The Two Formulations of Khovanov's Tangle Invariant

Hazel Browne

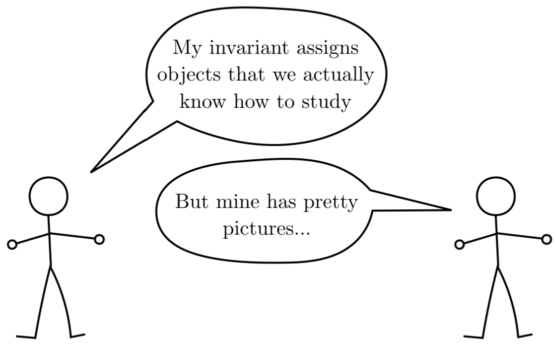
February 17, 2018

1 Introduction

In [2], Khovanov introduces a homology theory for links, which he calls a 'categorification' of the Jones polynomial. To an oriented link projection he assigns a chain complex of graded abelian groups, such that different projections of the same link have homotopy equivalent chain complexes. The graded Euler characteristic of the complex is the (un-normalised) Jones polynomial; however, Khovanov's homology is a strictly stronger invariant than the Jones polynomial, and is known to detect the unknot ([4]), while it remains unknown whether the Jones polynomial does.

Khovanov's homology for links has been extended to a tangle invariant in two ways: by Khovanov in [3], and by Bar-Natan in [1]. Both assign chain complexes to tangle diagrams, but in different categories. While Bar-Natan's invariant assigns a complex of direct sums of planar tangles and matrices of cobordisms, Khovanov's invariant assigns a complex of graded bimodules and bimodule maps.

In the contest for the best tangle invariant, both Khovanov and Bar-Natan have justifiable claims. While Bar-Natan's invariant is defined geometrically, Khovanov's invariant is algebraic, which immediately allows us to use the basic tools of linear algebra to study it. On the other hand, the Bar-Natan version has excellent composition properties, allowing tangles to be composed in all conceivable ways, whereas Khovanov's invariant only allows tangles to be stacked 'vertically', but there is no obvious way to compose tangles 'horizontally'.



This is an essay on the two formulations of Khovanov's tangle invariant, with a focus on Khovanov's version. We begin by explaining the construction of Khovanov's link invariant, and then discuss how this is adapted to make a tangle invariant. We finish by showing that it is injective on planar tangles with the same number of closed loops. This shows that there is sufficient information, given the bimodules corresponding to two planar tangle diagrams, to compute the bimodule corresponding to their horizontal composition, although it does not give a nice way to actually compute it without reconstructing the original tangles.

2 Khovanov alla Bar-Natan

In this section, we introduce Khovanov's link invariant from [3] as presented by Bar-Natan in [1] (with a few straggling conventions from [3]). We begin by assigning to each oriented link projection a chain complex in an additive category defined below. Then we apply a functor taking this to a complex of graded abelian groups. Different projections of the same link have homotopy equivalent complexes, and thus the homology groups are a link invariant.

Let L be an oriented link projection with n crossings. Each crossing in L can be replaced by a 0-smoothing – 'approach from the bottom and turn right' – or a 1-smoothing – 'approach from the bottom and turn left' (see Figure 1). Thus there are 2^n distinct ways to smooth all n crossings, called the *complete smoothings of* L, each of which is a disjoint union of circles (with all orientations forgotten).

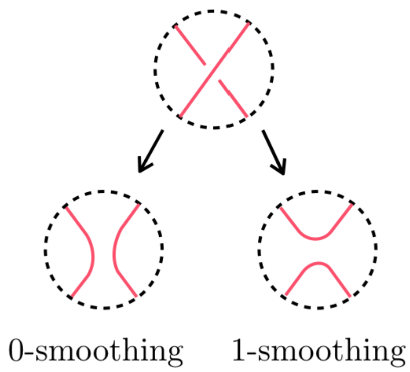


Figure 1: The two possible smoothings of a crossing.

Let $Cob^3(\emptyset)$ be the category whose objects are complete smoothings of links, and whose morphisms from smoothings $S_1 \to S_2$ are $\mathbb{Z}$-linear combinations of two-dimensional surfaces (called cobordisms) embedded in $\mathbb{R}^2 \times [0,1]$ whose boundary is the disjoint union of a copy of S_1 in $\mathbb{R}^2 \times \{0\}$ and a copy of S_2 in $\mathbb{R}^2 \times \{1\}$. Morphisms are regarded up to boundary-preserving isotopies. The identity on a smoothing S is the vertical surface $S \times [0,1]$, and composition of cobordisms is given by stacking the surfaces vertically and rescaling by $1/2$ (see Figure 2). Extending this composition in the natural bilinear manner, we have defined composition of morphisms in $Cob^3(\emptyset)$.

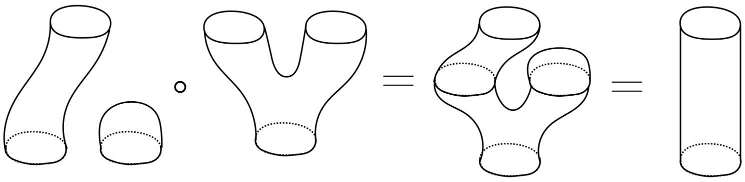


Figure 2: Composition of morphisms in $Cob^3(\emptyset)$.

Choose an arbitrary ordering of the crossings of L, so that the complete smoothings of L may be indexed by ordered strings of 1's and 0's. This allows us to assign the 2^n complete smoothings to vertices on an n-dimensional cube. An example is shown in Figure 3.

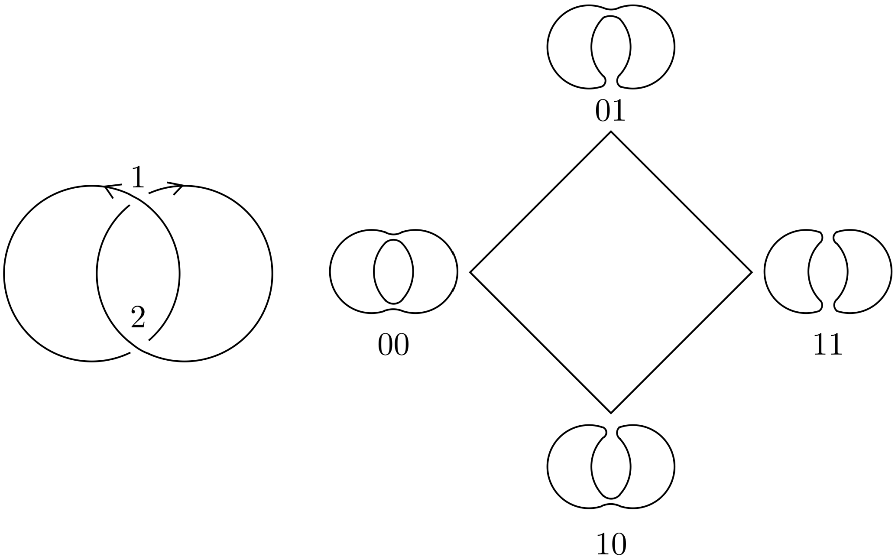


Figure 3: The Hopf link and its cube of complete smoothings are shown.

Each edge connects vertices whose smoothings differ in exactly one crossing, which is a 0-smoothing on one side of the edge (the 'tail') and a 1-smoothing on the other (the 'head'). We wish to assign to each edge ξ a morphism d_ξ in $Cob^3(\emptyset)$ from the smoothing at the tail to the smoothing at the head. We take this to be the identity on all but a small region surrounding the crossing whose smoothing changes along the edge, on which it is taken to be a simple saddle $)(\to \asymp$, which we will write as $)\!($. This is illustrated in Figure 4.

We now define the chain complex $\widetilde{C}(L)$ associated to L. The r'th chain group is the formal direct sum

$$[\widetilde{C}(L)]^r = \bigoplus \text{(complete smoothings with exactly } r \text{ 1-smoothings)} \{r + n_+ - 2n_-\},$$

where $.\{\,\}$ is the formal degree shift operation, n_+ is the number of positive crossings, and n_- is the number of negative crossings (see Figure 5). If we simply take the r'th differential to be the sum of the edge maps whose tail is a direct summand of $[\widetilde{C}(L)]^r$, our complex will not satisfy $d^2 = 0$. Each square of our cube commutes

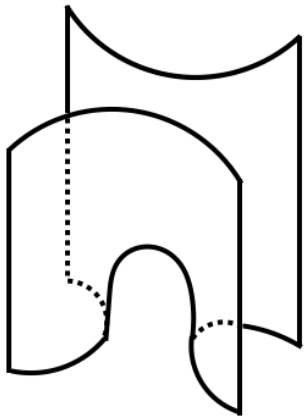


Figure 4: The simple saddle cobordism that maps from a 0-smoothing to a 1-smoothing.

(since the order of two spatially separated cobordisms does not matter), and thus if we assign negative signs to certain edges such that each square has an odd number of negative signs, every square will anti-commute. Then summing the edge maps, the complex will satisfy $d^2 = 0$.

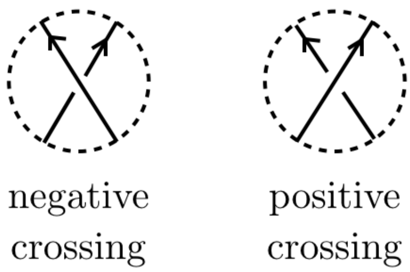


Figure 5: Positive and negative crossings.

If ξ connects vertices which differ in the smoothing of the i'th crossing, let $\mathrm{s}(\xi)$ be the number of 1-smoothings occuring before the i'th position in the complete smoothings of the vertices on either side of the edge, which is odd for an odd number of edges around each square. Then the r'th differential is then given by

$$\sum_{\xi}(-1)^{\mathrm{s}(\xi)}d_{\xi}.$$

Finally, let $C(L)$ be $\widetilde{C}(L)$ with a height shift of $-n_-$; that is,

$$(C(L))^r = (\widetilde{C}(L))^{r+n_-}.$$

We are now ready to assign a TQFT. A TQFT, or Topological Quantum Field Theory, is a functor from $Cob^3(\emptyset)$ to the category of graded abelian groups and group homomorphisms, such that disjoint unions are mapped to tensor products.

Let $\mathcal{A}$ be the free abelian group spanned by $\mathbf{1}$ and X, where $\mathbf{1}$ has degree 1 and X has degree -1. Our functor $\mathcal{F}$ assigns to a circle the abelian group $\mathcal{A}$, and to the generating cobordisms it assigns the following group homomorphisms:

$$\mathcal{F}(\ ) = \epsilon : \begin{cases} \mathbf{1} \mapsto 0 \\ X \mapsto 1 \end{cases}$$

$$\mathcal{F}(\ ) = \iota : \begin{cases} 1 \mapsto \mathbf{1} \end{cases}$$

$$\mathcal{F}(\ ) = m : \begin{cases} \mathbf{1} \otimes \mathbf{1} \mapsto \mathbf{1} \\ \mathbf{1} \otimes X \mapsto X \\ X \otimes \mathbf{1} \mapsto X \\ X \otimes X \mapsto 0 \end{cases}$$

$$\mathcal{F}(\ ) = \Delta : \begin{cases} \mathbf{1} \mapsto \mathbf{1} \otimes X + X \otimes \mathbf{1} \\ X \otimes X \mapsto X \end{cases}$$

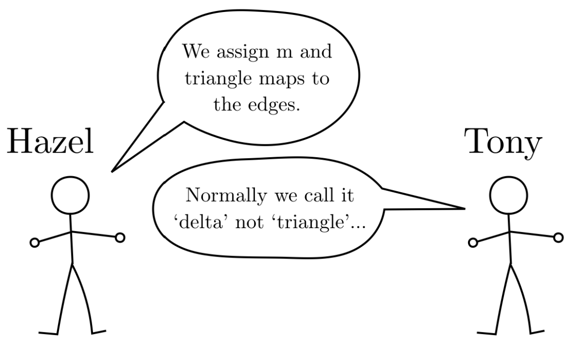


Figure 6: First meeting with Tony.

This functor is known to be well-defined, since the maps ϵ, ι, Δ and m satisfy the relations of a *Frobenius algebra*.

Appyling $\mathcal{F}$ to all objects and morphisms in our n-dimensional cube, and sending formal direct sums to honest direct sums, we obtain a chain complex of graded abelian groups, which is an object in the category of complexes of graded abelian groups and chain maps between them. Different projections of the same link have homotopy equivalent chain complexes (see [1]). Thus if we set homotopic maps to be equal, the isomorphism class in the resulting *homotopy category* of complexes of graded abelian groups is a link invariant, and it is constructed so that its graded Euler characteristic is the Jones polynomial.

3 Extending to a Tangle Invariant

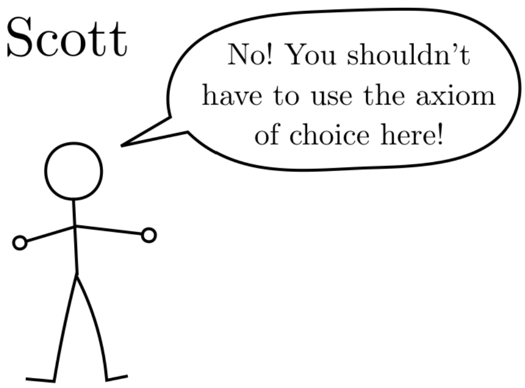
Suppose that we now wish to extend this link invariant to a tangle invariant. We can smooth the crossings of a tangle projection and add edge maps to construct a cube as in the previous section. The challenge we face is turning this geometric construction into something easier to work with. Unlike with links, we cannot apply a TQFT, since the objects assigned to each vertex are now disjoint unions of both circles and arcs.

Bar-Natan avoids this issue by only considering chain complexes of smoothings and cobordisms, modulo some 'local relations', without assigning any algebraic structure. Essentially, we follow the construction of the previous section, but stop immediately before applying a TQFT, and instead we quotient out by certain 'local relations' (see Section 4 of [1]). The isomorphism class in the homotopy category is a tangle invariant, although given two complexes, it may be difficult to see whether they are homotopy equivalent. This construction does, however, have the advantage that tangles can be composed in all possible ways (see Section 5 of [1]). Thus, if the ultimate goal is a local method for computing the chain complex of a link, to which we can then apply a TQFT, Bar-Natan's construction is ideal.

If, however, we want a tangle invariant with some algebraic structure, Khovanov shows how to assign graded bimodules to tangle smoothings, by considering all possible 'planar closures' of the tangle.

An (oriented) (n, m)-tangle is an (oriented) tangle in $\mathbb{R}^2 \times [0, 1]$ with $2n$ boundary points in $\mathbb{R}^2 \times \{0\}$ and $2m$ boundary points in $\mathbb{R}^2 \times \{1\}$. These are positioned uniformly along $\mathbb{R} \times \{0\} \times \{0\}$ and $\mathbb{R} \times \{0\} \times \{1\}$ to allow (n, m)- and (m, k)-tangles to be composed by stacking vertically (provided the orientations match). A projection of an (oriented) (n, m)-tangle onto $\mathbb{R} \times [0, 1]$ is called an (oriented) (n, m)-tangle diagram.

Let $\widetilde{B}^n_m$ be the set of all complete smoothings of (n, m)-tangle diagrams, which we will call flat (n, m)-tangles, or simply flat tangles. For a flat tangle $T \in \widetilde{B}^n_m$, let $W(T) \in \widetilde{B}^m_n$ be the reflection of T in $\mathbb{R} \times \{1/2\}$. Choose a set B^n_m with one representative from each isotopy class of flat tangles without circles, such that $W(B^n_m) = B^m_n$ for all n and m.



We begin by defining the rings H^n for each non-negative integer n. We will then assign an (H^n, H^m)-bimodule to each flat (n, m)-tangle.

As an abelian group, H^n is given by

$$H^n = \bigoplus_{a,b \in B^n} \mathcal{F}(W(a)b)\{n\},$$

where $\mathcal{F}$ is the functor given in Section 2 and $\{n\}$ denotes a grading shift of n. For example, in B^2 has two elements: either the two arcs connect adjacent points, or the two outer points are connected and the two inner points are connected. Thus there are four direct summands in H^2, as shown in Figure 7.

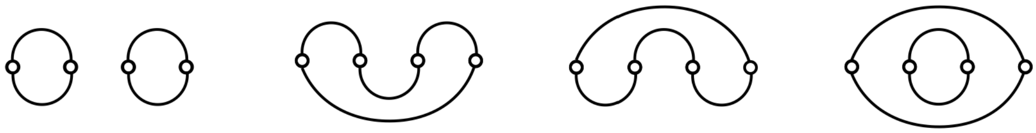


Figure 7: The four direct summands in the ring H^2 correspond to the above diagrams. From left to right, these are $\mathcal{A}^{\otimes 2}\{2\}$, $\mathcal{A}\{2\}$, $\mathcal{A}\{2\}$ and $\mathcal{A}^{\otimes 2}\{2\}$.

We now need to define multiplication in H^n, and we do so using cobordisms and the functor $\mathcal{F}$ from Section 2. Between direct summands $\mathcal{F}(W(a)b)$ and $\mathcal{F}(W(b')c)$, we define the multiplication to be zero if $b \neq b'$.

If $b = b'$, the idea is to find a canonical cobordism between $bW(b)$ and the flat tangle consisting of $2n$ vertical lines, which we will denote Vert_{2n}. By taking the identity cobordism on $W(a)$ and c, this will give a map from

$$W(a)bW(b)c \to W(a)c.$$

But since $W(a)b$ and $W(b)c$ are disjoint unions of circles,

$$W(a)bW(b)c = W(a)b \sqcup W(b)c.$$

Applying $\mathcal{F}$, this will give us a group homomorphism from

$$\mathcal{F}(W(a)b) \otimes \mathcal{F}(W(b)c) \to \mathcal{F}(W(a)c)$$

which defines the ring multiplication xy of elements in $x \in \mathcal{F}(W(a)b)$ and $y \in \mathcal{F}(W(b)c)$.

The canonical cobordism from $bW(b) \to \mathrm{Vert}_{2n}$ is the composition of n saddles given by successively 'saddling off' outer arcs in b with their counterpart in $W(b)$. The resulting cobordism, which we will call $S(b)$, is diffeomorphic to the disjoint union of n discs. An example is shown in Figure 8.

We are now ready to define the (H^n, H^m)-bimodule associated to a flat (n,m)-tangle T, which we will denote $\mathcal{F}(T)$. The underlying abelian group of the bimodule is given by

$$\mathcal{F}(T) = \bigoplus_{a \in B^m, b \in B^n} \mathcal{F}(W(a)Tb)\{m\},$$

where $\mathcal{F}$ is the functor given in Section 2. Just like multiplication within the ring, the ring action is defined on pairs of direct summands. For $r \in \mathcal{F}(W(a)b) \subseteq H^n$ and $m \in \mathcal{F}(W(b')Tc) \subseteq \mathcal{F}(T)$, rm is defined to be zero if $b \neq b'$. For $b = b'$, consider the cobordism consisting of the canonical cobordism $S(b)$ on $bW(b)$ and the identity cobordism on $W(a)$ and Tc. Since

$$W(a)bW(b)Tc = W(a)b \sqcup W(b)Tc,$$

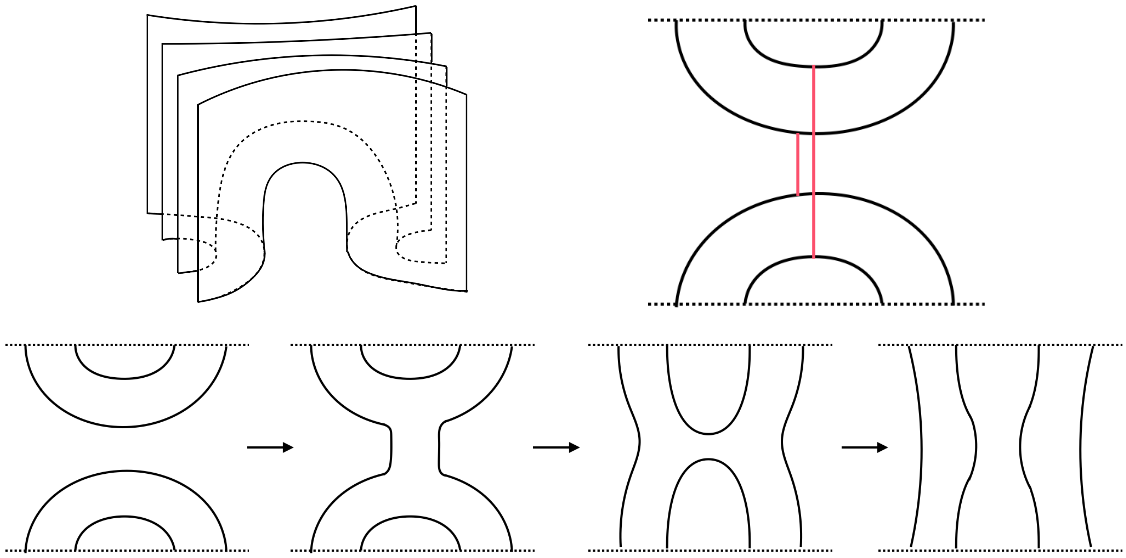


Figure 8: The canonical cobordism $S(b) : bW(b) \to \mathrm{Vert}_4$ is shown for $b =$. The surface is drawn out explicitly on the top left; the top right gives a diagrammatic form where vertical lines show where two arcs are joined in a saddle; the bottom is a 'movie representation', where each picture is a cross-section of the cobordism.

this gives a cobordism from

$$W(a)b \sqcup W(b)Tc \to W(a)c.$$

Applying $\mathcal{F}$ gives a map from

$$\mathcal{F}(W(a)b) \otimes \mathcal{F}(W(b)Tc) \to \mathcal{F}(W(a)c),$$

which defines the left multiplication rm. The right action of H^m is defined analogously, completing the definition of the bimodule associated to each vertex of the cube of smoothings of an oriented (n, m)-tangle diagram.

For an edge ξ joining smoothings T_1 and T_2, consider the cobordism that is a simple saddle on the small neighbourhood of the crossing whose smoothing differs between T_1 and T_2 and the identity on the rest of T_1. This induces a bimodule map from

$$\mathcal{F}(W(a)T_1b) \to \mathcal{F}(W(a)T_2b)$$

for each $a \in B^n$, $b \in B^m$, since it commutes with the spatially separated cobordism defining the ring action. By taking the sum of all such maps, it induces a bimodule map from

$$\mathcal{F}(T_1) \to \mathcal{F}(T_2).$$

This is the map we assign to ξ.

We have thus defined the vertices and edges of the cube of resolutions of an oriented (n, m)-tangle T. The chain complex corresponding to the cube is then given as in Section 2: the r'th chain group is the direct sum of vertices with r 1-smoothings, with a grading shift of $\{r + n_+ - 2n_-\}$; the edges are given a sign and summed to ensure that $d^2 = 0$; and a height shift of $-n_-$ is applied. An example is shown in Figure 9.

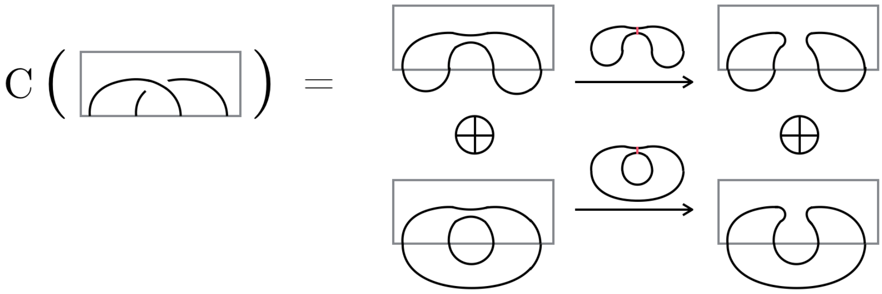
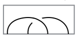


Figure 9: The (geometric) complex of bimodules and bimodule maps for the $(0, 2)$-flat tangle  is shown. By applying $\mathcal{F}$ and height and grading shifts, we recover Khovanov's invariant.

4 Injectivity of Khovanov's functor

Suppose that somebody hands us the bimodule $\mathcal{F}(T)$ for some flat tangle T. Can we reconstruct T up to planar isotopy?

We first note that knowing the abelian group alone is in general not sufficient to determine the isotopy class of a flat (n, m)-tangle T, since there are non-isotopic flat tangles (such as those shown in Figure 10) to which $\mathcal{F}$ assigns the same abelian group.

Figure 10: Non-isotopic flat $(2, 1)$-tangles for which the associated abelian groups are the same.

However, only a little more information is required to uniquely determine the isotopy class of the flat tangle up to repositioning of closed loops.

Let $a_n \in B^n$ be the flat tangle with arcs connecting the $2i - 1$'st and $2i$'th points for all i (see Figure 11). We have:

Proposition 1. *Let T be a flat (n, m)-tangle. Given the abelian group $\mathcal{F}(T)$, the left ring action of the direct summand $\mathcal{F}(W(a_n)a_n) \subseteq H^n$, and the right ring action of the direct summand $\mathcal{F}(W(a_m)a_m) \subseteq H^m$, the isotopy class of T is uniquely determined up to repositioning of circles.*

Proof. The proof is in two parts. First, we show how that ring actions determine T', where T' denotes T with all circles removed. Then it easily follows that the number

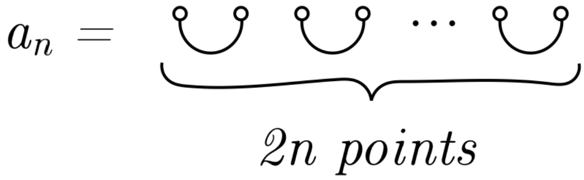


Figure 11: The flat tangle a_n is shown. Arcs join the $2i-1$'st and $2i$'th points.

of circles in T can be determined given the abelian group $\mathcal{F}(T)$.

Elements of $\mathcal{F}(W(a_n)a_n) \subseteq H^n$ are of the form $x_1 \otimes x_2 \otimes \ldots \otimes x_n\{n\}$, with $x_i \in \mathcal{A}$ the element corresponding to the i'th circle from the left in $W(a_n)a_n$. Let $k_i \in \mathcal{F}(W(a_n)a_n)$ be of this form with $x_i = X$ and $x_{\neq i} = \mathbf{1}$.

We now define the map $\mu_i : \mathcal{F}(T) \to \mathcal{F}(T)$ as follows:

- For $1 \leq i \leq n$, μ_i is defined as left-multiplication by $k_i \in \mathcal{F}(W(a_n)a_n)$ and right-multiplication by $\mathbf{1}^{\otimes m} \in \mathcal{F}(W(a_m)a_m)$.
- For $n+1 \leq i \leq n+m$, μ_i is defined as left-multiplication by $\mathbf{1}^{\otimes n} \in \mathcal{F}(W(a_n)a_n)$ and right-multiplication by $k_{m-i} \in \mathcal{F}(W(a_m)a_m)$.

The map μ_i acts non-trivially only on the direct summand $W(a_n)Ta_m$. This is shown in Figure 12.

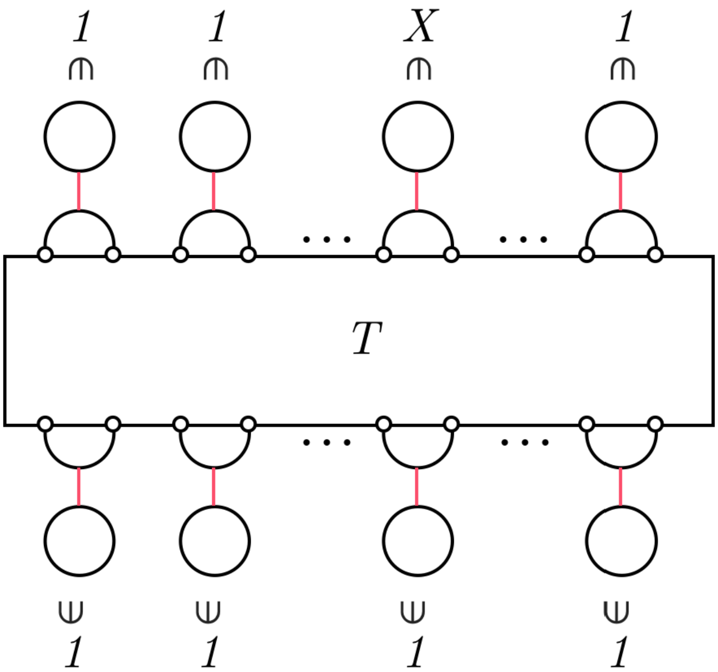


Figure 12: The maps μ_i are given by the above cobordism, where the red lines represent saddles, with X in the copy of $\mathcal{A}$ corresponding to the i'th circle counting clockwise from the top left.

Numbering the arcs of $W(a_n)$ and a_m clockwise from the top left, we note that μ_i is the identity on all tensor factors except the tensor factor corresponding to the circle in $W(a_n)Ta_m$ containing the i'th arc. Thus $\mu_i = \mu_j$ if and only if the i'th and j'th arcs are connected by T. The maps μ_i therefore partition the arcs into subsets S_c of arcs contained in the same circle $c \subseteq W(a_n)Ta_m$.

For example, Figure 13 has three circles: one containing the first and fourth arcs, one containing the second arc and one containing the third arc. This corresponds to the fact that the only equality amongst the multiplication maps is $\mu_1 = \mu_4$.

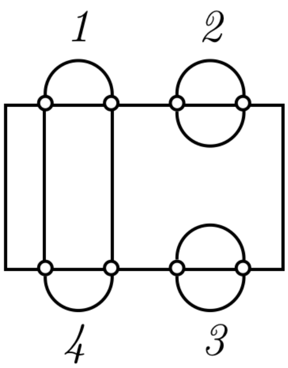


Figure 13: The multiplication maps μ_1 and μ_4 are equal, since the flat tangle connects the first and fourth arcs into the same circle.

If we trust whoever provided us with the bimodule, we know it corresponds to some tangle T, so we know that there is at least one way to construct a tangle that connects exactly the arcs that the μ_i's tell us must be connected and has no circles, namely T', defined as T with any circles removed. We now show that the maps μ_i allow us to determine T' up to planar isotopy.

Choose a set S_{c_1} of arcs contained in some circle $c_1 \subseteq W(a_n)Ta_m$. If we consider the rectangle containing T now as a disk and the endpoints of the arcs in S_{c_1} as points on the boundary of the disk, we see that T must connect each point to an adjacent point, since any other arc would split the disk into two regions with points in either region. It would then be impossible to achieve a single circle. There is a unique way, up to planar isotopy, to connect every point to an adjacent point and end up with a single circle. Thus, up to planar isotopy, we have determined the arcs of T' that connect the arcs of S_{c_1}; call this set of arcs A_{c_1}, so that

$$c_1 = S_{c_1} \cup A_{c_1}.$$

Now choose another set S_{c_2} of arcs contained in a circle $c_2 \subseteq W(a_n)Ta_m$. The arcs in A_{c_1} have already split the rectangle containing T into various sections, but if we still trust whoever provided us with the bimodule, it must be possible to connect S_{c_2} into a single circle regardless, meaning that the boundary points of the arcs of S_{c_2} must all lie within the same section of the rectangle. Again, we can think of this section of the rectangle as a disk, and by the same argument as before, there is a unique way to connect them into a single circle up to planar isotopy. Thus we have determined A_{c_2}, the set of arcs in T' connecting the arcs of S_{c_2} into a single circle.

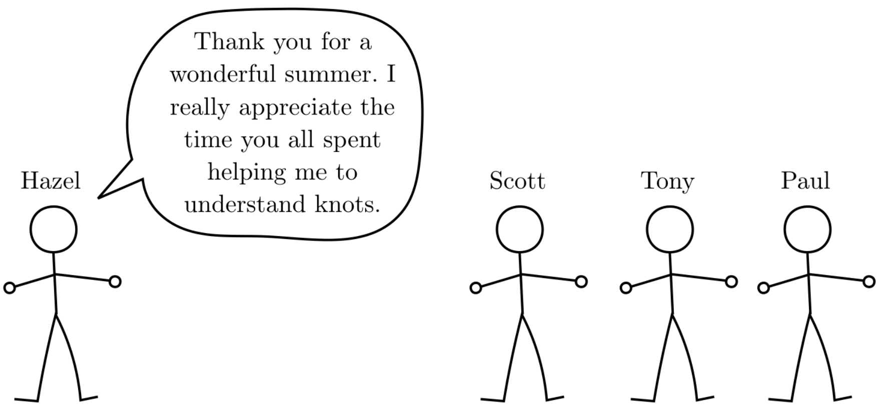
Continuing in this way, for each circle $c_i \subseteq W(a_n)Ta_m$, there is a unique way, up to planar isotopy, to connect the arcs in S_{c_i} into a single circle. Thus, completing the process for all circles c_i, T' is uniquely determined up to planar isotopy.

All that remains is to deduce how many circles we need to add to T' to get T. This is simply the difference in the tensor factors in each direct summand of $\mathcal{F}(T')$ and $\mathcal{F}(T)$.

$\square$

Unfortunately, the bimodule structure alone tells us nothing about where we should place the circles. This isn't so bad though, because no matter where we place them on the tangle diagram, they represent the same tangle, since we can move circles between the various sections using Reidemeister II.

Acknowledgements



Appendix A Alternative Ways to Write Zero

In knot theory, there can be considerable ambiguity caused by the similarity in appearance of the circle $\bigcirc$ and the number 0. This problem can be averted by writing the word ‘zero’. However, when the author posed this dilemma to the LG104 office, they came up with the following notation convention. Every time a zero is required, choose an arbitrary element of the following list:

- $\lim_{n\to\infty} 1/n$
- $\chi($$)$
- $dx \wedge dx$
- $E(X), X \sim N(0,1)$
- $\int_{\mathbb{S}^3} dw$
- $\nabla \cdot (\nabla \times \underline{u})$
- $\frac{1}{12} + \sum_{n=1}^{\infty} n$
- $\pi - \frac{22}{7}$

The present paper contained few zeroes and thus the solution was not utilised; however, the author is confident that this notation will gain widespread use in the field.

References

[1] D. Bar-Natan. "Khovanov's homology for tangles and cobordisms". In: *Geom. Topol.* 9 (2005), pp. 1443–1499. URL: https://doi.org/10.2140/gt.2005.9.1443.

[2] M. Khovanov. "A categorification of the Jones polynomial". In: *Duke Math. J.* 101.3 (2000), pp. 359–426. URL: https://doi.org/10.1215/S0012-7094-00-10131-7.

[3] M. Khovanov. "A functor-valued invariant of tangles". In: *Algebr. Geom. Topol.* 2 (2002), pp. 665–741. URL: https://doi.org/10.2140/agt.2002.2.665.

[4] P. B. Kronheimer and T. S. Mrowka. "Khovanov homology is an unknot-detector". In: *Publ. Math. Inst. Hautes Études Sci.* 113 (2011), pp. 97–208. URL: https://doi.org/10.1007/s10240-010-0030-y.