\pi i}{9}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{8\pi i}{9}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -e^{\frac{8\pi i}{9}} - e^{\frac{14\pi i}{9}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{\frac{14\pi i}{9}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{\frac{8\pi i}{9}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Appendix C

GAP code

```
# Returns dual of the ith simple object

FindDual := function(i, smatrix)

local Cmatrix, no_of_simples, j;

Cmatrix := smatrix * smatrix;

no_of_simples:= DimensionsMat(smatrix)[1];

for j in [1 .. no_of_simples] do

if Cmatrix[i][j] = 1 then

return j;

else

continue;

fi;

od;
```

```
end;

# Return the Verlinde's formula N^k_{lj}

Verlinde := function(k, l, j, smatrix)
    local r, sum, dual_k, no_of_simples;
    no_of_simples:= DimensionsMat(smatrix)[1];
    sum := 0;
    dual_k := FindDual(k, smatrix);
    for r in [1 .. no_of_simples] do
        sum := sum + smatrix[l][r] * smatrix[j][r] * smatrix[dual_k][r]/smatrix[1][r];
    od;
    return sum;
    end;

# Gives the order of the T matrix

GiveOrderOfT := function(tmatrix)
    local i, no_of_simples;
    i := 1;
    no_of_simples:= DimensionsMat(tmatrix)[1];
    while not (tmatrix^i = IdentityMat(no_of_simples)) do
        i := i + 1;
    od;
```

```
        return i;
    end;


# Calculates the FS indicator \rho_n^L(X)

FSIndicator := function(smatrix,tmatrix, n, L, X)

local sum, no_of_simples , J , K, R, dual_X, sum_1, sum_2;

no_of_simples:= DimensionsMat(smatrix)[1];

dual_X := FindDual(X, smatrix);

sum := 0;

    for R in [1 .. no_of_simples] do

    sum_2 := 0;

    for K in [1 .. no_of_simples] do

    sum_1 := 0;

        for J in [1 .. no_of_simples] do

        sum_1 := sum_1 + smatrix[L][J]*smatrix[J][R]*tmatrix[J][J]^n;

        od;

        sum_2 := sum_2 + sum_1 * smatrix[1][K] *smatrix[K][R]  / tmatrix[K][K]^n;

    od;

    sum := sum + sum_2 * smatrix[dual_X][R]/ smatrix[1][R];

    od;

    return sum;

end;
```

```
# Gives a nth root of cyc
# Cyc has to be a root of unity

NthRootOfCyc := function(cyc, n)

local list;

list:= DescriptionOfRootOfUnity(cyc);

return E(list[1]*n)^list[2];

end;


# Given the a number in the nth cyclotomic field, computes all of the Frobenius maps and returns a list
# We need n prime for the error message to work

FrobeniusMap := function(ind, n)

local list , i, higher;

list := [ind];

if Conductor(ind) = n or Conductor(ind) = 1 then

for i in [2 .. n] do

higher := GaloisCyc( ind, i );

Append( list, [higher]);

od;

else

Print("SOMETHING WRONG: Your FS indicator does not live in the right number field, or maybe n is not a prime");

fi;
```

5

oid

79

```
    return list;
  end;


# Performs the Discrete Fourier transform from a list of sums of power and n

GiveWeights := function(list, n)

local i, j, sum, divide_sum, outlist;

outlist := [];


for i in [1 .. n] do

sum := 0;

for j in [1 .. n] do

sum := sum + list[j]*(E(n)^(-j*i));

od;

divide_sum := sum/ n;

Append( outlist, [divide_sum]);

od;

return outlist;

end;


# Calculate T_{n,X}^m

# We require n to be prime
```

```
TorusInvariant := function(smatrix, tmatrix, n , m, X)

local no_of_simples, sum, subsum, L, l, indicator_scaled, weights, cat_dim;

cat_dim := 1/smatrix[1][1];

sum := 0;

no_of_simples:= DimensionsMat(smatrix)[1];

for L in [1 .. no_of_simples] do

subsum := 0;

indicator_scaled := FSIndicator(smatrix,tmatrix, n, L, X) * (NthRootOfCyc(tmatrix[L][L], n));

weights := GiveWeights(FrobeniusMap(indicator_scaled, n), n);

for l in [1 .. n] do

subsum := subsum + E(n)^(l*m)*weights[l];

od;

sum := sum + NthRootOfCyc(tmatrix[L][L], n)^(-m)*smatrix[L][1]*subsum;

od;

return sum*(tmatrix[X][X]^m)*(cat_dim);

end;


# Give (n,m) torus knots invariants from all simple objects in the category
# We require n to be prime

GiveInvariants := function(smatrix, tmatrix, n, m)

local no_of_simples, j , list;
```

```
no_of_simples:= DimensionsMat(smatrix)[1];

list := [];

for j in [1 .. no_of_simples] do

Add( list , TorusInvariant(smatrix, tmatrix, n , m, j) );

od;

return list;

end;


# Gives all torus link invariants T_{n,X}^m

# Returns a matrix where the ij-th entry is T_{n,j}^i

# We require n to be prime

GiveAllInvariants := function(smatrix, tmatrix, n)

local m , list;

list := [];

for m in [1 .. n-1] do

Add( list , GiveInvariants(smatrix, tmatrix, n , m) );

od;

return list;

end;
```

```
# Calculate normalized T_{n,X}^m

# We require n to be prime

Unframed_TorusInvariant := function(smatrix, tmatrix, n , m, X)
return TorusInvariant(smatrix, tmatrix, n , m, X)*(tmatrix[X][X])^((n-1)*m);
end;


# Give (n,m) normalized torus invariants from all simple objects in the category

# We require n to be prime

GiveUnframedInvariants := function(smatrix, tmatrix, n, m)
local no_of_simples, j , list;
no_of_simples:= DimensionsMat(smatrix)[1];
list := [];
for j in [1 .. no_of_simples] do
Add( list , Unframed_TorusInvariant(smatrix, tmatrix, n , m, j ) );
od;
return list;
end;


# Give the list of (n,n-1) normalized torus invariants for simple objects in the category

# where n ranges from the first p no. of primes
```

```
GiveUnframedInvariantsAlt  :=  function(smatrix, tmatrix, p, j)

local n ,list;

list := [];

for n in [1 .. p] do

    Add( list ,  Unframed_TorusInvariant(smatrix, tmatrix, Primes[n] , Primes[n]-1 , j ) );

    od;

    return list;

end;


# Gives all normalized torus invariants T_{n,X}^m

# Returns a matrix where the ij-th entry is normalized T_{n,j}^i

# We require n to be prime

GiveUnframedAllInvariants := function(smatrix, tmatrix, n)

local m , list;

list := [];

for m in [1 .. n-1] do

    Add( list ,  GiveUnframedInvariants(smatrix, tmatrix, n , m) );

    od;

    return list;

end;
```

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