Understanding the Lee Spectral Sequence

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To Dad, who would have liked to read this.
And to Mum, for always encouraging my learning.

Declaration

The work in this thesis is my own except where otherwise stated.

Jack Brand

Acknowledgements

Of course, I emphatically thank my supervisor Scott Morrison for teaching me such interesting mathematics not just this year, but over the past three years. Your teaching has been both engaging and inspirational.

I must also thank Paul Wedrich for also taking on a supervisory role. Thankyou for your time and for your tireless enthusiasm for Khovanov Homology.

I won't name names because I will inevitably forget someone, but I thank my friends for the company during a hard year — especially while doing the more fun things in life such as running, cycling, mountain biking and rogaining.

I especially thank Alice Patterson-Robert for her support throughout the year, it would have been much more difficult otherwise.

I would also like to thank Ben Thompson - I drew most of the diagrams, but some of the diagrams were drawn by Ben and used with his permission. Explicitly, the pictures drawn by Ben are: pictures 1 and 2 on page 1; picture 2 on page 17; picture 1 on page 21; picture 1 on page 23; picture 1 and 3 on page 24.

Notation and quick reference

A link may be referred to by numbers, for example 3_1 denotes the trefoil knot. This is from Rolfsen's knot classification and can be found in the Knot Atlas online [BNMea].

For a crossing  in a tangle diagram,  is its zero smoothing and  is its one smoothing.

For an oriented tangle diagram,  is a positive crossing and  is a negative crossing. The one and zero smoothings of oriented crossings are the same as their non-oriented counterparts.

$[\![-]\!]$ is the 'Bar-Natan bracket', which is a chain complex in Cob^3. See Proposition 2.1.

$BN(-)$ is $[\![-]\!]$ with gradings. See Definition 2.19.

$CKh(-)$ is the Khovanov chain complex. See Definition 1.10, Remark 2.23, or Example 2.25.

$Kh(-)$ is Khovanov homology (the homology of the Khovanov complex).

$CKh^{Lee}(-)$ is the Lee chain complex. See Example 2.26.

$Kh^{Lee}(-)$ is Lee homology (the homology of the Lee complex).

$CKh^{\alpha}(-)$ is the chain complex for 'universal Khovanov homology'. See Definition 4.10.

$Kh^{\alpha}(-)$ is 'universal Khovanov homology' (the homology of CKh^{α}).

Contents

Chapter 1

Introduction

Khovanov homology is a (co)homology theory that gives invariants of tangles. It was originally described by Khovanov as a categorification of the Jones polynomial in [Kho00]. This original construction, however, was only defined for links, but in a series of papers [BN02] [BN05] [BN07] Bar-Natan generalised the theory to give an account for tangles, as did Khovanov in [Kho01]. At about the same time, Lee [Lee05] defined another homology theory, appropriately dubbed 'Lee homology' that is 'interestingly boring'. In fact, as we will explain below, Khovanov homology is naturally viewed as the second page of a spectral sequence that converges to Lee homology. This spectral sequence was then skilfully used by Rasmussen in [Ras10] to define the 's-invariant', which is a knot invariant that provides an obstruction to 4-dimensional smooth structure. More sepcifically, the s-invariant $s(K)$ of a knot K gives a lower bound on the slice (4-ball) genus of such a knot,

$$|s(K)| \leq 2g_4(K).$$

That is, any surface Σ in B^4, with $\partial\Sigma = K \subset S^3 = \partial B^4$ has minimal genus at least $1/2|s(K)|$. This is great news since we have very few invariants in 4 dimensional topology, and of those the interesting ones come from gauge theory which are hard to compute. For example, one consequence of the s-invariant is a purely combinatorial proof of the Milnor Conjecture [Ras10] that was originally proved using gauge theory by Kronheimer and Mrowka [KM93].

In [FGMW10], Freedman, Gompf, Morrison, and Walker showed that for a certain 4-manifold W^4, where W^4 is homeomorphic to B^4, but where it is unknown whether W^4 and B^4 are diffeomorphic, there is a knot K in S^3 which is slice in W so it 'bounds a disk in W'. In other words, if the slice genus $g_4(K) > 0$, then $W^4 \ncong_{\text{diffeo}} B^4$. Thus, motivating this thesis is that we need to find more 4-manifold invariants out of Khovanov homology, not just the s-invariant. We would like to have, for example, another invariant $\hat{s}_W$ for knots $K \subset \partial W \neq S^3$, so that $|\hat{s}(K)| \leq \text{slice\_genus}_W(K)$. To do this, we will investigate the Lee spectral sequence, but this method seems much harder to generalise from ∂B^4 to ∂W^4. Hopefully what is written below should at least provide a clear account of the subject, and assist in future progress in the area. Most of what is written below (besides perhaps Bar-Natan's seminal papers) is often talked about in research, but not clearly written down, and the literature is inaccessible to the non-expert.

The breakdown is as follows:

- In chapter 1, we will start with an introduction to Khovanov homology, and briefly explain Khovanov's orginal categorification of the Jones polynomial. The material here covers [BN02] and [Kho00] closely, and hopefully provides a good introduction to the subject and lays down some preliminary ideas and definitions.
- In chapter 2, we will then exposit Bar-Natan's generalisation of Khovanov homology to 'local Khovanov homology', which gives an extension of the theory from link to tangles. Bar-Natan

introduces this theory in a series of three papers [BN02] [BN05] [BN07], which is so clear, that the account below provides almost nothing new. If something is unclear, refer back to these papers. We will then focus on the possible TQFTs that arise from Bar-Natan's construction, as these are then used to describe the Lee spectral sequence which will be of primary importance in later chapters.

- In chapter 3, we will digress and give a brief description of spectral sequences. If you are familiar with spectral sequences, then skip this chapter, however we use this chapter to specify the conventions for spectral sequences that we will be using thereafter. The material here is based on [Wei94].
- In chapter 4, we introduce Kh^α, which is a generalisation of Khovanov homology and Lee homology, and see how you can get bounds on the slice genus from this construction as proved by Rasmussen in [Ras10]. In this chapter, we compare Kh^α to the Lee spectral sequence. The correspondence is very close, thus giving us a way to bypass discussion of spectral sequences, and perhaps giving us a better way of viewing the relation between the homology theories.

1.1 Preliminaries

A lot of what follows in this paper stems from Khovanov's original categorification of the Jones polynomial which we will briefly describe. Although this construction is not strictly essential to what follows, and we could alternatively dive right into Bar-Natan's generalisation, it gives an introduction to the subject and helps provide examples and computations. Moreover, it allows us to set up some of the conventions for what comes afterwards. In the following chapter we will then go on to understand Bar-Natan's generalisation of Khovanov homology to tangles, from which we can easily recover Khovanov's original construction. But before we discuss into the homology theories, we will spell out a few preliminary definitions.

Definition 1.1. A *knot* is a smooth embedding of S^1 into S^3, $K : S^1 \to S^3$. A *link* is a disjoint union of knots, $L : S^1 \sqcup \cdots \sqcup S^1 \to S^3$.

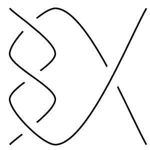
Definition 1.2. A tangle is a proper smooth embedding of a compact 1-manifold X, possibly with boundary, into a 3-ball B^3 such that the 1-manifold X and the boundary of the 3-ball B^3 intersect transversely.

We will want to consider links up to smooth isotopy, so we can compare links by a deformation without breaking the link, or allowing the link to pass through itself. The smoothness condition rules out 'pulling the knot tight' that would mean that all knots are isotopic to the unknot.

Definition 1.3. Let L_1 and L_2 be two links, regarded as the images of the maps $f_1 : S^1 \to S^3$ and $f_2 : S^1 \to S^3$, respectively. Then the links are smoothly *isotopic* if there is a smooth homotopy $H : S^1 \times [0,1] \to S^3$ from f_1 to f_2 such that $H(S^1, t)$ is an embedding for all fixed $t \in [0,1]$.

For the rest of this paper, we will refer to smooth isotopy simply as isotopy. For tangles, we consider a tangle up to isotopy relative its boundary, so we can't move around the endpoints.

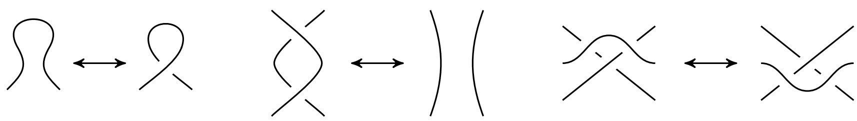
We will often consider a link's or tangle's *diagram*, which is a (regular) projection of the link onto a plane where we take note of strands going 'over' or 'under':



For tangles, we call a tangle a (n, m)-tangle if it has n 'input' strands (in the bottom) and m 'output strands', so the above picture would be for a $(2, 2)$-tangle. In particular, a link is a $(0, 0)$-tangle.

Projections of the same tangle are not unique, we could isotope the link so that it looks totally different, and then consider a different projection of the tangle. However, we have the following very useful theorem that allows us to compare diagrams:

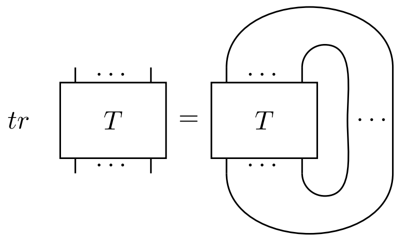
Theorem 1.4. *(Reidemeister). Two tangle diagrams D_1 and D_2 correspond to the same tangle up to isotopy if D_2 can be obtained from D_1 by the following 'Reidemeister moves' and planar isotopies (relative boundary):*



R1 R2 R3

Thus to find link and tangle invariants, we wish to find properties of the link or tangle diagrams that do not depend on the Reidemeister moves.

Throughout this paper, we will take the trace of an (n,n)-tangle T, denoted $tr(T)$. This is just joining the top and bottom strands to make a tangle T into a link L.



1.2 Khovanov's Original Categorification of the Jones Polynomial

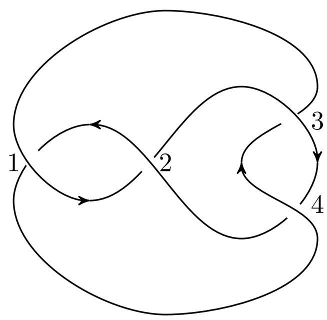
This section will describe Khovanov's original categorification of the Jones polynomial. The original formulation of Khovanov homology is a direct result of 'categorifying' the Jones polynomial, as we will describe below.

The Jones polynomial is a function J : {oriented link diagrams} $\to \mathbb{Z}[q^{\pm 1}]$ that can be defined via the Kauffman bracket, $\langle - \rangle$. The Kauffman bracket has the following properties: $\langle \emptyset \rangle = 1$, $\langle \bigcirc L \rangle = (q+q^{-1})\langle L \rangle$ and $\langle \times \rangle = \langle \asymp \rangle - q\langle)(\rangle$ for some formal variable q, where $\asymp$ is called a '1-smoothing' or '1-resolution' and $)($ is called a '0-smoothing' or '0-resolution'. For example,

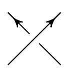
$$\begin{aligned} \langle \text{ᘯ} \rangle &= \langle \text{ᘯ} \rangle - q\langle \text{ᘯ} \rangle \\ &= (q+q^{-1})\langle \text{ᘯ} \rangle - q\langle \text{ᘯ} \rangle \\ &= q^{-1}\langle \text{ᘯ} \rangle. \end{aligned}$$

(So the Kauffman bracket gains a factor of $q^{\pm 1}$ under the first Reidemeister move!)

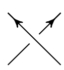
Consider an *oriented* link L embedded in S^3, and consider its projection, for example consider a diagram for the oriented 4_1 knot (with numbered crossings)



A crossing that locally looks like



is called a positive crossing, and a crossing that locally looks like

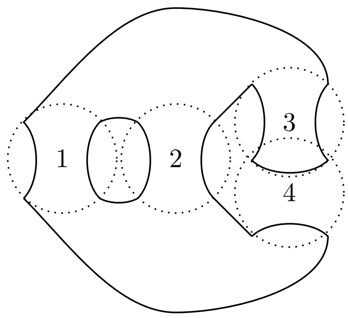


is called a negative crossing. Their 0- and 1- smoothings are the same as their non-oriented versions. Denote (n_+, n_-) to be the number of positive crossings and negative crossings, respectively. So in the above example $(n_+, n_-) = (2, 2)$.

Definition 1.5. The unnormalized Jones polynomial is defined by $\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$. The normalized Jones polynomial is defined to be $J(L) = \hat{J}(L)/(q + q^{-1})$.

Theorem 1.6. *The Jones polynomial is an oriented link invariant. That is, it is invariant under the Reidemeister moves.*

Note that the Jones polynomial is defined for oriented links, whereas the Kauffman bracket does not see this orientation. An easy way to compute the Jones polynomial is via the following algorithm as explained in Bar-Natan [BN02]. Suppose we have a link diagram with n crossings labelled 1 to n. From this diagram we can apply a 0- or 1- smoothing to every crossing to obtain a 'complete smoothing' of the diagram. For example, below we have a complete smoothing of the 4_1 knot. We record which smoothing we have applied to which crossing by a string of n digits comprised of 0s and 1s (so $v \in \{0, 1\}^n$) which records in the ith position whether we have applied a 0- or 1-smoothing on the ith crossing. Thus the complete smoothing below is a 1110 smoothing of the 4_1 knot.

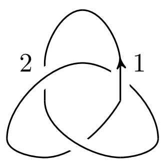


1110 smoothing of 4_1

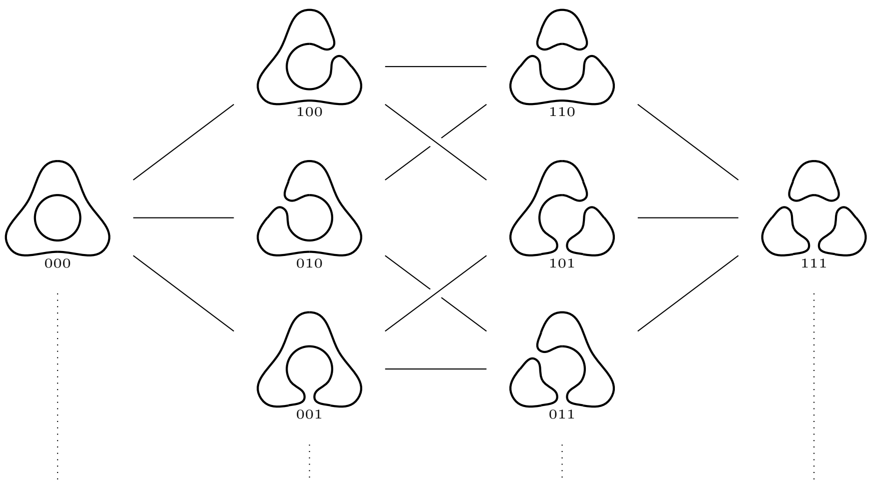
Each complete resolution is then going to be a disjoint collection of circles. For example, in the case of the above diagram, we have two circles.

From all possible complete smoothings we then can form the 'cube of resolutions' for this knot, which at each vertex of this (hyper)cube has a complete smoothing of the knot diagram. We assemble the vertices so that smoothings with the same 'height' (i.e. number of 1-smoothings in the complete smoothing) are in the same column, and there is an edge between two smoothings if their resolution differs by changing one digit. To each vertex of the cube we assign a polynomial $(-1)^r q^r (q + q^{-1})^k$, where k is the number of disjoint circles at the vertex, and r is the height of the complete smoothing. We sum up all of the polynomials at each vertex and then multiply by a normalisation term $(-1)^{n_-} q^{n_+ - 2n_-}$.

Example 1.7. Take the oriented trefoil 3_1, with crossings numbered as below:



Then the cube of resolutions for such a diagram is given by



$$(-1)^0q^0(q+q^{-1})^2 \;+\; 3(-1)^1q^1(q+q^{-1}) \;+ 3(-1)^2q^2(q+q^{-1})^2 + \;(-1)^3q^3(q+q^{-1})^3$$

Adding the polynomials above, the polynomial is equal to $p(q) = q^{-2} + 1 + q^2 - q^6$, and since $(n_+, n_-) = (3, 0)$, $(-1)^{n_-}(q^{n_+-2n_-})p(q) = q + q^3 + q^5 - q^9$, which is the unnormalised Jones polynomial. This method gives a version of the Jones polynomial that may differ from other sources by a relabelling and change of variables, but we will use this algorithm since it will be easier to 'lift' to Khovanov homology.

1.2.1 Categorification

Categorification is a loosely defined term, that we will convey through some examples. In its most simple description, categorification lifts mathematical structures from sets to categories, where these categorified structures provide additional information that their decategorified counterparts do not. Perhaps the opposite process of decategorification is simpler to describe. One way to decategorify is to take the Grothendieck group $K_0(-)$, which, in the case of pre-additive categories (a category enriched in abelian groups, see Definition 2.2) , amounts to looking at the isomorphism classes of the objects modulo $[A \oplus B] = [A] \oplus [B]$, and in categories of complexes in pre-additive categories amounts to taking the Euler characteristic of the complex valued in the Grothendieck group of the underlying category. The Jones polynomial is a Laurent polynomial, so let's try to categorify $\mathbb{Z}[q, q^{-1}]$. The steps below are shown using graded vector spaces, as Bar-Natan's exposition does [BN02], but this works more generally — Khovanov does this for graded free $\mathbb{Z}$-modules [Kho00].

- The category $\mathsf{Vect}_{\Bbbk}$ of finite dimensional vector spaces over the field $\Bbbk$ categorifies $\mathbb{Z}_+$, and is decategorified by $\dim(-)$. Indeed, via decategorification (where the Grothendieck group amounts to considering isomorphism classes of vector spaces which are classified up to positive integers), $V \mapsto \dim(V)$. Moreover, the operations on the natural numbers are categorified, too, where the operations $\oplus$ and $\otimes$ descend to $+$ and $\times$ respectively. As you would hope, these intertwine with $\dim(-)$ in the following way:

$$\dim(V \oplus W) = \dim(V) + \dim(W)$$
$$\dim(V \otimes W) = \dim(V) \times \dim(W).$$

  For example, the distributive law for the operations in natural numbers, $(a + b)c = ac + bc$, for all $a, b, c \in \mathbb{Z}_{\geq 0}$, also hold for $\mathsf{Vect}_{\Bbbk}$:

$$(V \oplus W) \otimes U \cong (V \otimes U) \oplus (W \otimes U),$$

for all $V, W, U \in \mathsf{Vect}_{\Bbbk}$.

- To extend to all of $\mathbb{Z}$, we consider the category $\mathsf{Kom}(\mathsf{Vect}_{\Bbbk})$ of finite length complexes of finite dimensional vector spaces over $\Bbbk$. This category categorifies $\mathbb{Z}$, and is decategorified by Euler characteristic. Recall that for a chain complex $V_\bullet$ of vector spaces,

$$\cdots \to V_{n+1} \to V_n \to V_{n-1} \to \cdots,$$

the Euler characteristic is given by

$$\sum_n (-1)^n \dim(H_n(V_\bullet)) = \sum_n (-1)^n \dim(V_n),$$

where H_n is taking the nth homology of the chain complex and where the equality is since V is finite dimensional and by the Rank-Nullity Theorem. Thus for $V, W \in \mathrm{ob}(\mathsf{Kom}(\mathsf{Vect}_{\Bbbk}))$,

$$\begin{aligned} \chi(V \oplus W) &= \chi(V) + \chi(W) \\ \chi(V \otimes W) &= \chi(V) \times \chi(W) \\ \chi(C(f)) &= \chi(W) - \chi(V). \end{aligned}$$

where $C(f)$ is the cone of a chain map $f : V \to W$.

- The category $\mathsf{GVect}_{\Bbbk}$ of finite dimensional $\mathbb{Z}$-graded vector spaces over $\Bbbk$ categorifies $\mathbb{Z}_+[q, q^{-1}]$ (without subtraction), decategorified by the graded dimension. The graded dimension for a graded vector space $V = \bigoplus_{m \in \mathbb{Z}} V^m$ is defined by

$$\mathrm{qdim}(V) = \sum_{m \in \mathbb{Z}} q^m \dim(V^m).$$

It can be easily checked that this decategorifies the operations $\oplus$ and $\otimes$ in the desired way.

- Lastly, consider the category $\mathsf{Kom}(\mathsf{GVect}_{\Bbbk})$ of finite chain complexes of finite dimensional graded vector spaces (so the differentials between objects preserve gradings: $d(V_n^m) \subset V_{n-1}^m$, where n is the nth piece of the chain complex and m is the grading in each space). The homology of such a complex is then a bigraded vector space

$$H(V) = \bigoplus_{n,m \in \mathbb{Z}} H_n^m(V)$$

where each $H_n^m(V)$ is the nth homology of the subcomplex

$$\cdots \to V_{n+1}^m \to V_n^m \to V_{n-1}^m \to \cdots$$

Taking the graded Euler characteristic of such a complex gives a Laurent polynomial with $\mathbb{Z}$ coefficients. Here, the graded Euler characteristic is given by

$$\chi(V) = \sum_{m \in \mathbb{Z}} q^m \chi(V^m) = \sum_{n,m \in \mathbb{Z}} (-1)^n q^m \dim(H_n^m) = \sum_{n,m \in \mathbb{Z}} (-1)^n q^m \dim(V_n^m).$$

So, in hindsight, although a surprise at the time Khovanov published his original paper [Kho00], the natural thing to do to categorify the Jones polynomial is to lift the Jones polynomial to the category of finite complexes of finite dimensional graded vector spaces. Khovanov gives us a construction that enables us to lift the Jones polynomial to such a category.

1.2.2 Computing Khovanov Homology

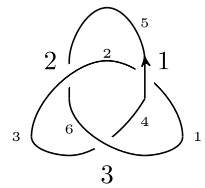
We will compute Khovanov homology for the trefoil knot 3_1, as does Bar-Natan [BN02]. Compare with the computation of the Jones polynomial for 3_1 in Example 1.7 which is almost identical. We will also provide some extra detail, that involves indexing the vector spaces to help work out the differentials between those vector spaces.

For this computation, we introduce the following definitions:

Definition 1.8. For a graded vector space $W = \bigoplus W_m$, we define the 'degree shift' operation $\cdot\{r\}$, where $W\{r\}_m := W_{m-r}$. So $\text{qdim}(W\{r\}) = q^r \text{qdim}(W)$.

Definition 1.9. For a chain complex $C_\bullet$, we have the 'height shift' operation $\cdot[s]$ so that $C[s]_m = C_{m-s}$ with differentials shifting accordingly.

As before, we index the crossings so that we can keep track of the smoothings in the cube of resolutions. We also index the arcs between the crossings in the diagram (in smaller text), as below:



The crossings are all positive, so $(n_+, n_-) = (3, 0)$. From this diagram, we get the following cube of resolutions, where, as with computing the Jones polynomial, each vertex of the diagram will be a collection of disjoint circles. The circles are indexed by the smallest index of the arcs it contains:

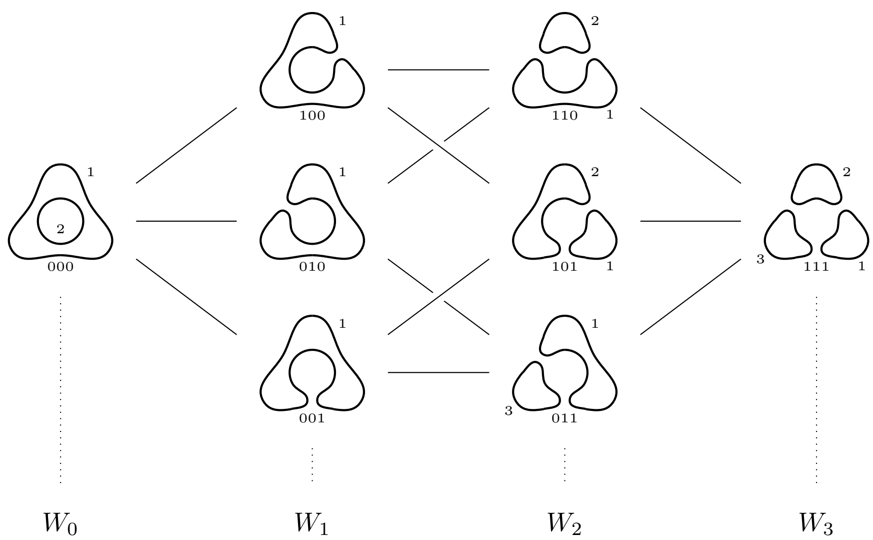


Figure 1.1: Cube of resolutions for 3_1

where

$$\begin{aligned} W_0 &= V_1 \otimes V_2 \\ W_1 &= (V_1 \oplus V_1 \oplus V_1)\{1\} \\ W_2 &= ((V_1 \otimes V_2) \oplus (V_1 \otimes V_2) \oplus (V_1 \otimes V_3))\{2\} \\ W_3 &= (V_1 \otimes V_2 \otimes V_3)\{3\}. \end{aligned}$$

For each disjoint circle we assign a graded vector space $V_i = V = \Bbbk\langle v_-, v_+\rangle$, where $\deg(v_\pm) = \pm 1$, so $\mathrm{qdim}(V) = q^{-1} + q$. Moreover, we index each V by the index of the circle that it corresponds to. At a given vertex of the cube of resolutions, we tensor the vector spaces corresponding to the circles in a smoothing diagram and apply a degree shift according to the number of 1 smoothings at the vertex. So at each vertex we have $V^{\otimes k}\{r\}$ and $\mathrm{qdim}(V^{\otimes k}\{r\}) = q^r(q^{-1}+q)^k$. We then take the direct sum down the columns to get the spaces $CKh'(L)_s = W_s$.

Next we need to define differentials to make this into a complex. Each edge of the cube corresponds to a differential (of a subcomplex in homogeneous q-grading). We label the differentials d_α where $\alpha \in \{0,1,*\}^n$, where there is a single $*$, and where n is the number of crossings in the link diagram. The $*$ records which digit in the vertex's index changes: for example, $0010 \xrightarrow{d_{0*10}} 0110$ since the second index in the two smoothings changes. We then sprinkle minus signs on the differentials to make the cube faces anticommute according to $(-1)^s$ where s is the number of 1's before the star $*$. For the maps d_α, we want them to have degree zero so that the graded Euler characteristic returns the Jones polynomial. Since each column has a degree shift according to the number of 1 smoothings at the vertex, and this will increase by $+1$ from column to column, we must have the differentials to have degree -1. From vertex to vertex, either two circles will merge into one, or one circle will split into two circles. Since $\deg(v \otimes w) = \deg(v) + \deg(w)$, Khovanov defines the differentials to be:

$$\begin{aligned} (\bigcirc\ \bigcirc \to \bigcirc) = m : V \otimes V &\to V \\ v_+ \otimes v_+ &\mapsto v_+ \\ v_+ \otimes v_- &\mapsto v_- \\ v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto 0 \\ (\bigcirc \to \bigcirc\ \bigcirc) = \Delta : V &\to V \otimes V \\ v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- &\mapsto v_- \otimes v_-, \end{aligned}$$

and we set d_α to be the identity on the circles that don't participate. Since there is no order on the circles in the diagram at each vertex, this forces m and Δ to be commutative and cocommutative, respectively. So, in particular, this forces $m(v_+ \otimes v_-) = m(v_- \otimes v_+)$ and m and Δ to be defined in the above way up to scalars. By the indexing of vector spaces we can keep track of which circles are merged or split. So for the vector space $V_i \otimes V_j$, which has basis $\{v_+^i \otimes v_+^j, v_+^i \otimes v_-^j, v_-^i \otimes v_+^j, v_-^i \otimes v_-^j\}$, we define $m_{ij} : V_i \otimes V_j \to V_{\min\{i,j\}}$ and $\Delta_{ij} : V_{\min\{i,j\}} \to V_i \otimes V_j$. Thus, for example, in Figure 1.1, $d_{1*1} = -\mathrm{id}_{V_1} \otimes \Delta_{23}$, so

$$\begin{aligned} -d_{1*1} : V_1 \otimes V_2 &\to V_1 \otimes V_2 \otimes V_3 \\ v_+^1 \otimes v_+^2 &\mapsto v_+^1 \otimes v_+^2 \otimes v_-^3 + v_+^1 \otimes v_-^2 \otimes v_+^3 \\ v_+^1 \otimes v_-^2 &\mapsto v_+^1 \otimes v_-^2 \otimes v_-^3 \\ v_-^1 \otimes v_+^2 &\mapsto v_-^1 \otimes v_+^2 \otimes v_-^3 + v_-^1 \otimes v_-^2 \otimes v_+^3 \\ v_-^1 \otimes v_-^2 &\mapsto v_-^1 \otimes v_-^2 \otimes v_-^3 \end{aligned}$$

An edge map will be of the following form: ordering the basis of $V_1 \otimes V_2$ to be $(v_+^1 \otimes v_+^2, v_+^1 \otimes v_-^2, v_-^1 \otimes v_+^2, v_-^1 \otimes v_-^2)$ and the basis of $V_1 \otimes V_2 \otimes V_3$ to be (omitting $\otimes$)

$$(v_+^1 v_+^2 v_+^3, v_+^1 v_+^2 v_-^3, v_+^1 v_-^2 v_+^3, v_-^1 v_+^2 v_+^3, v_+^1 v_-^2 v_-^3, v_-^1 v_+^2 v_-^3, v_-^1 v_-^2 v_+^3, v_-^1 v_-^2 v_-^3),$$

we get the differential

$$d_{1*1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The differentials of our complex $CKh(L)'_\bullet$ will be the sum of the edge maps

$$\begin{aligned} d_0 &= d_{*00} + d_{0*0} + d_{00*} \\ d_1 &= d_{1*0} + d_{10*} + d_{*10} \\ d_2 &= d_{*01} + d_{01*} + d_{0*1} \\ d_3 &= d_{11*} + d_{1*1} + d_{*11} \end{aligned}$$

We now have a complex $CKh'(L)$ with rth space $CKh(L)'_r$. This is a complex because $d^2 = 0$ because the faces of the cube anticommuted.

Definition 1.10. We define the Khovanov complex to be $CKh(L) = CKh'(L)[-n_-]\{n_+ - 2n_-\}$.

In the case of the trefoil above, the Khovanov complex is thus

$$0 \to W_0 \xrightarrow{d_0} W_1 \xrightarrow{d_1} W_2 \xrightarrow{d_2} W_3 \to 0.$$

(Since $n_- = 0$, W_0 is in homological height 0). Indeed, with these degree and height shifts, the graded Euler characteristic of this complex is, by construction, the unnormalised Jones polynomial: $\chi_q(Kh(L)) = \hat{J}(L)$.

Knowing the differentials, we can take homology. Sparing you the details, and letting our vector spaces be over the field $\mathbb{Q}$,

$$\begin{aligned} H_0 &= \mathbb{Q}\{v_- \otimes v_-, v_+ \otimes v_- - v_- \otimes v_+\} \\ H_1 &= 0 \\ H_2 &= \mathbb{Q}\left\{\begin{pmatrix} v_- \otimes v_+ \\ v_- \otimes v_+ \\ v_- \otimes v_+ \end{pmatrix}\right\} \\ H_3 &= \mathbb{Q}\{v_+ \otimes v_+ \otimes v_+\} \end{aligned}$$

Thus, with grading shifts,

$$\begin{aligned} \mathrm{qdim}(H_0) &= q + q^3 \\ \mathrm{qdim}(H_1) &= 0 \\ \mathrm{qdim}(H_2) &= q^5 \\ \mathrm{qdim}(H_3) &= q^9 \end{aligned}$$

So the Poincaré polynomial is

$$K(q,t) = \sum_k t^k \mathrm{qdim}(H_k) = q + q^3 + t^2 q^5 + t^3 q^9.$$

Setting $t = -1$ returns the graded Euler characteristic, which is the unnormalised Jones polynomial:

$$K(q, -1) = q + q^3 + q^5 = q^9 = (q + q^{-1})(q^2 + q^6 - q^8).$$

It turns out that this construction gives us a knot invariant $Kh(K)$ of a knot K that is stronger than the Jones polynomial. For example, $\hat{J}(\bar{5}_1) = \hat{J}(10_{132})$, but $Kh(\bar{5}_1) \neq Kh(10_{132})$. To show that it is a knot invariant, we should check that for two links L and $\tilde{L}$ that differ by a Reidemeister move, that $Kh(K) \cong Kh(\tilde{L})$ (the map $CKh(L) \to CKh(\tilde{L})$ is a quasi-isomorphism of complexes). The interested reader can find this [Kho00] and [BN02].

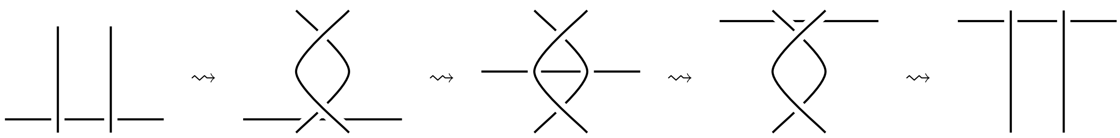
1.2.3 A note on Functoriality

In addition to the quasi-isomorphisms $CKh(L) \to CKh(\tilde{L})$ between two link diagrams that differ by a Reidemeister move, we also have chain maps for 'Morse moves'

$$\emptyset \to \bigcirc, \bigcirc \to \emptyset, \ )\,(\to \asymp,$$

called birth, death and fusion moves, respectively.

Given a link cobordism Σ, we can decompose it as a union of elementary cobordisms $\Sigma_1 \cup \cdots \cup \Sigma_k$ given by the Morse and Reidemeister moves, and we would like to associate chain maps to each of these moves so $CKh(\Sigma) : CKh(L) \to CKh(\tilde{L})$ is the composition $CKh(\Sigma) = CKh(\Sigma_1) \circ \cdots \circ CKh(\Sigma_k)$. Unfortunately, different ways of decomposing a cobordism gives different answers. For example,

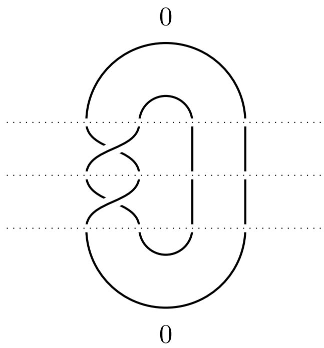


is just the identity, but however we pick our homotopy equivalences for Reidemeister moves, this composition is assigned $-\mathrm{id}$. In fact, this is the best we can do: the description of Khovanov homology in this chapter is only functorial up to a minus sign. This was independently fixed by Clark, Walker and Morrison in [CMW09], Caprau in [Cap08] and Blanchet in [Bla10].

Thus, to summarise, Khovanov homology is a categorical link invariant. It sends objects in $C_\bullet = \mathsf{Kom}(\mathsf{GVec}_{\Bbbk})$ and isotopies Σ between links L and L' to isomorphisms in that category. The Grothendieck group $K_0(C_\bullet)$ is $\mathbb{Z}[q, q^{-1}]$, and the image there is exactly the (unnormalised) Jones polynomial.

$$\begin{tikzcd}
L \arrow[r, "Kh"] \arrow[d, "\Sigma"] & Kh(L) \arrow[r, "K_0"] \arrow[d, "Kh(\Sigma)"] & {[Kh(L)] = \hat{J}(L)} \\
L' \arrow[r, "Kh"] & Kh(L') \arrow[ur] &
\end{tikzcd}$$

In fact, a braided tensor category $\mathcal{C}$ (or just a tensor category with a functor from tangles) will always give a link invariant, since we can think of the $L \in \mathrm{Hom}_{\mathcal{C}}(0 \to 0)$. For example in string diagrams $L \in \mathrm{Hom}_{\mathcal{C}}(0 \to 0)$ looks like



This motivates the following chapter's generalisation of Khovanov homology to tangles: if we start with a tangle, then its cube of resolutions will have at each vertex a complete smoothing that will be an element of $\mathrm{Hom}_{\mathsf{TL}}$, where TL is the Temperley-Lieb category, which is a braided tensor category defined by

- objects are the natural numbers $\mathbb{N}$
- $\mathrm{Hom}_{\mathsf{TL}}(n, m)$ is the free $\mathbb{Z}[q, q^{-1}]$-module of planar diagrams with n 'input' strands and m 'output' strands with no crossings
- composition is by stacking diagrams with common endpoints, replacing loops with $q + q^{-1}$
- tensor product is by juxtaposing diagram horizontally.

In fact, the Hom space of TL can be viewed as a category (so TL is a tensor 2-category) and we can look at the morphisms between the Hom spaces of TL. This will be the content of Chapter 2.

It turns out that Khovanov homology is associated to the braided tensor category $\mathsf{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ (which is equivalent as a category to the idempotent completion of TL) and there are more general Khovanov homology theories that are related to other Lie algebras, such as $\mathfrak{sl}_n(\mathbb{C})$. An introduction can be found in [Lau12].

1.3 Topological Quantum Field Theories

Before progressing onto the next chapter, the reader who is familiar with TQFTs (topological quantum field theories) may have noticed that we have applied a TQFT to the cube of resolutions. For those who are not familiar we will briefly describe this more general theory. This section is intentionally brief, since the the term TQFT is used throughout this paper but only its basic definition is used. For further introductory explanation of TQFTs, there are plenty of references such as [Koc04]. TQFTs have a functorial axiomatisation due to Atiyah on which we will base our discussion.

Definition 1.11. An n-dimensional TQFT is a symmetric monoidal functor from $(\mathsf{nCob}, \sqcup, \emptyset, \tau)$ to $(\mathsf{Vect}_{\mathbb{k}}, \otimes, \mathbb{k}, \sigma)$. This is also called a linear representation of nCob.

Here, nCob is the category whose objects are closed oriented $(n-1)$-manifolds and a morphism between two $(n-1)$-manifolds Σ and Σ' is an oriented n-dimensional manifold M whose 'in-boundary' is Σ and whose 'out-boundary' is Σ'. This manifold (a cobordism) M is defined up to diffeomorphism relative its boundary. It does not need to be embedded in any ambient space, and can be considered as an abstract manifold.

Composition of morphisms is by gluing common boundaries (this is just topological gluing and the smooth structure is well-defined up to diffeomorphism). The identity cobordisms are cylinders. The symmetric monoidal structure of nCob is as follows: the tensor product of two manifolds Σ and Σ' is their disjoint union $\Sigma \bigsqcup \Sigma'$ and the tensor unit zero is the empty manifold. The symmetry map $\tau_{\Sigma,\Sigma'} : \Sigma \bigsqcup \Sigma' \to \Sigma' \bigsqcup \Sigma$ that switches the two factors is a diffeomorphism that provides the symmetry structure. This extends to the disjoint union of cobordisms: given the cobordisms $M_0 : \Sigma_0 \to \Sigma_0'$ and $M_1 : \Sigma_1 \to \Sigma_1'$ we can form the disjoint union $M_0 \sqcup M_1 : \Sigma_0 \sqcup \Sigma_1 \to \Sigma_0' \sqcup \Sigma_1'$.

In the case $n = 2$, $\mathsf{2Cob}$ is generated by disjoint sets of circles and the morphisms are obtained by composing 2 and tensoring the following basic building blocks:

There is an equivalence of categories between 2-dimensional TQFTs and commutative Frobenius algebras given by sending the TQFT to its value on the circle. A Frobenius algebra is an algebra which is simultaneously a coalgebra with certain compatibility condition between multiplication and comultiplication:

Definition 1.12. A Frobenius algebra A is a vector space with multiplication map $\mu : A \otimes A \to A$, unit $\eta : \mathbb{k} \to A$, a comultiplication map $\Delta : A \to A \otimes A$, and counit $\epsilon : A \to \mathbb{k}$, such that the Frobenius relation holds:

$$(1 \otimes \mu) \circ (\Delta \otimes 1) = \Delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \Delta).$$

This is all we will need for the following chapters.

Chapter 2

Bar-Natan's Khovanov Homology

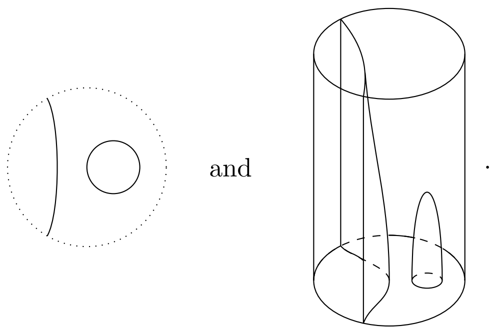
2.1 The Categorical Setting

Our goal is to apply different TQFTs to the n-dimensional cube of resolutions we've just described. From this, we can obtain link invariants from any knot (or tangle) diagram by taking homology. Already, we have seen one such TQFT, the Khovanov TQFT, which for a link L gives us a chain complex $CKh(L)$. The following theory was developed by Bar-Natan [BN05] in order to extend Khovanov homology to tangles, giving us a 'local' Khovanov homology. Furthermore, we wish to develop our understanding of the category where the cube of resolutions is an object, so we can simplify the cube (its objects and morphisms) *before* we apply the TQFT, via Bar-Natan's dotted theory. This will ultimately lead to savings in computational complexity.

As with Khovanov's original homology theory, we want to think of such a cube obtained from a tangle T as a chain complex $BN(T)$ in a certain graded category where we have direct sums. Each vertex of the cube of resolutions for a diagram has a complete smoothing. Thus, this category should have smoothings as objects and cobordisms for maps between these objects.

Definition 2.1. The category $\mathsf{Cob}^3(\emptyset)$ has the following data: it has smoothings (simple closed curves in the plane) as objects, and the morphisms are cobordisms between such smoothings regarded up to boundary preserving isotopies. We could also take a finite set of points B on a circle (such as the boundary ∂T of a tangle T) and get the category $\mathsf{Cob}^3(\mathsf{B})$ which has as objects smoothings with boundary B, and the morphisms again are cobordisms between such objects up to boundary preserving isotopy. In both cases, composition of morphisms is given by stacking cobordisms and then a rescaling.

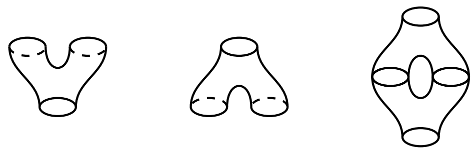
For example if $|B| = 2$, a smoothing and a boundary preserving cobordism will look like



Remarks.

- There is a minor difference here with the category 2Cob defined earlier, where we now allow objects to be not necessarily closed (i.e. compact without boundary).
- The 3 in Cob^3 refers to the fact that this category can be interpreted as a 3-category: there are 1-morphisms in $\mathbb{N}$ given by addition, 2-morphisms between pairs of numbers given by TL-diagrams, and then 3-morphisms that are surfaces between TL diagrams modulo isotopy.

We will stack morphisms from top to bottom, as below, which has g, f and $f \circ g$ from left to right:



You can think of the 'input' going into the bottom and the 'output' coming out of the top.

Going back to our cube of resolutions, and having defined the objects and morphisms of our category, we also want the rth chain space $[\![T]\!]^{r-n_-}$ of the complex $[\![T]\!]$ to be the 'direct sum' of the spaces (smoothings) at height r in the cube, and sum the maps (cobordisms) to get a differential for $[\![T]\!]$. To introduce direct sums into our category, we first make our category 'pre-additive'.

Definition 2.2. A *pre-additive category* $\mathcal{C}$ is a category enriched in the monoidal category of abelian groups. So the morphisms between two objects form an abelian group, and the composition maps are bilinear in the sense that composition distributes over the group operation: $f \circ (g+h) = f \circ g + f \circ h$. In particular, for any $A, B \in \mathrm{ob}(\mathcal{C})$, there is a zero map $0 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

If the category $\mathcal{C}$ is not pre-additive, we can consider another category $\tilde{\mathcal{C}}$ with the same objects and morphisms, but $\tilde{\mathcal{C}}$ is pre-additive by taking $\mathbb{Z}$-linear combinations of elements in $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for any two objects A and B, and by allowing composition to distribute $\mathbb{Z}$-linearly.

Example 2.3. The category Ab of abelian groups is pre-additive. Without the commutativity, Grp is not pre-additive since sums of group homomorphisms may not again be a group homomorphism, that is, it is not the case that we have the following for all $f, g \in \mathrm{Hom}_{\mathsf{Grp}}(A, B)$, and all $a, b \in A$, because we need commutativity for the third equality:

$$\begin{aligned}(f+g)(a+b) &= f(a+b) + g(a+b) \\ &= f(a) + f(b) + g(a) + g(b) \\ &= f(a) + g(a) + f(b) + g(b) \\ &= f(a+b) + g(a+b).\end{aligned}$$

Continuing in this vein,

Definition 2.4. An *additive category* is a pre-additive category with a zero object (a unique object that is both initial and terminal), and hence zero morphisms, and an additive category also has biproducts, denoted $\oplus$.

Lastly, we describe *abelian* categories.

Definition 2.5. *Abelian categories* are additive categories where each morphism has a kernel and cokernel, and every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

This is the most general category in which we can work with homology, or rather, it is the most general setting where the Snake lemma holds. Examples include the categories $\mathsf{Ab}, \mathsf{R\text{-}Mod}$ and $\mathsf{Vect}_{\mathbb{k}}$.

Definition 2.6. If $\mathcal{C}$ is a pre-additive category, then we can form a new additive category, called the *additive closure* of $\mathcal{C}$, $\mathsf{Mat}(\mathcal{C})$, via the following:

- The objects of $\mathsf{Mat}(\mathcal{C})$ are formal direct sums $\oplus_{i=1}^{n} A_i$ of $\mathcal{C}$.
- For two objects $A = \oplus_{i=1}^{m} A_i$ and $A' = \oplus_{j=1}^{n} A'_j$, a morphism between them $F : A \to A'$ is an $n \times m$ matrix $F = (F_{ij})$ where each $F_{ij} : A_i \to A'_j$ is a morphism in $\mathcal{C}$.
- Morphisms in $\mathsf{Mat}(\mathcal{C})$ are added via matrix addition.
- Composition of morphisms is by matrix multiplication; i.e. $(F \circ G)_{ik} := \sum_j F_{ij} \circ G_{jk}$.

The category $\mathsf{Mat}(\mathcal{C})$ is additive, and if $\mathcal{C}$ was additive with biproduct $\oplus'$, then no harm was done: $X \oplus Y \cong X \oplus' Y$.

At this point, our cube of resolutions can be interpreted as a tower of morphisms between formal direct sums of smoothings. For example, for an n-crossing tangle T, we can interpret it as a length n tower of morphisms between formal direct sums of smoothings

$$[\![T]\!] = \left([\![T]\!]^{-n_-} \to [\![T]\!]^{-n_-+1} \to \cdots \to [\![T]\!]^{n_+}\right).$$

In fact, these towers will be complexes, for which we provide the following definition.

Definition 2.7. Let $\mathcal{C}$ be a pre-additive category. Then $\mathsf{Kom}(\mathcal{C})$, 'the category of complexes over $\mathcal{C}$', is the category with the following information:

- objects are complexes of finite length $\ldots \to C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \to \ldots$ where $d_n d_{n-1} = 0$ for all n.
- morphisms between such complexes $F : C \to C'$ are the set of maps $\{F_n\}$ that give commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "F_{n-1}"] & C_n \arrow[r, "d_n"] \arrow[d, "F_n"] & C_{n+1} \arrow[r] \arrow[d, "F_{n+1}"] & \cdots \\
\cdots \arrow[r] & C'_{n-1} \arrow[r, "d'_{n-1}"] & C'_n \arrow[r, "d'_n"] & C'_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

  where all arrows are morphisms in $\mathcal{C}$. This is just the formal analogue of maps of chain complexes in homological algebra. Composition $F \circ G$ is defined as in ordinary homological algebra, $(F \circ G)_n = F_n \circ G_n$.

We can consider these formal complexes up to homotopy, which works the same as the homological algebra case: let $\mathcal{C}$ be a category, and $\mathsf{Kom}(\mathcal{C})$ its category of complexes. Then two morphisms $f, g : C_\bullet \rightrightarrows D_\bullet$ between complexes $(C_\bullet, d^C_\bullet)$ and $(D_\bullet, d^D_\bullet)$ are *homotopic* if there exist diagonal maps maps $h_n : C_n \to D_{n+1}$ so that $f_n - g_n = d^D_{n+1} h_n + h_{n-1} d^C_n$, i.e. their difference is null homotopic).

Given the previous definitions, and by firstly making our category pre-additive, we aim to consider $[\![T]\!]$ as an object in $\mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3))$. For this we will have to check that $[\![T]\!]$ is indeed a chain complex (we need to check that the differentials square to be zero), which we will do below in Proposition 2.1.

Now we are in a position to define the Khovanov complex $[\![T]\!]$ of an oriented tangle diagram T, this is going to be very reminiscent of Khovanov's original construction, but now we think of the cube of resolutions as an object in $\mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3))$.

Definition 2.8. Consider the diagram of a tangle T with $n = n_+ + n_-$ crossings. As we did in Khovanov's original construction, label each of the crossings from 1 to n. We will again construct the cube of resolutions, where at each vertex there is complete smoothing of the original tangle, described completely by a length n word of 0s and 1s. Two vertices are connected by an edge if and only the words at the vertices differ by one letter. We orient these edges by having the arrow directed towards the larger word (when the word is viewed as an integer). We then label the arrows by an n letter word in $\{0, 1, *\}$ where the $*$ is in the position where the words at the top and tail of the arrow differ:

$$0010 \xrightarrow{0{*}10} 0110.$$

Lastly, to make each face of the cube anticommute, we sprinkle minus signs according to multiplying each differential with label $\xi_1\xi_2\dots\xi_n$ by $(-1)^{\sum_{i<j}\xi_j}$ where $\xi_j = *$ (this is just to the power of k, where k is the number of 1s before the $*$). Recall that in $\mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3))$ each differential is a matrix of cobordisms. The cobordism corresponding to an arrow is, in the cyclinder between the two smoothings that differ, the obvious saddle. The kth column in the cube of resolutions (which is a direct sum of objects in $\mathsf{Mat}(\mathsf{Cob}^3)$) has homological height $k - n_-$.

Proposition 2.1. *For any tangle diagram T, the chain $[\![T]\!]$ is a complex in $\mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3(\partial T)))$. That is, $d^2 = 0$ for these chains.*

Proof. Take any square in our cube of resolutions. We have to show that this square anticommutes. Firstly, as stated above, the cube of resolutions has minus signs sprinkled according to $(-1)^{\sum_{i<j}\xi_j}$. Thus in each square, the edges will have an odd number of minus signs. So we just need to show that morphisms (without minus signs) commute. Each vertex is labelled by a string of 0's and 1's, so as we traverse around the square, then, for example, the i and j index will switch to $i+1 \mod 2$ and $j+1 \mod 2$, respectively. To make it clear, the indices change $+1 \mod 2$ as follows

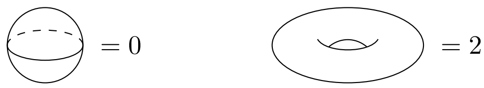
$$\begin{tikzcd}
 & \{\dots i+1\dots j\dots\} \arrow[dr, "d_j"] & \\
\{\dots i\dots j\dots\} \arrow[ur, "d_i"] \arrow[dr, "d_j"'] & & \{\dots i+1\dots j+1\dots\} \\
 & \{\dots i\dots j+1\dots\} \arrow[ur, "d_i"'] &
\end{tikzcd}$$

The maps $i \to i+1$ and $j \to j+1$ are disjoint saddles, so the compositions d_jd_i, d_id_j are equal since they are isotopic relative boundary (they are just a time reordering of saddles). $\square$

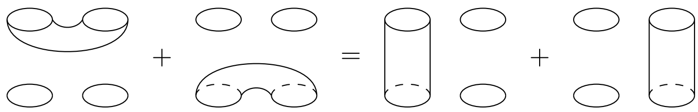
Remark 2.9. It is always possible to sprinkle minus signs on a cube so that each of the faces anticommute (so each face has an odd number of minus signs), and the space of choices in contractible. To see this, suppose you have a solid cube $X = [0,1]^n$. We want a map $\sigma : \{\text{edges}\} \to \{\pm 1\}$, i.e. an element σ in the 1-cocyles of X with coefficients in $\{\pm 1\} = \mathbb{Z}/2$, so $\sigma \in C^1(X; \mathbb{Z}/2)$, such that for all faces A, $\Pi_{\text{edge } e \text{ in } \partial A}\sigma(e) = -1$. We also want σ such that, $\delta^1\sigma = M$ in $C^2(X; \mathbb{Z}/2)$, where $M(A) = -1$ for all faces A of X. Now, $\delta^2 M \in C^3(X; \mathbb{Z}/2)$ and $(\delta^2 M)(X) = (-1)^6 = 1$ (and 1 is our 'zero' in $\mathbb{Z}/2$). So M is a 2-cocycle. But $H^2(X; \mathbb{Z}/2) = 0$ because X is contractible, so $\ker\delta = \operatorname{im}\delta$, so there exists some σ such that $\delta^1\sigma = M$ as desired. Moreover, suppose we had two sprinkling of signs σ and σ'. Then $\delta^1(\sigma\cdot\sigma') = 0$ which implies $\sigma\cdot\sigma'$ is a 1-cocycle. But $H^1(X; \mathbb{Z}/2) = 0$, so there exists a $\tau \in C^0(X; \mathbb{Z}/2)$ (so $\tau : \{\text{vertices}\} \to \pm 1$) such that $\delta^0\tau = \sigma\cdot\sigma'$, and $\tau(v)\cdot\tau(v') = \sigma(e)\sigma'(e)$ where e is an edge between vertices v and v'. Therefore $\sigma(e) = \sigma'(e)$ if and only if $\tau(v) = \tau(v')$. Define $\phi : [\![T]\!] \to [\![T]\!]'$ to be the map of complexes with different sprinkling of signs such that all faces anticommute, where ϕ is induced by $\tau(v)\cdot\mathrm{id}$. Then ϕ is an isomorphism of chain complexes.

Given this category, we can consider the morphisms modulo some local relations, written $\mathsf{Cob}^3_{/l}$

so that $[\![T]\!]$ in $\mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3_{/l}))$ is invariant up to homotopy of formal complexes. These local relations are described below, denoted S, T and $4Tu$ respectively:



S T



4Tu

That is, when a cobordism contains a component isotopic to a 2-sphere, that cobordism is equal to zero. When a cobordism contains a component isotopic to a torus, it is equal to the same cobordism without the torus, but multiplied by 2. And the last relation says that you can change the cobordisms locally in the specified way.

We will write $\mathsf{Kom}_{/h}(\mathcal{C})$ for $\mathsf{Kom}(\mathcal{C})$ modulo homotopies. Moreover, we will use the shorthands $\mathsf{Kob}(B) := \mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3_{/l}(B)))$ and $\mathsf{Kob}_{/h}(B) := \mathsf{Kom}_{/h}(\mathsf{Mat}(\mathsf{Cob}^3_{/l}(B)))$.

Theorem 2.10. *The isomorphism class of $[\![T]\!]$ in $\mathsf{Kob}_{/h}$ is an invariant of the tangle T. This means that $[\![T]\!]$ does not depend on the ordering of the column vectors obtained from the cube of resolutions, and that it does not depend on the ordering of the crossings, and that it is invariant under the Reidemeister moves.*

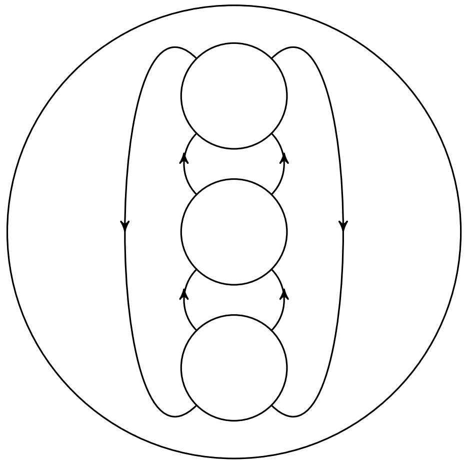
A proof of Theorem 2.10 can be found in [BN05].

This functor has excellent composition features by which we mean it is a *planar algebra*.

Definition 2.11. A d-input planar arc diagram D is an 'output' disk with the following data:

- d smaller 'input' disks removed. The input disks are labelled 1 to d, each with a basepoint marked (this basepoint is often omitted from pictures).
- a collection of disjoint embedded arcs with endpoints intersecting the boundary of either the input or output disks transversely.

Planar diagrams can be either oriented or unoriented. Below is an example of an oriented planar diagram, where if we insert three positive crossings into the input disks, we get the right-handed trefoil.



Definition 2.12. A collection of sets $\mathcal{P}(k)$ along with operations defined for each (oriented) unoriented planar arc diagram is an *(oriented) unoriented planar algebra* if

- the radial planar arc diagrams act as identities,
- associativity condition: if D_i is the result of plugging D' into the ith hole of D (provided this is possible), then as operations $D_i = D \circ (I \times \cdots \times D' \times \cdots \times I)$

Note. In the categorical context, the axiom where the radial arc diagrams act as identities needs to be weakened, but can be safely ignored in this paper. For example, see [ETW17].

We usually consider the sets as topological objects, as in the above diagram, where we can place positive crossings into the operator that is the output disk. Let's see how planar algebras come into play in this context. Firstly, consider the collection of all unoriented tangle diagrams in a based disk with k endpoints on the boundary, considered up to boundary preserving isotopies, $T^0(k)$. Let $T(k)$ be the quotient of $T^0(k)$ by the three Reidemeister moves. Then both $T^0(k)$ and $T(k)$ are planar algebras: let D be an output disk with k_i arcs intersecting the ith input disk and k arcs intersecting the boundary of the output disk. Then we define the operator

$$D : T^0(k_1) \times T^0(k_2) \times \cdots \times T^0(k_d) \to T^0(k)$$

by placing the d input tangles into the holes of D. The radial arc operators clearly act as identities, and the associativity condition also holds. We can define the planar algebra $T(k)$ in the same way.

More importantly for us, the collection $\mathrm{ob}(\mathsf{Cob}^3_{/l})$ and $\mathrm{Hom}(\mathsf{Cob}^3_{/l})$ are both planar algebras. The objects form a sub-planar algebra of $T(k)$. For the morphisms, consider a cylinder over the output disk D, $D \times [0,1]$. This will have d cylindrical holes in it, connected by vertical curtains joining the top and bottom arcs in $D \times \{0\}$ and $D \times \{1\}$. We can simply place cobordisms in the holes, defining an operation $D : (\mathrm{Hom}(\mathsf{Cob}^3_{/l}))^d \to \mathrm{Hom}(\mathsf{Cob}^3_{/l})$.

Now suppose we wish to place tangle *complexes* in the holes of a planar arc diagram, that is, put elements of Kob into planar arc diagrams. This will work like the tensor product of chain complexes, which we will define below for the unfamiliar reader.

Definition 2.13. Given two chain complexes $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$, we can form the tensor product double complex, where the (p,q)th position of the lattice is $(C \otimes D)_{p,q} = C_p \otimes D_q$. The vertical maps are $d_v = (-1)^p \otimes d^D$ and the horizontal maps are $d_h = d^C \otimes 1$. From the tensor product double complex, we can form the tensor product total (chain) complex, where the nth space is $\mathrm{Tot}(C \otimes D)_n = \bigoplus_{p+q=n} (C \otimes D)_{p,q}$. The differential on such a complex is simply $d = d_v + d_h$.

In Example 3.8 we show as an example how spectral sequences can be used to compute the homology of $\mathrm{Tot}(C \otimes D)$ where C and D are complexes of vector spaces.

For $\mathsf{Kob}(B_k)$ and $\mathsf{Kob}_{/h}(B_k)$ (where B_k is a fixed placement of k points on a based circle), we define the operator D, where D is a d input planar arc diagram with k_i arc endpoints on the ith holes and k arc endpoints on the boundary of the disk, as follows: for a collection of complexes $(C^i, d^i) \in \mathsf{Kob}(k_i)$, $D(C^1, \ldots, C^d)$ is the complex (D, d) with components

$$D_j := \bigoplus_{j_1+\cdots+j_d=j} D(C^1_{j_1}, \ldots, C^d_{j_d})$$

and differentials

$$d\big|_{D(C^1_{j_1},\ldots,C^d_{j_d})} = \sum_{i=1}^{d} (-1)^{\sum_{n<i} j_n} D(I_{C^1_{j_1}}, \ldots, d_i, \ldots, I_{C^d_{j_d}}).$$

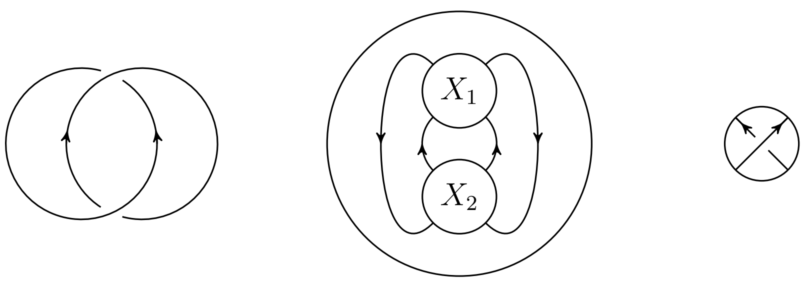
Properties of honest tensor products of chain complexes still hold in this case of tensor products of formal chain complexes. For example, morphisms $f : C^i \to C^{i'}$ induce morphisms $D(I, \ldots, f, \ldots I) : D(C^1, \ldots C^i, \ldots, C^d) \to D(C^1, \ldots C^{i'}, \ldots, C^d)$.

Proposition 2.2. *The collection* $\mathsf{Kob}(B_k)$ *is a planar algebra. Moreover, the operators* D *acting on* $\mathsf{Kob}(B_k)$ *send homotopy equivalent complexes to homotopy equivalent complexes, so* $\mathsf{Kob}_{/h}(B_k)$ *is also a planar algebra.*

Definition 2.14. A morphism Φ of planar algebras $(P(k)) \to (Q(k))$ is a collection of maps $\Phi : P(k) \to Q(k)$ such that for any operator D of $P(k)$, $\Phi \circ D = D \circ (\Phi \times \Phi \times \cdots \times \Phi)$.

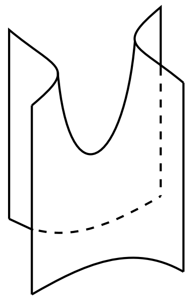
Definition 2.15. The Bar-Natan bracket $[\![-]\!]$ is an oriented planar algebra morphism $[\![-]\!] : (T(k)) \to \mathsf{Kob}_{/h}(B_k)$.

We are now due for an example which will hopefully make this all clear. Consider the oriented Hopf link, denoted 2^2_1, which can be viewed as the operator D with input tangles that are a pair of positive crossings:



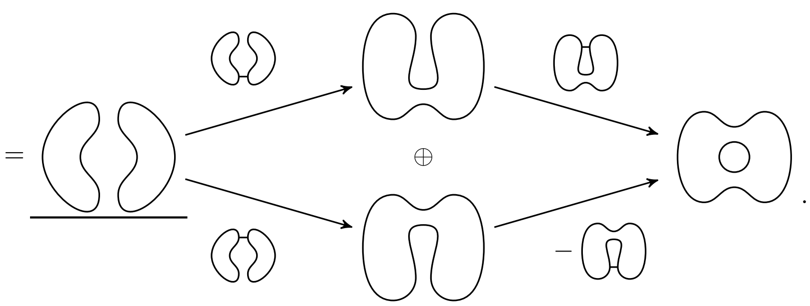
$2^2_1 = D(X_1, X_2)$ $\qquad$ D $\qquad$ $X_1 = X_2$

Thus, $[\![2^2_1]\!] = [\![D(X_1, X_2)]\!] = D([\![X_1]\!], [\![X_2]\!]) = tr([\![X_1]\!] \otimes [\![X_2]\!])$, where $[\![X_i]\!] = \underline{)\,(} \xrightarrow{s} \asymp$. Here, and thoughout this paper, the underline in the first space in the complex $[\![X_i]\!]$ denotes homological height zero, and s is the appropriate saddle



Computing this tensor product,

$$
\begin{tikzcd}
tr(\llbracket X_1 \rrbracket \otimes \llbracket X_2 \rrbracket) = & \underline{D()\,(,)\,()} \arrow[r, "D(1,s)"] \arrow[r, "D(s,1)"'] \arrow[d, dotted] & D(\asymp,)\,() \oplus D()\,(,\asymp) \arrow[r, "D(s,1)"] \arrow[r, "-D(1,s)"'] \arrow[d, dotted] & D(\asymp,\asymp) \arrow[d, dotted] \\
& \llbracket 2^2_1 \rrbracket^0 \arrow[r] & \llbracket 2^2_1 \rrbracket^1 \arrow[r] & \llbracket 2^2_1 \rrbracket^2
\end{tikzcd}
$$



Next we want to add some gradings to this category just we did in the original Khovanov theory. The rewards for this are great as it then allows us to read off the Jones polynomial as before.

Definition 2.16. A *graded category* is a pre-additive category $\mathcal{C}$ with the following properties:

- for any two objects $C_1, C_2 \in \text{ob}(\mathcal{C})$, the set of morphisms $\text{Hom}_{\mathcal{C}}(C_1, C_2)$ form a graded abelian group. Under composition, $\deg(f \circ g) = \deg(f) + \deg(g)$ for all f and g where this makes sense. Moreover, all identity maps have degree zero.
- there is an action by $\mathbb{Z}$ that 'shifts degrees': for an object $C \in \text{ob}(\mathcal{C})$, and $m \in \mathbb{Z}$, we can define the same object but with a degree shift $C\{m\}$. This does not change the morphisms between objects, i.e. $\text{Hom}_{\mathcal{C}}(C_1\{m_1\}, C_2\{m_2\}) = \text{Hom}_{\mathcal{C}}(C_1, C_2)$, but changes the degrees of the morphisms so that $\deg(f : C_1\{m_1\} \to C_2\{m_2\}) = \deg(f : C_1 \to C_2) - m_1 + m_2$.

Later, we will use the convention of recording the grading of an object A with grading m by $q^m A$ rather than $A\{m\}$, referred to as the q-grading.

So we can make Cob^3 and $\mathsf{Cob}^3_{/l}$ (and thus Kob and $\mathsf{Kob}_{/h}$) graded categories, where the grading is given by the following:

Definition 2.17. Let $C \in \mathrm{Hom}(\mathsf{Cob}^3(B))$ be a cobordism in a cylinder, with $|B|$ vertical boundary components on the side of the cylinder. Then we define the degree of the morphism to be $\deg(C) := \chi(C) - \frac{1}{2}|B|$.

This is also known as the relative Euler characteristic, which is computed by taking the Euler characteristic of the relative (simplicial) homology.

Example 2.18. The degree of a (upside down) pair of pants is -1 since it has Euler characteristic of -1 (it is homotopy equivalent to a twice punctured disk) and has no boundary components. The degree of a cap $\ominus$ or cup Θ is $+1$ since it is homotopy equivalent to a disk and also has no boundary components. The degree of a saddle s is -1 since it is homotopy equivalent to a disk ($\chi(s) = 1$) and has four vertical boundary components so $|B| = 4$.

For our next definition, we will use '$\bullet$' to denote the zero object in $\mathsf{Kob}_{/h}$.

Definition 2.19. Let T be a tangle diagram with n_+ positive crossings and n_- negative crossings. Define $BN(T)$ to be the complex with each space in the complex

$$BN^r(T) := [\![T]\!]^r\{r + n_+ - n_-\}$$

and whose differentials are the same as the differentials of $[\![T]\!]$:

$$\begin{array}{rc} [\![T]\!]: & \bullet \to [\![T]\!]^{-n_-} \to \cdots \to [\![T]\!]^{n_+} \to \bullet \\ BN(T): & \bullet \to [\![T]\!]^{-n_-}\{n_+ - 2n_-\} \to \cdots \to [\![T]\!]^{n_+}\{2n_+ - n_-\} \to \bullet \end{array}$$

So $BN(-)$ is just the complex $[\![T]\!]$ decorated with grading shifts.

Thus, using the above definitions, we have

$$\begin{aligned} BN(\;) &= \bullet \to q\ )(\ \xrightarrow{\ s\ } q^2 \asymp \to \bullet \\ BN(\;) &= \bullet \to q^{-2} \asymp \xrightarrow{\ s\ } q^{-1}\ )(\ \to \bullet \end{aligned}$$

where s is the saddle cobordism.

Theorem 2.20. *With the above gradings, $BN(-)$ has the following properties:*

- *All differentials for $BN(T)$ for any tangle T have degree zero.*
- *The bracket $BN(T)$ is a tangle invariant up to degree-zero homotopy equivalence. That is, if T_1 and T_2 differ by a sequence of Reidemeister moves, then there is a degree-zero homotopy equivalence $F : BN(T_1) \to BN(T_2)$.*
- *The $BN(-)$ induces a planar algebra morphism $(T(k)) \to \mathsf{Kob}(k)$, and all the planar algebra operations are of degree zero.*

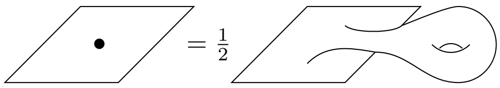
At this point, we have extended the original Khovanov homology of links to tangles. However, this category is clearly difficult to work with. To compare two tangles whose diagrams differ by a sequence of Reidemeister moves we need to construct an explicit homotopy equivalence between two complexes in Kob. To manage this, we will take a functor (a TQFT) from Kob into some abelian category where we can take homology. Through this process we lose information but will be able to detect homotopy inequivalence much more easily.

Note. Bar-Natan denotes the graded complex obtained from a tangle T via this method $Kh(T)$ (which we call $BN(T)$). We do this to emphasis that we are working in the topological category, and we reserve Kh for the homology of the complex CKh which is obtained by a certain TQFT from BN to vector spaces that is isomorphic to Khovanov's original homology theory.

2.1.1 Adding dots

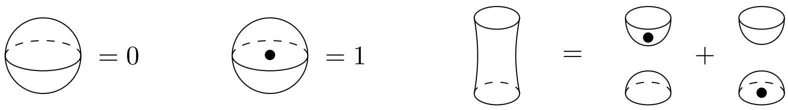
Next we will describe Bar-Natan's dotted theory, which is how we will think of Khovanov homology and its variants from here on. We change our category Kob to its dotted version $\mathsf{Kob}_\bullet = \mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}_{\bullet/l}))$, which will allow us to consider new homotopy equivalences.

This begins by extending the category Cob^3 to a category of dotted cobordisms $\mathsf{Cob}^3_\bullet$, where the objects are the same as Cob^3, but morphisms can carry dots '$\bullet$' on them of degree -2. If 2 is invertible, dots are really a shorthand for

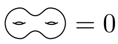


The specified degree really makes sense since $\bullet$ is a genus 1 hole in our surface, and a punctured torus has relative Euler characteristic -2.

Dotted cobordisms are isotopic if the underlying cobordisms are isotopic and contain the same number of dots per component, and we consider morphisms up to boundary preserving isotopy. We then consider $\mathsf{Cob}^3_\bullet$ up to local relations $\mathsf{Cob}^3_{\bullet/l}$ defined by

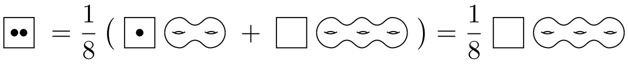


where the last relation is called the 'neck cutting' relation. Using these relations,



Notably, we also have the following computation,

Example 2.21.



So for example, if we additionally set the three holed torus to be zero, we have a sheet with two dots on it is also zero. As we will see below, this extra condition will return the original Khovanov homology.

2.2 Taking Homology of $BN(T)$

Up to now, we have extended Khovanov homology to the case of tangles by looking at a topological complex rather than an algebraic one. However, as a link/tangle invariant, it is difficult to work with, as it is difficult to construct explicit homotopy equivalences between complexes in Kob. Instead, we take a functor from Kob into an abelian category, where if we take homology, we can find whether two complexes are not homotopy equivalent.

Let $\mathcal{A}$ be an abelian category. A functor $\mathcal{F} : \mathsf{Cob}^3_{/l} \to \mathcal{A}$ extends to a functor $\mathcal{F} : \mathsf{Mat}(\mathsf{Cob}^3_{/l}) \to \mathcal{A}$ by sending the formal direct sums to honest direct sums in the abelian category. This again extends to a functor $\mathcal{F} : \mathsf{Kob} \to \mathsf{Kom}(\mathcal{A})$, since $\mathcal{F}(d) \circ \mathcal{F}(d) = \mathcal{F}(d \circ d) = 0$ because $\mathcal{F}$ is additive. Thus for a tangle T, $\mathcal{F}BN(T)$ is an ordinary complex in $\mathsf{Kom}(\mathcal{A})$. If we apply $\mathcal{F}$ to all (formal) homotopies in $\mathsf{Kob}(T)$, $\mathcal{F}BN(T)$ is an invariant of tangles up to homotopy, and thus $H_\bullet(\mathcal{F}BN(T))$ is an invariant of T. If $\mathcal{A}$ is graded, and $\mathcal{F}$ is degree respecting, then $H_\bullet(\mathcal{F}BN(T))$ is a graded invariant of T.

2.2.1 Choices of functor $\mathcal{F}$

We will outline two choices for the monoidal functor on $\mathsf{Cob}^3(\emptyset)$ valued in the abelian category of graded $\mathbb{Z}$-modules $\mathbb{Z}\mathsf{Mod}$ (we will also later work with $\mathbb{Q}[\alpha]$-modules) that maps disjoint unions to tensor products. For this we define the functor on the generators of $\mathsf{Cob}^3(\emptyset)$. The objects of $\mathsf{Cob}^3(\emptyset)$ are generated by a single circle $\bigcirc$ (i.e. they are disjoint unions of circles) and the morphisms are generated by the cup , cap , pair of pants and upside down pair of pants .

Definition 2.22. Let V be the graded $\mathbb{Z}$-module freely generated by the two generators $\{v_\pm\}$ with grading $\deg(v_\pm) = \pm 1$. Let $\mathcal{F}$ be the TQFT defined via $\mathcal{F}(\bigcirc) = V$, $\mathcal{F}() = \epsilon : V \to \mathbb{Z}$ $(v_- \mapsto 1, v_+ \mapsto 0)$, $\mathcal{F}() = \eta : \mathbb{Z} \to V$ $(1 \mapsto v_+)$ and most importantly, we have 'multiplication'

$$\begin{aligned} \mathcal{F}() = m : V \otimes V &\to V \\ v_+ \otimes v_+ &\mapsto v_+ \\ v_+ \otimes v_- &\mapsto v_- \\ v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto 0. \end{aligned}$$

and 'comultiplication'

$$\begin{aligned} \mathcal{F}() = \Delta : V &\to V \otimes V \\ v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- &\mapsto v_- \otimes v_-. \end{aligned}$$

It can be checked that this indeed gives us a Frobenius algebra structure.

Proposition 2.3. *The functor $\mathcal{F}$ is well-defined and the degree of the morphisms is preserved. Moreover $\mathcal{F}$ induces a functor $\mathsf{Cob}^3_{/l}(\emptyset) \to \mathbb{Z}\mathsf{Mod}$ (so the local relations are preserved).*

Proof. To see that $\mathcal{F}$ is well-defined, we have to see that it respects the relations between the generators of Cob^3, or in other words the image of the functor has the structure of a Frobenius algebra. To see that $\mathcal{F}$ is preserves the degrees of the morphisms, recall that a pair of pants (and an upside down pair of pants) has degree -1. This indeed is the case for m and Δ, which lower the degrees by 1. The degrees $\deg(\mathcal{F}()) = 1 = \deg(\mathcal{F}())$ also checks out. To see that $\mathcal{F}$ also induces a functor on the category modulo local relations, we note those relations:

$$S \,:\, 1 \mapsto v_+ \mapsto 0,$$

and

$$T \,:\, 1 \mapsto v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+ \longmapsto 2v_- \mapsto 2.$$

The relation 4Tu also holds:

$$(\mathcal{F}())(1) + (\mathcal{F}())(1) = (\mathcal{F}())(1) + (\mathcal{F}())(1)$$

because

$(\mathcal{F}($$))(1) = \Delta\epsilon 1 \otimes \epsilon 1 \otimes \epsilon 1$

$= v_- \otimes v_+ \otimes v_+ \otimes v_+ + v_+ \otimes v_- \otimes v_+ \otimes v_+$

$:= v_{-+++} + v_{+-++}$

$(\mathcal{F}($$))(1) = v_{++-+} + v_{+++-}$

$(\mathcal{F}($$))(1) = v_{+++-} + v_{+-++}$

$(\mathcal{F}($$))(1) = v_{++-+} + v_{-+++}.$

□

Remark 2.23. It can be seen that for a link L, $H_\bullet(\mathcal{F}(BN(L)))$ is equal to Khovanov's categorification of the Jones polynomial. That is, $CKh(L) = \mathcal{F}(BN(L))$.

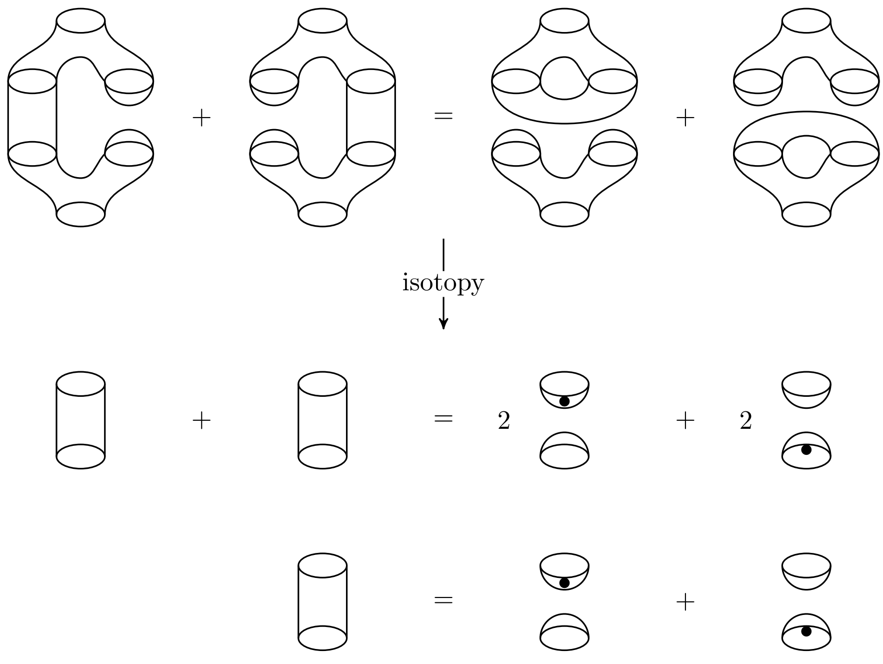
So now we have a functor (TQFT) that defines Khovanov's categorification of the Jones polynomial. Now we want to construct more functors, in particular the Lee TQFT. More generally, the Khovanov and Lee TQFTs are specialisations of the tautological functor:

Definition 2.24. Let B be a set of points on a circle S^1 and let $\mathcal{O}$ be an object in $\mathsf{Cob}^3_{/l}(B)$. So $\mathcal{O}$ is a smoothing with boundary B. The *tautological functor* $\mathcal{F}_{\mathcal{O}} : \mathsf{Cob}^3_{/l} \to \mathbb{Z}\mathsf{Mod}$ is defined on an object $\mathcal{O}'$ by $\mathcal{F}_{\mathcal{O}}(\mathcal{O}') = \mathrm{Hom}(\mathcal{O}, \mathcal{O}')$ and on a morphism $F : \mathcal{O}' \to \mathcal{O}''$ by $\mathcal{F}_{\mathcal{O}}(F) : \mathrm{Hom}(\mathcal{O}, \mathcal{O}') \to \mathrm{Hom}(\mathcal{O}, \mathcal{O}'')$ where $\mathcal{F}_{\mathcal{O}}(F)\left(\mathcal{F}_{\mathcal{O}}(\mathcal{O}')\right) = F\left(\mathcal{F}_{\mathcal{O}}(\mathcal{O}')\right)$.

Example 2.25. Take $B = \emptyset$ and $\mathcal{O} = \emptyset$. To make 2 invertible, tensor with $\mathbb{Z}[\frac{1}{2}]$, and quotient out by all surfaces with genus $g > 1$, so

$$\mathcal{F}_1(\mathcal{O}) := \mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \mathrm{Hom}(\emptyset, \mathcal{O})/((g > 1) = 0).$$

Taking two twice punctured disks (or equivalently, pairs of pants), and since 2 is invertible, the 4Tu relation becomes the neck cutting relation in Bar-Natan's dotted theory.



This again returns the original Khovanov homology theory:

Proposition 2.4. *The functors $\mathcal{F}_1(-) \cong \mathcal{F}(BN(-)) = CKh(-)$ are isomorphic.*

Proof. Firstly we will see how this works on objects:

$$\mathrm{Hom}(\emptyset, \bigcirc) \cong \mathbb{Z}\langle\ ,\ \rangle \cong \mathbb{Z}\langle v_-, v_+\rangle = V = CKh(\bigcirc),$$

where $\mapsto v_-$ and $\mapsto v_+$ in the second isomorphism. Next, consider $\mathrm{Hom}(\emptyset, \bigcirc\bigcirc) \cong \mathbb{Z}\langle\ ,\ ,\ ,\ \rangle$. Then the pair of pants takes $\mathrm{Hom}(\emptyset, \bigcirc\bigcirc) \to \mathrm{Hom}(\emptyset, \bigcirc)$, by

$$\mapsto$$
$$\mapsto$$
$$\mapsto$$
$$\mapsto 0$$

where the last map is because two dots are zero and neck cutting. With the above isomorphism on objects, we see this is exactly the Khovanov homology m multiplication. A similar argument shows the isomorphism on the comultiplication, unit, and counit maps. $\square$

Example 2.26. (The Lee TQFT) Take the same as before, but instead of modding out by all surfaces of genus greater than 1, quotient out by the relation that all genus 3 surfaces are equal to 8:

$$\mathcal{F}_2(\mathcal{O}) := \mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \mathrm{Hom}(\emptyset, \mathcal{O})/(\quad = 8).$$

Then as a $\mathbb{Z}[\frac{1}{2}]$-module, $\mathcal{F}_2(\mathcal{O}) = \mathcal{F}_1(\mathcal{O})$ on objects, but Δ and m are modified:

$$\begin{aligned} m_2 : V \otimes V &\to V \\ v_+ \otimes v_+ &\mapsto v_+ \\ v_+ \otimes v_- &\mapsto v_- \\ v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto v_+ \end{aligned}$$

and

$$\begin{aligned} \Delta_2 : V &\to V \otimes V \\ v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- &\mapsto v_- \otimes v_- + v_+ \otimes v_+. \end{aligned}$$

To see how these maps are derived, take for example, $\in \mathrm{Hom}(\emptyset, \bigcirc\bigcirc)$, then

$$m_2(v_- \otimes v_-) = \mathcal{F}_2(\ )(\mathcal{F}_2(\ ) \otimes \mathcal{F}_2(\ )) = \mathcal{F}_2(\ \circ (\ )) = \mathcal{F}_2(\ ) = v_+.$$

Remark 2.27. We could set the three-holed torus to be any invertible number (if we are in $\mathbb{Q}$, then any $\alpha \in \mathbb{Q}^*$) but we choose $\alpha = 8$ so we can drop some constants. This was shown in Example 2.21.

Remark 2.28. Here, we make the important remark that the gradings of the maps have been broken (they are no longer homogeneous) and thus $\mathcal{F}_2$ is not degree respecting. Thus, when we apply the Lee TQFT to the complex in Kob, the spaces in the chain complex are graded vector spaces (the same as the Khovanov TQFT spaces), but the induced differentials are *not graded*. In fact, the differentials are filtered where the filtration is induced by the grading

$$\cdots \bigoplus_{i>k+1} V_i \subset \bigoplus_{i>k} V_i \subset \cdots$$

and $d\left(\bigoplus_{i>k+1} V_i\right) \subset \bigoplus_{i>k} V_i$ giving us a filtered chain complex of graded vector spaces.

Thus we have two isomorphic ways to think of Khovanov homology and Lee homology:

- For a tangle T, Khovanov homology can be obtained via taking $\mathrm{Hom}(\emptyset, BN(T))$ and by quotienting out by all closed surfaces of genus greater than 1 (i.e. two dots on a surface is zero), and Lee homology can be obtained by taking $\mathrm{Hom}(\emptyset, BN(T))$ and by quotienting out by a genus 3 surface is equal to 8 (i.e. two dots can be pulled off a surface).
- This is the same as taking the Khovanov and Lee TQFTs ,$CKh(-)$ and $CKh^{Lee}(-)$, into the free modules $\mathbb{Z}\langle v_-, v_+\rangle$ which do the same thing to objects, but the multiplication and comultiplication maps between the objects differ according to Definition 2.22 and Example 2.26.

We will denote the graded chain complex of vector spaces obtained by the Khovanov TQFT applied to a tangle T by $CKh(T)$ and its homology to be $Kh(T)$. Similarly, we will denote the filtered chain complex of vector spaces obtained by the Lee TQFT by $CKh^{Lee}(T)$ and the Lee homology by $Kh^{Lee}(T)$. We have for each space $CKh^{Lee}(T)_r = CKh(T)_r$, so the chain spaces are the same but the differentials for the Khovanov and Lee complexes are graded and filtered respectively.

A result by Lee in [Lee05] is that Lee homology is interestingly boring. To show this, she introduces a new basis $\{a, b\}$ for V where $a = v_- + v_+$ and $b = v_- - v_+$. The multiplication and comultiplication maps become

$$\begin{aligned} m(a\otimes a) &= 2a \\ m(a\otimes b) &= m(b\otimes a) = 0 \\ m(b\otimes b) &= -2b \end{aligned} \qquad \begin{aligned} \Delta(a) &= a\otimes a \\ \Delta(b) &= b\otimes b \end{aligned}$$

By using this basis, Lee proves that $Kh^{Lee}(L)$ has rank 2^n, where n is the number of components of L. Thus, working over $\mathbb{Q}$, $Kh^{Lee}(K) \cong \mathbb{Q}\oplus\mathbb{Q}$ for any knot K.

As we will show in Theorem 4.1, our focus on these two homology theories is because Khovanov homology can be naturally viewed as the second page of a spectral sequence (defined in the following chapter) that converges to Lee homology which is a surprisingly simple object.

2.3 Tools and Computation

There are a couple of tools from homological algebra which are useful in simplifying the chain complexes in Kob. The idea is that when we make these simplifications before we take the tensor product of the chain complexes, we can reduce the number of computations.

Proposition 2.5. *(Gaussian elimination) Suppose we have a segment of a chain complex in the additive category* $\mathsf{Mat}(\mathcal{C})$ *of the form*

$$\cdots \longrightarrow A \xrightarrow{\begin{pmatrix}\xi\\ \beta\end{pmatrix}} B\oplus C \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} D\oplus E \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} F \longrightarrow \cdots$$

where ϕ is an isomorphism. Then the above segment of chain complex is isomorphic to the direct sum of complexes:

$$\begin{tikzcd} & B \arrow[r, "\phi"] & D & & \\ & \oplus & \oplus & & \\ \cdots \arrow[r] & A \arrow[r, "\beta"'] & C \arrow[r, "\epsilon-\gamma\phi^{-1}\delta"'] & E \arrow[r, "\nu"'] & F \arrow[r] & \cdots \end{tikzcd}$$

Both of the above complexes are then homotopy equivalent to

$$\cdots \longrightarrow A \xrightarrow{\beta} C \xrightarrow{\epsilon-\gamma\phi^{-1}\delta} E \xrightarrow{\nu} F \longrightarrow \cdots$$

This is because the complex $B \xrightarrow{\phi} D$ is contractible because ϕ is an isomorphism.

Proof. Consider the change of basis matrices

$$T_1 = \begin{pmatrix} 1 & \phi^{-1}\delta \\ 0 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 0 \\ -\gamma\phi^{-1} & 1 \end{pmatrix}$$

and denote

$$M = \begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}.$$

Then we have the 'row (and column) reduction' of M,

$$T_2 M T_1^{-1} = \begin{pmatrix} \phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}$$

as desired. Furthermore,

$$T_1 \begin{pmatrix} \xi \\ \beta \end{pmatrix} = \begin{pmatrix} \xi + \phi^{-1}\delta\beta \\ \beta \end{pmatrix}$$
$$(\mu \quad \nu)T_2^{-1} = (\mu + \nu\gamma\phi^{-1} \quad \nu)$$

But since the original chain is a chain complex, $d^2 = 0$, so

$$\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix} \begin{pmatrix} \xi \\ \beta \end{pmatrix} = 0$$

and thus $\phi\xi + \delta\beta = 0$, and

$$(\mu \ \nu) \begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix} = 0$$

and thus $\mu\phi + \nu\gamma = 0$. This gives $\xi = -\phi^{-1}\delta\beta$ and $\mu = -\nu\gamma\phi^{-1}$, so

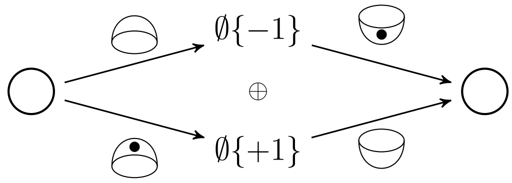
$$T_1 \begin{pmatrix} \xi \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \text{ and } (\mu \quad \nu)T_2 = (0 \quad \nu).$$

□

Proposition 2.6. *(Delooping) If an object S in $\mathsf{Cob}^3_{\bullet/l}$ contains a closed loop ◯, then it is isomorphic in $\mathsf{Mat}(\mathsf{Cob}^3_{\bullet/l})$ to the direct sum of copies $S'\{+1\}$ and $S'\{-1\}$ of S in which ◯ is removed, one taken with a degree shift of $+1$ and one with a degree shift of -1. Symbolically, ◯ $\cong \emptyset\{+1\} \oplus \emptyset\{-1\}$.*

Recall that if an object $\mathcal{O}$ has a grading $\mathcal{O}\{m\}$, we will write the gradings using a q instead, so $\mathcal{O}\{m\}$ will be recorded as $q^m\mathcal{O}$.

Proof. The isomorphisms are as follows:



To verify this is an isomorphism, this uses all of the relations in $\mathsf{Cob}^3_{\bullet/l}$. The map $\bigcirc \to \bigcirc$ is an isomorphism since

$$\bigcirc \to \bigcirc = \left(\ \ \ \right)\begin{pmatrix} \\ \end{pmatrix} = \ + \ = \ = 1$$

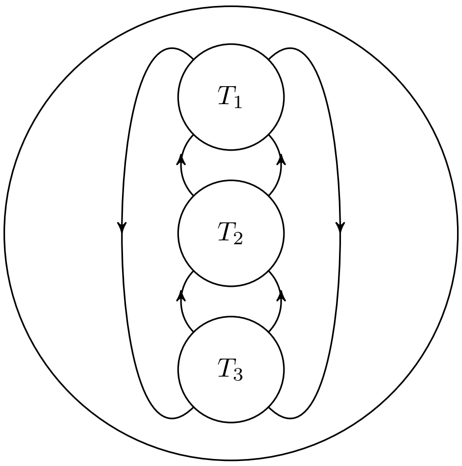
where the third equality is the neck cutting relation. To see $f : \emptyset\{-1\} \oplus \emptyset\{+1\} \to \emptyset\{-1\} \oplus \emptyset\{+1\}$ is an isomorphism,

$$f = \begin{pmatrix} \\ \end{pmatrix}\left(\ \ \ \right) = \begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Example 2.29. Let's see these tools in action. In our previous example for finding the Khovanov complex for the Hopf link, we could have simplified the complex using these tools. We will now find the Khovanov complex for the right-handed trefoil 3_1. Importantly, we will also keep track of the gradings in this example.

Consider the oriented planar arc diagram from Definition 2.11:



and take $T_1 = T_2 = T_3 = \otimes$ so $D(T_1, T_2, T_3) = 3_1$. Then

$$BN(3_1) = BN(D(T_1, T_2, T_3)) = D(BN(T_1), BN(T_2), BN(T_3)).$$

To compute this, we will build this up step by step using the properties of planar algebras, simplifying as we go. Firstly, the concatenated positive crossings is computed by taking the tensor product of the complex $BN(\times)$. Then to obtain the knot, we then take the trace: we join the top right strand to the bottom right strand, and then join the top left to the bottom left. For $BN(\times)^{\otimes 3}$, we have the simplified expressions

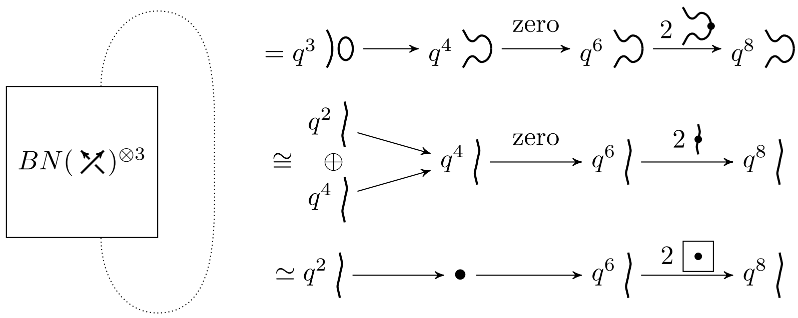
$$\begin{aligned} BN(\times) &= \underline{q\,)(} \xrightarrow{s} q^2 \asymp \\ BN(\times)^{\otimes 2} &\simeq \underline{q^2\,)(} \xrightarrow{s} q^3 \asymp \xrightarrow{\ \ -\ \ } q^5 \asymp \\ BN(\times)^{\otimes 3} &\simeq \underline{q^3\,)(} \xrightarrow{s} q^4 \asymp \xrightarrow{\ \ -\ \ } q^6 \asymp \xrightarrow{\ \ +\ \ } q^8 \asymp. \end{aligned}$$

More generally, we have for a two strand tangle with n twists:

$$BN(\times)^{\otimes n} \simeq \underline{q^n\,)(} \xrightarrow{s} q^{n+1} \asymp \xrightarrow{\ \ -\ \ } q^{n+3} \asymp \xrightarrow{\ \ +\ \ } q^{n+5} \asymp \to \cdots \to q^{3n-1} \asymp$$

where $a = \ \ -\ \ $ and $\sigma = \ \ +\ \ $ alternate (for example, see [Tho17]).

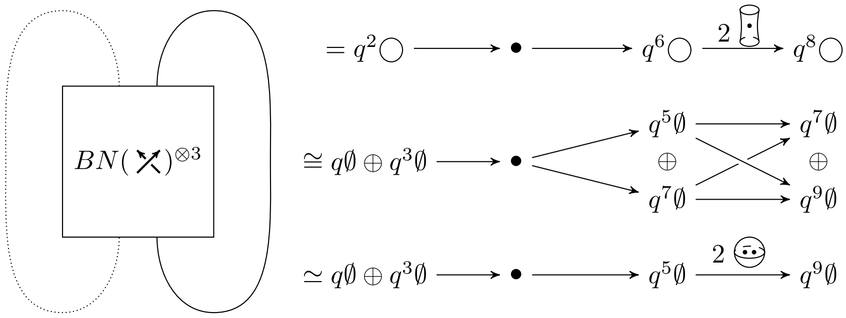
Joining the top right to the bottom right, take



where the isomorphism is by delooping and the $\simeq$ (homotopy equivalence) is by Gaussian elimination. The above tangle is isotopic to the cut trefoil which we denote $c(3_1)$. Thus

$$c(3_1) \simeq \underline{q^2} \to \bullet \to q^6 \xrightarrow{2\,\boxed{\bullet}} q^8$$

And then lastly, we join the top left to the bottom left:



Again, the isomorphism is by delooping and the homotopy equivalence is by Gaussian elimination. Since two dots on a cobordism is zero in Khovanov homology, we have the following if we work over $\mathbb{Q}$:

$$CKh(3_1) = \mathrm{Hom}_{\mathbb{Q}}(\emptyset, BN(3_1))/(\quad = 0) \simeq \underline{q\mathbb{Q} \oplus q^3\mathbb{Q}} \to \bullet \to q^5\mathbb{Q} \xrightarrow{\text{zero}} q^9\mathbb{Q},$$

and Poincaré polynomial

$$K(q,t) = q + q^3 + t^2 q^5 + t^3 q^9.$$

In Lee homology, $\boxed{\bullet\bullet} = \alpha \boxed{}$, where $\alpha \in \mathbb{Q}^*$, so the portion of chain complex $q^5\mathbb{Q} \to q^9\mathbb{Q}$ is contractible, and thus

$$CKh^{Lee}(3_1) = \mathrm{Hom}_{\mathbb{Q}}(\emptyset, BN(3_1))/(\quad = \alpha) \simeq \underline{q\mathbb{Q} \oplus q^3\mathbb{Q}}.$$

As we will show below, the Khovanov and Lee TQFTs are related by a spectral sequence. In order to describe this, we will define spectral sequences for filtered chain complexes and investigate some of their immediate properties in the next chapter.

Chapter 3

Spectral Sequences of Filtered Chain Complexes

3.1 Definitions and Basic Properties

Spectral sequences for filtered chain complexes are a tool for computing the homology of such chain complexes by using the chain homology of the associated graded chain complex, which in general is much simpler. From the associated graded chain complex we can form a sequence of chain complexes that 'converges' to the homology of the original filtered chain complex. We will make this precise below.

Spectral sequences are notoriously confusing with indices. For the most part, the indexing here follows Weibel's treatment [Wei94], and at the very least we will be consistent throughout this paper. A lot of the discussion may seem confusing, and the best way to understand would be for the reader to work out the introductory definitions and theorems for herself. Moreover, some good introductory texts start with a heuristic treatment of spectral sequences before specifying the precise definitions. We will attempt to do the same, and we hope this is not too difficult to follow.

Suppose we have a filtered chain complex

$$\cdots \hookrightarrow F_{p-1}C_\bullet \hookrightarrow F_pC_\bullet \hookrightarrow F_{p+1}C_\bullet \hookrightarrow \cdots ,$$

where $F_pC_\bullet$ (also written as $C_{p,\bullet}$) is the pth filtration of the chain complex $C_\bullet$. More explicitly, this is a chain complex of filtered vector spaces $C_n = C_{p,n} \supseteq C_{p-1,n} \supseteq \cdots \supseteq C_{1,n} \supseteq C_{0,n} = 0$,

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

where the differentials respect the filtration:

$$d(F_pC_n) \subseteq F_pC_{n-1}.$$

Of course, since $C_\bullet$ is a chain complex, $d^2 = 0$.

From a filtered vector space, we can form the 'associated graded' vector space. For example, if we have a vector space V and a subspace W of V, we can decompose V (non-canonically) into $V \cong V/W \oplus W$. If there were again a subspace U of W, we could iterate this process again so that $V \cong V/W \oplus W/U \oplus U$.

Thus in the above situation for filtered vector spaces, for each space in the chain complex

$$C_n \cong \frac{C_{p,n}}{C_{p-1,n}} \oplus \frac{C_{p-1,n}}{C_{p-2,n}} \oplus \cdots \oplus \frac{C_{1,n}}{C_{0,n}}$$

where, since $C_{0,n} = 0$, we have for the last term $C_{1,n}/C_{0,n} = C_{1,n}$. This is the associated graded vector space for the filtered vector space, and denoted $G_pC_n := C_{p,n}/C_{p-1,n}$. The differentials d for the filtered chain complex then induce differentials on the associated graded vectors spaces, giving us a chain complex of graded vector spaces. Explicitly, take $x \in G_pC_n$. Then $x = x' + C_{p-1,n}$ for some $x' \in C_{p,n}$. So $d(x) = d(x' + C_{p-1,n}) = d(x') + d(C_{p-1,n})$, and since the differentials d respect the filtration, $d(x') \in C_{p,n-1}$ and $d(C_{p-1,n}) \subseteq C_{p-1,n-1}$. Thus $d(x) \in G_pC_{n-1}$. Moreover, it is clear that again $d^2 = 0$.

Let us make some preliminary definitions:

Definition 3.1. For a filtered chain complex $F_\bullet C_\bullet$, we define

- G_pC_{p+q} to be the the (p, q)- or $(p+q)$-chains of filtering degree p (so $p+q$ is the homological degree).
- $Z^r_{p,q} = \{x \in G_pC_{p+q} \mid dx = 0 \mod F_{p-r}C_\bullet\} = \{x \in F_pC_{p+q} \mid dx \in F_{p-r}C_{p+q-1}\}/F_{p-1}C_{p+q}$, called the '$r$-almost (p, q)-cycles',
- $B^r_{p,q} = d(F_{p+r-1}C_{p+q+1})$, called the '$r$-almost (p, q)-boundaries'.

Next, we set $Z^\infty_{p,q}$ to be the actual cycles of the associated graded pieces, $Z^\infty_{p,q} = Z(G_pC_{p+q})$ and we similarly define $B^\infty_{p,q} = d(C_{p,p+q+1})$.

Proposition 3.1. *The differentials of the original filtered complex $C_\bullet$ restrict to differentials $d^r : Z^r_{p,q} \to Z^r_{p-r,q+r-1}$ on the r-almost cycles.*

Proof. The subquotient $Z^r_{p,q}$ by definition contains elements in filtering degree p where the differential d shifts the elements down by r degrees to $p - r$. Moreover, the differential d by definition shifts the homological degree $p + q$ down by one. Thus d restricted to $Z^r_{p,q}$ lands in $Z^\bullet_{p-r,q+r-1}$. Now, the image $d(Z^r_{p,q})$ consists of actual boundaries, that is, $d(Z^r_{p,q}) \subset B_{s,t}$ for some s, t where $s + t = p + q - 1$. But since actual boundaries are in particular r-almost boundaries for some r, we can take the codomain to be $Z^r_{p-r,q+r-1}$. □

Proposition 3.2. *We have the sequence of inclusions*

$$B^0_{p,q} \hookrightarrow B^1_{p,q} \hookrightarrow \cdots \hookrightarrow B^\infty_{p,q} \hookrightarrow Z^\infty_{p,q} \hookrightarrow \cdots \hookrightarrow Z^1_{p,q} \hookrightarrow Z^0_{p,q}.$$

Proof. This is just by checking the definitions. □

Proposition 3.3. $Z^{r+1}_{p,q} = \ker(d^r : Z^r_{p,q} \to Z^r_{p-r,q+r-1})$.

Proof. Take an element $x \in F_pC_{p+q}$. Then $x \in Z^r_{p,q}$ if $dx \in F_{p-r}C_{p+q-1}$. For the same x, $x \in Z^{r+1}_{p,q}$ if $dx \in F_{p-r-1}C_{p+q-1}$. But $F_{p-r-1}C_{p+q-1} \hookrightarrow F_{p-r}C_{p+q-1}$, so $x \in Z^{r+1}_{p,q}$ if and only if $dx = 0$ in the quotient $F_{p-r}C_{p+q-1}/F_{p-r-1}C_{p+q-1}$, and $Z^r_{p-r,q+r-1} \subset F_{p-r}C_{p+q-1}/F_{p-r-1}C_{p+q-1}$, proving the claim. □

Definition 3.2. We define the (p, q)th entry on the rth page of the spectral sequence to be the quotient of the r-almost cycles by the r-almost boundaries.

$$E^r_{p,q} := Z^r_{p,q}/B^r_{p,q}.$$

Since the differentials on the filtered complex $C_\bullet$ restrict to differentials $d^r : Z^r_{p,q} \to Z^r_{p-r,q+r-1}$, the differentials also restrict on the r-almost homology groups $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. Also, by the previous proposition, we have that $E^{r+1}_{p,q}$ is the d^r-chain homology at $E^r_{p,q}$:

$$E^{r+1}_{p,q} = \frac{\ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\operatorname{im}(d^r : E^r_{p+r,q-r+1} \to E^r_{p,q})}.$$

For low dimensional pages, we have the following:

Proposition 3.4. *The zeroth page is the associated graded complex*

$$E^0_{p,q} = G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}$$

and the first page is the chain homology of the associated graded

$$E^1_{p,q} = H_{p+q}(G_pC_\bullet).$$

Proof. Setting $r = 0$ in the definition of $E^r_{p,q}$ gives the first statement. For $r = 1$,

$$E^1_{p,q} = \frac{\{x \in G_pC_{p+q} \mid dx = 0 \in G_pC_{p+q}\}}{d(F_pC_{p+q})} = H_{p+q}(G_pC_\bullet).$$

□

So the associated graded vector space is the zeroth page of the spectral sequence. We can think of this page as a lattice of vector spaces $E^1_{p,q} = G_{p,q}$ and the morphisms between objects induced by the filtration go horizontally $d : E^1_{p,q} \to E^1_{p,q-1}$.

Remark 3.3. These definitions and propositions are the easy way out. What we do is define our r-almost cycles and r-almost boundaries so that the differentials on the rth page that are induced by the original filtered complex land in the appropriate space. The harder option would be to define spectral sequences in the more intuitive way, that is, by taking successive quotients:

$$E^{r+1}_{p,q} = \frac{\ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\operatorname{im}(d^r : E^r_{p+r,q-r+1} \to E^r_{p,q})}.$$

Then there is the task defining the induced differentials. For example, lets take $[x] \in E^2_{p,q}$, and define our pages as the successive quotients, so

$$[x] \in E^1_{p,q} = \frac{\ker(d^1 : E^0_{p,q} \to E^0_{p,q-1})}{\operatorname{im}(d^1 : E^0_{p,q+1} \to E^0_{p,q})},$$

So what are our differentials d^1 on the first page? For $[x] \in E^1_{p,q}$, take a representative x^1 of $[x]$, where $x^1 \in \ker(d^1 : E^0_{p,q} \to E^0_{p,q-1})$. So $x^1 \in E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}$, and so write $x^1 = x^0 + F_{p-1}C_{p+q}$. Applying the differential from the filtered vector spaces, $d(x^1) = d(x^0 + F_{p-1}C_{p+q}) = d(x^0) + d(F_{p-1}C_{p+q})$ and note that $d(F_{p-1}C_{p+q}) \subset F_{p-1}C_{p+q-1}$. Since $x^1 \in \ker(d^1 : E^0_{p,q} \to E^0_{p,q-1})$, we also have $d^1(x^1) \in F_{p-1}C_{p+q-1}$. Thus $d(x^0) \in F_{p-1}C_{p+q-1}$. So $x^0 \in \{z \in F_pC_{p+q} \mid dz \in F_{p-1}C_{p+q-1}\}/F_{p-1}C_{p+q}$. So we see that indeed $d^1 : Z^1_{p,q} \to Z^1_{p-1,q}$. This is the idea behind defining the r-almost cycles.

3.2 Convergence of Spectral Sequences

Next we make 'convergence' of spectral sequences:

Definition 3.4. Let $\{E^r_{p,q}\}_{r,p,q}$ be a spectral sequence, where, for all p, q, there is an $r(p, q)$ such that for all $r \geq r(p, q)$, $E^r_{p,q} \cong E^{r(p,q)}_{p,q}$. Then we say that the bigraded object $E^\infty := \{E^\infty_{p,q}\}_{p,q} := \{E^{r(p,q)}_{p,q}\}_{p,q}$ is the limit term (or 'page') of the spectral sequence.

If the limit page E^∞ exists, then the spectral sequence is said to 'converge' or 'collapse' at E^∞.

Definition 3.5. A spectral sequence $\{E^r_{p,q}\}$ is said to be bounded if for all $n, r \in \mathbb{Z}$, the diagonal $E^r_{k,n-k}$ has finitely many nonzero terms. For example, a first quadrant spectral sequence $\{E^r_{p,q\geq 0}\}$ is a bounded spectral sequence.

Proposition 3.5. *A bounded spectral sequence has a limit page.*

Proof. Firstly, if a spectral sequence has at most N non-vanishing terms of total degree n on page r, then the following page must have *at most* N non-vanishing terms again. Thus, for a bounded spectral sequence and for each n, there are integers $U(n), L(n) \in \mathbb{Z}$ such that $E^r_{p,n-p} = 0$ for all $p \leq L(n)$ and $E^r_{n-q,q} = 0$ for all $q \leq U(n)$. Then for

$$r > \max\{p - L(p+q-1), q + 1 - U(p+q+1)\},$$

the term $E^r_{p,q}$ is the limit term of the bounded spectral sequence, since for such r:

- $E^r_{p-r,q+r-1} = 0$, because $p - r < L(p+q-1)$, and so $\ker(d^r_{p-r,q+r-1}) = 0$,
- $E^r_{p+r,q-r+1} = 0$ because $q - r + 1 < U(p+q+1)$, and so $\mathrm{im}(d^r_{p,q}) = 0$.

Thus $E^{r+1}_{p,q} = \ker(d^r_{p-r,q+r-1})/\mathrm{im}(d^r_{p,q}) \cong E^r_{p,q}$. □

Definition 3.6. A filtration $F_\bullet C_\bullet$ on a chain complex $C_\bullet$ is called a bounded filtration if for all $n \in \mathbb{Z}$, there are integers $U(n), L(n) \in \mathbb{Z}$ such that $F_{p<L(n)}C_n = 0$ and $F_{p>U(n)}C_n = C_n$.

Proposition 3.6. *The spectral sequence of a complex of bounded filtration has a limit page.*

Proof. Since the filtration is bounded, the spectral sequence is bounded and thus converges by the previous proposition. □

Proposition 3.7. *If the spectral sequence of a filtered complex $F_\bullet C_\bullet$ has a limit page, then that page is isomorphic to the chain homology of the original filtered complex.*

Proof. Since there is a limit page, there is for each p, q an $r(p,q)$ such that for all $r \geq r(p,q)$, the r-almost cycles and r-almost boundaries in F_pC_{p+q} are in fact the ordinary cycles and boundaries. So for $r \geq r(p,q)$, we indeed have $E^r_{p,q} \cong G_pH_{p+q}(C_\bullet)$. □

Everything we do concerning Khovanov/Lee homology will involve finite chain complexes of finite dimensional graded vector spaces with bounded filtration (with possibly filtered chain maps only), so the relevant spectral sequences will converge.

3.3 Examples Using Spectral Sequences

Let $C_{\bullet,\bullet}$ be a double chain complex with horizontal differentials d^h and vertical differentials d^v. Then the total chain complex $\mathrm{Tot}(C)$ is defined to be

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

with differentials $d = d^h + (-1)^p d^v$. Then we define the horizontal filtration $F^h_\bullet \mathrm{Tot}(C)$ on $\mathrm{Tot}(C)$ to be the filtration

$$F^h_p(\mathrm{Tot}(C))_n := \bigoplus_{\substack{n_1+n_2=n \\ n_1 \leq p}} C_{n_1,n_2}.$$

Similarly, the vertical filtration is defined to be

$$F^v_q(\mathrm{Tot}(C))_n := \bigoplus_{\substack{n_1+n_2=n \\ n_2 \leq q}} C_{n_1,n_2}.$$

Proposition 3.8. *Let $\{E^r_{p,q}\}_{r,p,q}$ be a spectral sequence of the double complex $C_{\bullet,\bullet}$ with respect to the horizontal filtration (this is exactly the same for vertical filtration). Then*

- $E^0_{p,q} \cong C_{p,q}$,
- $E^1_{p,q} \cong H_q(C_{p,\bullet})$, *i.e. the homology of the columns,*
- $E^2_{p,q} \cong H_p(H^v_q(C_{\bullet,\bullet}))$, *i.e. homology of the rows of the previous page.*

Moreover, if $C_{\bullet,\bullet}$ is a first quadrant double complex ($0 \leq p, q$), then the spectral sequence converges to the chain homology of the total complex:

$$E^\infty_{p,q} \cong G_p H_{p+q}(\mathit{Tot}(C)_\bullet).$$

Thus the chain homology of the total chain complex can be computed by either using a horizontal or vertical filtration, and we will get the same answer on the limit page.

Proof.

$$\begin{aligned} E^0_{p,q} &:= G_p \mathrm{Tot}(C)_{p+q} \\ &= \frac{F_p \mathrm{Tot}(C)_{p+q}}{F_{p-1}\mathrm{Tot}(C)_{p+q}} \\ &= \frac{\bigoplus_{\substack{n_1+n_2=p+q \\ n_1 \leq p}} C_{n_1,n_2}}{\bigoplus_{\substack{n_1+n_2=p+q \\ n_1 \leq p-1}} C_{n_1,n_2}} \\ &\cong C_{p,q}. \end{aligned}$$

$$\begin{aligned} E^1_{p,q} &\cong H_{p+q}(G_p \mathrm{Tot}(C)_\bullet) \\ &\cong H_{p+q}(C_{p,\bullet}) \\ &\cong H_q(C_{p,\bullet}) \end{aligned}$$

For the next page, $[x] \in E^1_{p,q} = H_q(C_{p,\bullet})$, so pick a representative $x \in \ker(C_{p,q} \to C_{p,q-1})$. So $x \in \{z \in C_{p,q} \,|\, d^v x = 0\}$, i.e. the vertical maps are zero on x. So the differential d^1 which is induced by the original differential $d = d^h + d^v$ is just the horizontal differential d^h, so $d^1 : E^1_{p,q} \to E_{p-1,q}$. Thus,

$$E^2_{p,q} = \frac{\ker(d^1 : E^1_{p,q} \to E^1_{p-1,q})}{\mathrm{im}(d^1 : E^1_{p+1,q} \to E^1_{p,q})} \cong H_p(H^v_q(C_{\bullet,\bullet})).$$

By the convergence properties stated earlier, this bounded filtration means that this sequence will converge, and it will converge to the homology of the original filtered complex. □

Example 3.7. Special case when rows or columns are exact

Using the previous properties, it is clear that if a double complex has exact rows or columns, the limit page E^∞ will be zero. This can be very useful as we will see for an example easy proof of the Five-lemma:

Lemma 3.1. *Take the following diagram that has exact rows and α, β, δ and ϵ are isomorphisms:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

Then γ is an isomorphism.

Proof. Since the rows are exact by assumption, the spectral sequence converges on the first step in the horizontal direction immediately, so that $E^{\infty}_{p,q} = 0$. Now we compute the spectral sequence in the vertical direction, so the first page is

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \ker\gamma \arrow[r] & 0 \arrow[r] & 0 \\
\\
0 \arrow[r] & 0 \arrow[r] & \operatorname{coker}\gamma \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since the limit page is zero, and in the spectral sequence, there will be no point where we can take homology to get rid of the final remaining terms, we must have $\ker(\gamma) = 0$ and $\operatorname{coker}(\gamma) = 0$, and thus γ is an isomorphism. Looking at this page we can see that if we let $\ker(\alpha)$ and $\operatorname{coker}(\epsilon)$ be nonzero, the result still holds. In other words, we only need β and δ to be isomorphisms and $\operatorname{coker}(\alpha) = 0$ and $\ker(\epsilon) = 0$. This is the same as saying that we only need α to be surjective and ϵ to be injective. □

Example 3.8. Homology of tensor product of chain complexes

We can compute the homology of the tensor product of chain complexes using spectral sequences. More specifically, two chain complexes can be tensored together to give the tensor product double complex, from which we get the total chain complex for the tensor product, which is an honest chain complex (not a double complex). We will restrict the discussion to the case of chain complexes of vector spaces. More generally for R-modules the universal coefficient theorem for homology tells us we gets something a little more complicated than for vector spaces (see Remark 3.9). Consider two chain complexes of vector spaces $V_\bullet$ and $W_\bullet$ where V_i and W_i denote the ith component of the V and W complexes, respectively. Then the total chain complex $\operatorname{Tot}(V_\bullet \otimes W_\bullet)$, which we will write as $(V \otimes W)_\bullet$, has as the nth component

$$(V \otimes W)_n = \bigoplus_{n=i+j} V_i \otimes W_j = \bigoplus_{i \leq k} V_i \otimes W_{k-i}.$$

We define the differential to be

$$d^{V\otimes W}(v \otimes w) = d^V v \otimes w + (-1)^{\deg(v)} v \otimes d^W w.$$

This can be quickly verified to be a differential:

$$\begin{aligned}
(d^{V\otimes W})^2 &= d^{V\otimes W}\left(d^V v \otimes w + (-1)^{\deg(v)} v \otimes d^W w\right) \\
&= (d^V)^2 v \otimes w + \left((-1)^{\deg(d^V v)} + (-1)^{\deg(v)}\right)(d^V v \otimes d^W w) + v \otimes (d^W)^2 w \\
&= 0.
\end{aligned}$$

In order to use spectral sequences, we define a filtration as we did before on the total complex of $V_\bullet \otimes W_\bullet$ by

$$F_p(V \otimes W)_k = \bigoplus_{i \leq p} V_i \otimes W_{k-i}.$$

So, for example,

$$\begin{aligned}
F_0(V \otimes W)_k &= V_0 \otimes W_k \\
F_1(V \otimes W)_k &= (V_0 \otimes W_k) \oplus (V_1 \otimes W_{k-1})
\end{aligned}$$

and so on. The associated graded object is given by

$$G_p(V \otimes W)_k = V_p \otimes W_{k-p}.$$

In terms of our pages of spectral sequences,

$$E^0_{p,q} = G_p(V \otimes W)_{p+q} = V_p \otimes W_q.$$

The differential for this (the differential on the zeroth page) is given by $d^0 = (-1)^p \mathrm{id}_V \otimes d^W$, so $d^0(E^0_{p,q}) \subseteq E^0_{p,q+1}$. Thus for

$$\cdots \to E^0_{p,q-1} \to E^0_{p,q} \to E^0_{p,q+1} \to \cdots$$

we have

$$E^1_{p,q} = \frac{\ker(d : E^0_{p,q-1} \to E^0_{p,q})}{\mathrm{im}(d : E^0_{p,q} \to E^0_{p,q+1})} = \frac{V_p \otimes \ker(d^W : W_{q-1} \to W_q)}{V_p \otimes \mathrm{im}(d^W : W_q \to W_{q+1})} \cong V_p \otimes H_q(W).$$

The differential d^1 on the following page $E^1_{p,q}$ is given by $d \otimes \mathrm{id}_W$, so $E^2_{p,q} = H_p(V_\bullet \otimes H_q(W_\bullet)) = H_p(V) \otimes H_q(W)$. So each element of the the E^2 page is the tensor product of V-cycles and W-cycles and is thus a $V \otimes W$-cycle. Since the differentials of the spectral sequence come from the differential on $V \otimes W$, all the higher differentials must vanish, so $E^2 = E^\infty$. Thus,

$$H_k(V_\bullet \otimes W_\bullet) \cong \bigoplus_{i+j=k} H_i(V) \otimes H_j(W).$$

Remark 3.9. This differs from the more general case. The Künneth formula for complexes states that if V and W are right and left R-module complexes, and V_n and $d(V_n)$ are flat for each n, then there is an exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(V) \otimes H_q(W) \to H_n(V \otimes_R W) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(V), H_q(W)) \to 0$$

which is noncanonically split if $R = \mathbb{Z}$ and V is a complex of free abelian groups. But in the case of vector spaces, we always have $\mathrm{Tor}_1^{\mathbb{Z}} = 0$ because there is always a length zero free resolution of V, giving us the desired isomorphism by exactness. Further discussion can be found in [Wei94].

Chapter 4

The Lee Spectral Sequence

4.1 Comparison of Khovanov's and Lee's differentials

The Khovanov complex $CKh(L)$ of a link L is an object in the category $\mathsf{GKom}(\mathsf{GVec}_{\Bbbk})$ of finite length graded complexes of graded vector spaces, and thus the differentials in $CKh(L)$ are graded. On the other hand, the Lee complex $CKh^{Lee}(L)$ of a link L is an object in the category $\mathsf{FilKom}(\mathsf{GVec}_{\Bbbk})$ of finite length filtered complexes of graded vector spaces — each space $CKh^{Lee}(L)_r$ in the complex $CKh^{Lee}(L)$ is a graded vector space, but the differentials do not preserve the q-grading:

$$\begin{aligned} d^{Kh} &: CKh(L)_{q,n} \to CKh(L)_{q,n+1} \\ d^{Lee} &: CKh(L)_{q,n} \to CKh(L)_{q,n+1} \oplus CKh(L)_{q+4,n+1} \end{aligned}$$

The differentials for $CKh^{Lee}(L)$ are not even homogeneous, for example $\Delta^{Lee}(v_-) = v_- \otimes v_- + v_+ \otimes v_+$. However, given such filtered chain complex, its spectral sequence is closely related to Khovanov homology:

Theorem 4.1. *There is a spectral sequence with E^1 page $Kh(L)$ and E^∞ page $Kh^{Lee}(L)$.*

Proof. Define δ to be the differential so that $d^{Lee} = d^{Kh} + \delta$. So d^{Lee} is a differential for a filtered complex, where the d^{Kh} is the degree preserving part of the differential and δ shifts the degree up by 4. Explicitly, d^{Kh} is defined as in Definition 2.22, d^{Lee} is defined as in Example 2.26, and δ is defined via

$$\begin{aligned} &\delta_m(v_+ \otimes v_+) = \delta_m(v_+ \otimes v_-) = \delta_m(v_- \otimes v_+) = 0, \\ &\delta_m(v_- \otimes v_-) = v_+, \\ &\delta_\Delta(v_+) = 0, \\ &\delta_\Delta(v_-) = v_+ \otimes v_+. \end{aligned}$$

From this, it is clear that $\delta^2 = 0$. Because we have a chain complex of filtered vector spaces, there is an associated spectral sequence with E^∞ its homology, i.e. $E^\infty \cong Kh^{Lee}(L)$. Moreover, since the E^1 page is always the homology with respect to the degree preserving part of the original filtered complex, we conclude that the E^1 page is the homology with respect to d^{Kh}, or in other words, it is $Kh(L)$. $\square$

In fact, each page of the spectral sequence is also a link invariant. Following Rasmussen [Ras10], we can prove this via the following Lemma that can be found in [McC01]:

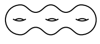
Lemma 4.1. *Suppose $F : C_1 \to C_2$ is a map of filtered complexes that respects the filtrations. Then F induces a map of spectral sequences $F_n : {}^{I}E^n \to {}^{II}E^n$, and if F_n is an isomorphism, F_m is an isomorphism for all $m \geq n$.*

In [Lee05], for link diagrams L and $\tilde{L}$ that differ by a Reidemeister move, Lee proves $Kh^{Lee}(L) \cong Kh^{Lee}(\tilde{L})$ by creating a chain map between $CKh^{Lee}(L)$ and $CKh^{Lee}(\tilde{L})$. Thus to show that the spectral sequence is a link invariant, we just need to check that these chain maps respect the filtration, and that they induce isomorphisms on Khovanov homology $Kh(L) \cong Kh(\tilde{L})$. For details, see [Ras10] [Lee05].

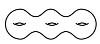
It also follows that each page of the spectral sequence is a link invariant by Theorem 4.4 below.

4.2 Equivariant Khovanov Homology

Recall that we described the Khovanov TQFT by

$\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \mathrm{Hom}(\emptyset, \mathcal{O})/($  $= 0),$

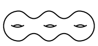
and the Lee TQFT by

$\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \mathrm{Hom}(\emptyset, \mathcal{O})/($  $= 8).$

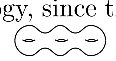
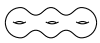
The above Hom spaces are $\mathbb{Z}$-modules, and then tensored with $\mathbb{Z}[\frac{1}{2}]$ to make 2 invertible to allow the local relations in $\mathsf{Cob}^3_{/l}$.

Now we choose to work over $\mathbb{Q}[\alpha]$, and we set the genus 3 surface to be 8α:

Definition 4.2. We define the 'universal Khovanov homology' or 'equivariant Khovanov homology'[1] $Kh^{\alpha}(T)$ of a tangle T to be the homology of the complex obtained via the TQFT

$\mathbb{Q}[\alpha] \otimes_{\mathbb{Z}} \mathrm{Hom}(\emptyset, \mathcal{O})/($  $= 8\alpha).$

The resulting complex by applying this TQFT to $BN(T)$ will be denoted $CKh^{\alpha}(T)$. Following Example 2.21, a surface carrying two dots is then equal to α times that surface. It follows that the formal Euler characteristic is $\deg(\alpha) = -4$. The Hom spaces are now $\mathbb{Q}[\alpha]$-modules, so we have lost information by killing all integer torsion, but having the coefficients invertible allows us to use a variant of Gaussian elimination that is essential in Theorem 4.10.

This can be considered in some ways as a generalisation of Khovanov and Lee homology, since those homology theories can be recovered from CKh^{α}: for a link L, the condition that  $= 0$ implies $\alpha = 0$ and  $= 8$ implies $\alpha = 8$. So if we do the calculation for the trefoil for Khovanov homology as we did earlier in Example 2.29, but working over $\mathbb{Q}[\alpha]$, we obtain the complex

$$BN(3_1) \simeq q\emptyset \oplus q^3\emptyset \to \bullet \to q^5\emptyset \xrightarrow{\alpha} q^9\emptyset.$$

Setting $\alpha = 0$, the above complex is quasi-isomorphic to the Khovanov complex $CKh(3_1)$, and setting $\alpha - 8 = 0$, the above complex is quasi-isomorphic to the Lee complex $CKh^{Lee}(3_1)$ (since $\mathbb{Q} \xrightarrow{8} \mathbb{Q}$ is contractible). Thus,

$$\begin{aligned} Kh(L)_{\bullet} &\cong H_{\bullet}\left(CKh^{\alpha}(L) \otimes \mathbb{Q}[\alpha]/(\alpha = 0)\right) \\ Kh^{Lee}(L)_{\bullet} &\cong H_{\bullet}\left(CKh^{\alpha}(L) \otimes \mathbb{Q}[\alpha]/(\alpha - 8 = 0)\right). \end{aligned}$$

[1] For comparison, Khovanov's definition of equivariant Khovanov homology can be found in [Kho04]

Note. We could alternatively recover Lee homology by tensoring with $\mathbb{Q}[\alpha]/(\alpha - 1 = 0)$ by the homotopy equivalence

$$CKh^{\alpha}(L) \otimes \mathbb{Q}[\alpha]/(\alpha - 8 = 0) \simeq CKh^{\alpha}(L) \otimes \mathbb{Q}[\alpha]/(\alpha - 1 = 0).$$

Digression. There is a subtlety involved when working with Hom above. When we take $\mathrm{Hom}(\emptyset, -)$, we are not taking the set of chain complex homomorphisms, rather we are taking the 'internal Hom' for the category of chain complexes. Explicitly, there are two categories under consideration:

- the category Kom whose objects are chain complexes, and whose morphisms are chain maps. The set of morphisms between two complexes forms a vector space.
- the category $\widehat{\mathsf{Kom}}$ whose objects are again chain complexes, but morphisms are linear maps $f : \bigoplus C_\bullet \to \bigoplus D_\bullet$, and these maps form a chain complex. That is, $\widehat{\mathsf{Kom}}(C_\bullet, D_\bullet)_i = \{f : C_j \to D_{j+i}\}$ with differential $(df)(x) = d(f(x)) + (-1)^{\deg f} f(dx)$. Since $\mathrm{Hom}(C_\bullet, D_\bullet)$ is itself a chain complex, i.e. it is an object in the category again, it is called 'internal Hom'. This is the same as taking the usual Hom cochain complex in homological algebra. Importantly, for homotopic complexes $D_\bullet \simeq D'_\bullet$,

$$\mathrm{Hom}_{\widehat{\mathsf{Kom}}}(C_\bullet, D_\bullet) \simeq \mathrm{Hom}_{\widehat{\mathsf{Kom}}}(C_\bullet, D'_\bullet),$$

  so taking homology will give us information about the original chain complexes.

Example 4.3. In Definition 4.10, we are taking the 'internal Hom' (think of $\emptyset$ as the chain complex concentrated in degree zero). Hence, taking $\mathrm{Hom}_{\mathsf{BN}}(\emptyset, -)$ into this complex (which is a covariant functor, that is only left exact, but that doesn't matter), we get

$$CKh^{\alpha}(3_1) = \mathrm{Hom}_{\mathsf{BN}}(\emptyset, BN(3_1)) \simeq q\mathbb{Q}[\alpha] \oplus q^3\mathbb{Q}[\alpha] \to \bullet \to q^5\mathbb{Q}[\alpha] \xrightarrow{\alpha} q^9\mathbb{Q}[\alpha].$$

This is because $\mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset) \cong \mathbb{Q}[\alpha]$ and

$$\begin{aligned} \mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset \xrightarrow{\alpha^n} \emptyset) &= \mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset) \xrightarrow{d} \mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset) \\ &\cong \mathbb{Q}[\alpha] \xrightarrow{\alpha^n} \mathbb{Q}[\alpha] \end{aligned}$$

where d is multiplication by α^n, by the rule $(df)(x) = d(f(x)) + (-1)^{\deg f} f(dx)$ which implies that for any polynomial $p(\alpha) \in \mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset)$, $dp(\alpha) = \alpha^n p(\alpha)$.

To see that $\mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset) \cong \mathbb{Q}[\alpha]$, we need to show that any $\Sigma \in \mathrm{Hom}_{\mathsf{BN}}(\emptyset, \emptyset) \cong \mathbb{Q}[\alpha]$ can be reduced, via local relations, to a polynomial function of a genus 3 surface (or a lower genus closed surface, which will be equal to a scalar). Note that Σ is a closed surface. Since BN is an additive category, $\mathrm{Hom}_{\mathsf{BN}}$ is closed under addition and we can take linear combinations of elements in $\mathrm{Hom}_{\mathsf{BN}}$. Thus we just need to show that we can reduce each surface Σ to a monomial in $\mathbb{Q}[\alpha]$. Let Σ' be a connected component of Σ and let g denote genus. Since all closed orientable surfaces are classified up to genus, and cobordisms are orientable, it suffices to check this for different genera. If $g(\Sigma') < 3$, then $\Sigma' \in \{0, 2\}$ by the local relations in $\mathsf{Cob}^3_{\bullet/l}$. If $g(\Sigma') = 3$, then clearly $\Sigma' = 8\alpha$ by Definition 4.10. Now suppose $g(\Sigma) = n > 3$, and think of this as an genus n surface

then iteratively use the neck cutting relation two holes down from the end (along dotted line) to reduce Σ' to some monomial in α. Multiplying the results of the connected components of Σ, we obtain a monomial in α.

Theorem 4.4. *The homology $Kh^\alpha(L)$ of a link L is a link invariant.*

The proof of this is exactly the same as in [BN07]: compute and simplify the complexes corresponding to each side of the Reidemeister move, and one gets the same result for both sides of any given Reidemeister move, and hence we have invariance.

The complex $Kh^\alpha(L)$ is an object in the category of graded complexes of $\mathbb{Q}[\alpha]$-modules $\mathsf{GKom}(\mathbb{Q}[\alpha]-\mathsf{Mod})$, for which we have the following Definition and Lemma which gives a version of 'Gaussian elimination' (see Proposition 2.5) for $\mathbb{Q}[\alpha]$-modules, found in [HKLM15].

Definition 4.5. Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is a Krull-Schmidt category if it is semisimple (i.e. can be written as a finite sum of indecomposable objects) and for any two decompositions into simple objects

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n = N_1 \oplus N_2 \oplus \cdots \oplus N_n,$$

there exists a permutation σ of $\{1, \ldots, n\}$ such that $N_i \cong M_{\sigma(i)}$.

Lemma 4.2. *The homotopy category $\mathsf{GKom}(\mathbb{Q}[\alpha] - \mathsf{Mod})$ of finite length graded complexes of $\mathbb{Q}[\alpha]$-modules is Krull-Schmidt.*

Proof. Take a complex $C_\bullet$ whose objects are direct sums of $\mathbb{Q}[\alpha]\{r\}$, i.e. a polynomial ring over the formal variable α with q-grading shifts, and whose differentials have degree zero. Since the maps are degree zero, the differentials must be matrices of monomials. Explicitly, a degree zero map

$$\bigoplus_{j\in J} \mathbb{Q}[\alpha]\{r_j\} \to \bigoplus_{k\in K} \mathbb{Q}[\alpha]\{r'_k\}$$

is a matrix whose (j,k) entry is a degree zero map $\mathbb{Q}[\alpha]\{r_j\} \to \mathbb{Q}[\alpha]\{r'_k\}$. That is, multiplication by $z\alpha^{r_j - r'_k}$ for some $z \in \mathbb{Q}$.

We will first show that we can break down $C_\bullet$ into a direct sum of copies of

$$0 \to \mathbb{Q}[\alpha] \to 0$$

and

$$0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^r} \mathbb{Q}[\alpha] \to 0$$

for $r \geq 0$ at various homological and q-gradings.

If the differentials of $C_\bullet$ are zero, then we are done. Suppose the differential d_n is nonzero (has a nonzero entry). Take the nonzero entry with lowest degree monomial. By permuting direct sums, we may assume that this entry is the top left entry,

$$d_n = \begin{pmatrix} z\alpha^{r_j - r'_k} & \delta \\ \gamma & \epsilon \end{pmatrix},$$

where $z\alpha^{r_j - r'_k}$ divides each other entry. Because $z\alpha^{r_j - r'_k}$ divides each other entry, we can do a version of Gaussian elimination described in Proposition 2.5. Hence, the following portion of chain complex

$$\cdots \longrightarrow A \xrightarrow{\begin{pmatrix} \xi \\ \zeta \end{pmatrix}} \mathbb{Q}[\alpha]\{r\} \oplus B \xrightarrow{\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}} \mathbb{Q}[\alpha]\{s\} \oplus C \xrightarrow{\begin{pmatrix} \mu & \nu \end{pmatrix}} D \longrightarrow \cdots$$

is isomorphic to the direct sum of complexes:

$$\begin{tikzcd}
 & & \mathbb{Q}[\alpha]\{r\} \arrow[rr, "z\beta^{r-s}"] & & \mathbb{Q}[\alpha]\{s\} & & \\
 & & \oplus & & \oplus & & \\
\cdots \arrow[r] & A \arrow[r, "\zeta"'] & B \arrow[rr, "\epsilon-\gamma(z^{-1}\alpha^{s-r})\delta"'] & & C \arrow[r, "\nu"'] & D \arrow[r] & \cdots
\end{tikzcd}$$

Thus we get the desired decomposition. Moreover, the resulting complex is homotopy equivalent to grading shifted copies of

$$0 \to \mathbb{Q}[\alpha] \to 0$$

and

$$0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^r} \mathbb{Q}[\alpha] \to 0$$

for $r > 0$, since

$$0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^0} \mathbb{Q}[\alpha] \to 0$$

is contractible because α^0 is invertible.

Lastly, to show that the homotopy category of $\mathsf{GKom}(\mathbb{Q}[\alpha] - \mathsf{Mod})$ is Krull-Schmidt, we need to show that each decomposition is unique up to homotopy. Suppose we have two homotopic complexes

$$C_\bullet = C_0 \otimes (0 \to \mathbb{Q}[\alpha] \to 0) \oplus \bigoplus_{i\geq 1} C_i \otimes \left(0 \to \mathbb{Q}[\alpha] \xrightarrow{\beta^i} \mathbb{Q}[\alpha] \to 0\right)$$
$$C'_\bullet = C'_0 \otimes (0 \to \mathbb{Q}[\alpha] \to 0) \oplus \bigoplus_{i\geq 1} C'_i \otimes \left(0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^i} \mathbb{Q}[\alpha] \to 0\right)$$

where C_i and C'_i are doubly graded vector spaces. Tensoring with $\mathbb{Q}[\alpha]/(\alpha - 1)$, we have

$$C_0 \cong H(C \otimes \mathbb{Q}[\alpha]/(\alpha - 1)) \cong H(C' \otimes \mathbb{Q}[\alpha]/(\alpha - 1)) \cong C'_0.$$

Next consider tensoring with $\mathbb{Q}[\alpha]/(\alpha^r)$. We firstly compute

$$H\left(\left(0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^i} \mathbb{Q}[\alpha] \to 0\right) \otimes \mathbb{Q}[\alpha]/(\alpha^r)\right) = H\left(0 \to \mathbb{Q}[\alpha](\alpha^r) \xrightarrow{\alpha^i} \mathbb{Q}[\alpha]/(\alpha^r) \to 0\right)$$
$$= \begin{cases} \mathbb{Q}^r \oplus t\mathbb{Q}^r & i \geq r \\ \mathbb{Q}^{r-i} \oplus t\mathbb{Q}^{r-i} & i < r \end{cases}$$

Thus,

$$H(C_\bullet \otimes \mathbb{Q}[\alpha]/(\alpha^r)) = C_0 \otimes \mathbb{Q}^r \oplus \bigoplus_{i\geq r} C_i \otimes (1+t)\mathbb{Q}^r \oplus \bigoplus_{r>i\geq 1} C_i \otimes (1+t)\mathbb{Q}^{r-i}$$

by using the result of Example 3.8. Hence, formally, we have

$$H(C_\bullet \otimes \mathbb{Q}[\alpha]/(\alpha^{r+1})) - H(C_\bullet \otimes \mathbb{Q}[\alpha]/(\alpha^r)) = C_0 \oplus (1+t) \bigoplus_{i\geq r} C_i,$$

and thus

$$-H(C_\bullet \otimes \mathbb{Q}[\alpha]/(\alpha^{r+2})) + 2H(C_\bullet \otimes \mathbb{Q}[\alpha]/(\alpha^{r+1})) - H(C_\bullet \otimes \mathbb{Q}[\alpha]/(\alpha^r)) = (1+t)C_r$$

which allows us to compute the graded dimensions of the C_i and this is independent of homotopy, thus $C_i \cong C'_i$. □

Remark 4.6. Note that in the above proof, we need z to be invertible, which is why we chose earlier to work over $\mathbb{Q}$.

4.3 Comparing Kh^α with the Lee Spectral sequence

Given $CKh^\alpha(L)$ for a link L, we have seen this specialises to the Khovanov and Lee complexes by tensoring with certain rings. So we may ask how $Kh^\alpha(L)$ is related to the Lee spectral sequence. In the case of the Lee chain complex, where the objects are graded but the differentials are only filtered, instead of using this to define a spectral sequence as we did before, we can correct the grading first by working over a polynomial ring where the formal variable has the desired degree to correct the grading.

4.3.1 Tensoring with a Polynomial Ring

Following Chapter 3, we denote filtrations by F_pC_q in this section, but there is a slight difference here: we will be starting with a graded vector space (rather than a filtered one) and then defining a filtration induced by this grading. That is, for a graded vector space $C_q = \bigoplus_i C_{i,q}$, it is filtered according to $F_pC_q := \bigoplus_{i\leq p} C_{i,q}$.

Suppose $(C_\bullet, d_\bullet)$ is a complex of filtered vector spaces, i.e. the differential $d_\bullet$ preserves the filtration:

$$d_q(F_pC_q) \subseteq F_pC_{q-1}.$$

From this, we can produce a spectral sequence $E^r_{p,q}$, and of course we have the $E^0_{p,q}$ page

$$C_{p,q} = \frac{F_pC_q}{F_{p-1}C_q}.$$

Another thing we can do is tensor the vector spaces C_q with a polynomial ring $\Bbbk[\alpha]$ (where $\Bbbk$ is a field — we will use $\mathbb{Q}$ later on) and then, using this α, we correct the grading of the maps between the graded objects so they are not only filtered but graded. To do this, we define a new differential which is a map of $\Bbbk[\alpha]$-modules,

$$d^\alpha_q : C_q \otimes \Bbbk[\alpha] \to C_{q-1} \otimes \Bbbk[\alpha].$$

Whether or not C_q was a graded vector space, $C_q \otimes \Bbbk[\alpha]$ is now a graded vector space. Choose α to have degree 1 (for what we will do with Lee homology, we will choose the degree of α to be -4). For example, if $x_p \in F_pC_q$, then $x \otimes \alpha^k \in F_{p+k}\left(C_q \otimes \Bbbk[\alpha]\right)$. Then we define d^α by the following. For $x_p \in F_pC_q$, where $d_q(x) = \sum_{i\leq p} y_{p-i}$ for $y_i \in F_iC_{q-1}$ (so $d_q(x_p) \in F_pC_{q-1}) = \bigoplus_{i\leq p} C_{i,q-1}$), then

$$d^\alpha(x \otimes 1) := \sum_{i\leq p} y_{p-i} \otimes \alpha^i.$$

Indeed, d^α preserves the gradings: for $y_{p-i} \in F_{p-i}C_{q-1}$,

$$\deg(y_{q-i} \otimes \alpha^i) = \deg(y_{q-i}) + \deg(\alpha^i) = q - i + i = q.$$

It is also a differential: suppose $d_q(x_p) = \sum_{i\leq p} y_{p-i}$ and $d_{q-1}(y_{p-i}) = \sum_{j\leq p-i} z_{p-i-j}$, so

$$\sum_{i\leq p}\sum_{j\leq p-i} z_{p-i-j} = 0.$$

Then

$$\begin{aligned}(d^\alpha)^2(x\otimes 1) &= d^\alpha\left(\sum_{i\le p} y_{p-i}\otimes\alpha^i\right)\\ &= \sum_{i\le p} d^\alpha(y_{p-i}\otimes\alpha^i)\\ &= \sum_{i\le p}\sum_{j\le p-i} z_{p-i-j}\otimes\alpha^{i+j}\\ &= 0.\end{aligned}$$

The last equality is by $d_{q-1}d_q = 0$ of the original differentials, because the composition of maps is the same, but we have just changed the gradings.

If we want α to have a different degree, say $\deg(\alpha) = s$ for some $s \in \mathbb{Z}$, then $\deg(x\otimes\alpha^i) = \deg(x)+is$.

Example 4.7. Suppose we have the following map $d : C_0 \to C_{-1}$ of graded vector spaces that does not preserve the grading,

$$\begin{array}{ccc}\mathbb{Q} & \xrightarrow{1} & \mathbb{Q}\\ \oplus & & \oplus\\ \mathbb{Q} & \xrightarrow{1} & q^{-1}\mathbb{Q}\end{array}$$

where $C_0 = \mathbb{Q}\oplus\mathbb{Q}$ and $C_{-1} = \mathbb{Q}\oplus q^{-1}\mathbb{Q}$ where q^i means the vector space is in degree i. So we have the following:

$$\begin{aligned}F_0C_0 &= \mathbb{Q}\oplus\mathbb{Q} &\qquad F_0C_{-1} &= \mathbb{Q}\oplus\mathbb{Q}\\ F_{-1}C_0 &= 0, &\qquad F_{-1}C_{-1} &= \mathbb{Q}\end{aligned}$$

Indeed, d really is a differential on $F_\bullet C_q$, since $d(F_{-1}C_0) \subseteq F_{-1}C_{-1}$ and $d(F_0C_0) \subseteq F_0C_{-1}$. Now take $d^\alpha : C_0\otimes\mathbb{Q}[\alpha] \to C_{-1}\otimes\mathbb{Q}[\alpha]$, and let $\deg(\alpha) = 1$. So, when C_0 and C_{-1} are decomposed by degrees as below, d^α preserves the gradings:

$$\begin{array}{lccc} & C_0\otimes\mathbb{Q}[\alpha] & \xrightarrow{d^\alpha} & C_{-1}\otimes\mathbb{Q}[\alpha]\\ & \| & & \|\\ \text{degree -1} & 0 & & \mathbb{Q}\\ & \oplus & & \oplus\\ \text{degree 0} & \mathbb{Q}\oplus\mathbb{Q} & & \mathbb{Q}\oplus\mathbb{Q}\langle\alpha\rangle\\ & \oplus & & \oplus\\ \text{degree 1} & \mathbb{Q}\langle\alpha\rangle\oplus\mathbb{Q}\langle\alpha\rangle & & \mathbb{Q}\langle\alpha\rangle\oplus\mathbb{Q}\langle\alpha^2\rangle\\ & \oplus & & \oplus\\ \text{degree 2} & \mathbb{Q}\langle\alpha^2\rangle\oplus\mathbb{Q}\langle\alpha^2\rangle & & \mathbb{Q}\langle\alpha^2\rangle\oplus\mathbb{Q}\langle\alpha^3\rangle\\ & \oplus & & \oplus\\ \vdots & \vdots & & \vdots\end{array}$$

Take (x, y) in the degree 0 piece $\mathbb{Q}\oplus\mathbb{Q}$ of $C_0\otimes\mathbb{Q}[\alpha]$ and $(1_x, 1_y)$ that sits in the degree 0 piece of $C_{-1}\otimes\mathbb{Q}[\alpha]$. So $d_\alpha(x) = x1_x$ and $d^\alpha(y) = y1_y\alpha$. Thus,

$$d^\alpha = \begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix}.$$

In general, after ordering the bases with respect to the grading, a filtered complex of graded spaces will be a differential that is upper (or lower) triangular, and d^α will be a block diagonal matrix.

4.3.2 Correcting the Grading of the Lee Complex

Following Theorem 4.1, since $d^{Lee} = d^{Kh} + \delta$, where δ has degree $+4$, we correct the grading with α, where $\deg(\alpha) = -4$, so $d^\alpha = d^{Kh} + \alpha\delta$. As usual, by setting $\alpha = 0$ we obtain the Khovanov differential and hence the Khovanov complex CKh, and by setting $\alpha = 1$ we obtain the Lee differential and hence the Lee complex CKh^{Lee}.

By the same argument as Lemma 4.2, whenever we have a complex in $CKh^\alpha(L)$, it will be a direct sum of E^αs and C_n^αs, where

$$E^\alpha = 0 \to \mathbb{Q}[\alpha] \to 0,$$

and

$$C_n^\alpha = 0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^n} q^{4n}\mathbb{Q}[\alpha] \to 0.$$

We can consider these summands of chain complexes as originally being a filtered chain complex and having had their grading corrected by α. So for

$$C_n^\alpha = 0 \to \mathbb{Q}[\alpha] \xrightarrow{\alpha^n} q^{4n}\mathbb{Q}[\alpha] \to 0$$

tensoring with $\mathbb{Q}[\alpha]/(\alpha - 1) = 0$ gives the summand of the Lee complex

$$0 \to \mathbb{Q} \xrightarrow{1} q^{4n}\mathbb{Q} \to 0$$

which is not graded, but it is filtered

$$\begin{array}{rccc}
 & 0 & \xrightarrow{0} & 0 \\
 & \cap & & \cap \\
\text{degree } 4n \dashrightarrow & 0 & \xrightarrow{0} & \mathbb{Q} \\
 & \cap & & \cap \\
 & \vdots & & \vdots \\
 & \cap & & \cap \\
 & 0 & \xrightarrow{0} & \mathbb{Q} \\
 & \cap & & \cap \\
\text{degree } 0 \dashrightarrow & \mathbb{Q} & \xrightarrow{1} & \mathbb{Q} \\
 & \cap & & \cap \\
 & \mathbb{Q} & \xrightarrow{1} & \mathbb{Q} \\
 & \cap & & \cap \\
 & \vdots & & \vdots
\end{array}$$

For this filtered complex, the associated graded page E^0 and subsequent pages are of the form

$$E^0 : \quad \begin{array}{rccc}
 & 0 & \longrightarrow & 0 \\
\text{degree } 4n \dashrightarrow & 0 & \longrightarrow & \mathbb{Q} \\
 & \vdots & & \vdots \\
 & 0 & \longrightarrow & 0 \\
\text{degree } 0 \dashrightarrow & \mathbb{Q} & \longrightarrow & 0 \\
 & 0 & \longrightarrow & 0 \\
 & \vdots & & \vdots
\end{array}$$

$$E^1 : \quad \begin{tikzcd}
 & 0 & 0 \\
\text{degree } 4n \arrow[r, dashed] & 0 \arrow[ur] & \mathbb{Q} \\
 & \vdots & \vdots \\
 & 0 \arrow[ur] & 0 \\
\text{degree } 0 \arrow[r, dashed] & \mathbb{Q} \arrow[ur] & 0 \\
 & 0 \arrow[ur] & 0 \\
 & \vdots & \vdots
\end{tikzcd}$$

The first nonzero differential appears on the E^{4n} page and it is of the form

$$0 \to \mathbb{Q} \to \mathbb{Q} \to 0,$$

and since the C_ns vanish in Lee homology (because they are null-homotopic) the above complex must be exact and $E^{4n+1} = E^{\infty} = 0$. In fact, by tracing our way through the definitions of spectral sequences, this must be the identity map because the differentials on the $4n$th page are induced by the restriction of the differentials of the original filtered complex to the $4n$-almost cycles. That is, $d^{4n} : Z^{4n}_{p,q} \to Z^{4n}_{p-4n,q+4n-1}$ is induced by d on the original complex, so for $[x] \in E^{4n}_{p,q}$, $d^{4n}([x]) = [dx] = [x]$.

Proposition 4.1. *There is an equivalence of categories between the category $\mathcal{C}$ of graded complexes over $\mathbb{Q}$ with only filtered differentials and the category $\mathcal{C}_\alpha$ of graded complexes of free $\mathbb{Q}[\alpha]$-modules.*

Proof. Denote the functors $(-)^{\alpha} : \mathcal{C} \to \mathcal{C}_\alpha$ and it's inverse $(-)^{\alpha^{-1}} : \mathcal{C}_\alpha \to \mathcal{C}$. The $(-)^{\alpha}$ functor is going to be the process of correcting the gradings using a formal variable α, and for the inverse functor $(-)^{\alpha^{-1}} = - \otimes \mathbb{Q}[\alpha]/\alpha = 1$. The map $(-)^{\alpha}$ is functorial because $(d^{\alpha})^2 = 0$. To see that this gives an equivalence of categories, take a complex $(V_\bullet, d_\bullet)$ in $\mathcal{C}$. Then $(V_\bullet, d_\bullet)^{\alpha} = (V_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\alpha], d^{\alpha}_\bullet)$, where

$$\begin{aligned} d^{\alpha}_\bullet = \sum d^{\alpha}_{\bullet,i,j} : V_{\bullet,i} \otimes \mathbb{Q}[\alpha] &\to V_{\bullet+1,j} \otimes \mathbb{Q}[\alpha] \\ v \otimes p &\mapsto \sum (d_{\bullet,i,j} v) \otimes \alpha^{j-i} p. \end{aligned}$$

Applying $(-)^{\alpha^{-1}}$,

$$(V_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\alpha], d^{\alpha}_\bullet)^{\alpha^{-1}} = (V_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\alpha] \otimes_{\mathbb{Q}[\alpha]} \mathbb{Q}[\alpha]/\alpha = 1, (d^{\alpha}_\bullet)^{\alpha^{-1}}) \cong (V_\bullet, (d^{\alpha}_\bullet)^{\alpha^{-1}})$$

where

$$(d^{\alpha}_\bullet)^{\alpha^{-1}} : v \otimes p \otimes q \mapsto d^{\alpha}_\bullet(v \otimes p) \otimes q = \sum d_{\bullet,i,j} v \otimes \alpha^{j-i} p \otimes q = d_{\bullet,i,j} v \otimes p \otimes q.$$

The last equality is since we are tensoring over $\mathbb{Q}[\alpha]$, so $\alpha^{j-i} p \otimes_{\mathbb{Q}[\alpha]} q = p \otimes_{\mathbb{Q}[\alpha]} \alpha^{j-i} q = p \otimes_{\mathbb{Q}[\alpha]} q$ because the last factor of the tensor product is in the ring $\mathbb{Q}[\alpha]/(\alpha - 1 = 0)$. So $(d^{\alpha}_\bullet)^{\alpha^{-1}}$ as a map from $V_\bullet \to V_{\bullet+1}$ is

$$v \mapsto v \otimes 1 \otimes 1 \mapsto d_{\bullet,i,j} v \otimes 1 \otimes 1 \mapsto 1.d_{\bullet,i,j} v.$$

Thus $((-)^{\alpha})^{\alpha^{-1}} \cong \mathrm{Id}_{\mathcal{C}}$. A similar argument shows $((-)^{\alpha^{-1}})^{\alpha} \cong \mathrm{Id}_{\mathcal{C}^{\alpha}}$, and hence the functors $(-)^{\alpha}$ and $(-)^{\alpha^{-1}}$ are inverse to one another. □

Proposition 4.2. *The category $\mathcal{C}$ is Krull-Schmidt, with summands $\tilde{E}$ and $\tilde{C}_n$ where $\tilde{E} = \mathbb{Q}$ and $\tilde{C}_n$ is the filtered complex $V_{j,0} \xrightarrow{id} V_{j,1}$, where*

$$V_{j,0} = \begin{cases} \mathbb{Q} & \text{if } j \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$V_{j,1} = \begin{cases} \mathbb{Q} & \text{if } j \leq n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Clearly, $(\tilde{C}_n)^{\alpha} = C^{\alpha}_n$. Do Gaussian elimination from Lemma 4.2 in $\mathcal{C}_\alpha$, and then apply $(-)^{\alpha^{-1}}$. □

Thus, as we can decompose $(V_\bullet, d_\bullet)^{\alpha}$ into a direct sum of E^{α}s and C^{α}_ns, the original complex $(V_\bullet, d_\bullet)$ is homotopic to the corresponding direct sum of $\tilde{E}$ and $\tilde{C}_n$. Since $CKh^{Lee} \in \mathcal{C}$ and $CK^{\alpha} \in \mathcal{C}^{\alpha}$ the correspondence between CKh^{Lee} and CKh^{α} is very close. In fact, we can stay in the category $\mathcal{C}^{\alpha}$ and read off the Lee spectral sequence, which is the content of the next section.

4.3.3 The Comparison

As we observed earlier $CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} \mathbb{Q}[\alpha]/(\alpha = 0)$ is the Khovanov complex $CKh(L)$, and $CKh(L)^\alpha \otimes_{\mathbb{Q}[\alpha]} \mathbb{Q}[\alpha]/(\alpha - 1 = 0)$ is the Lee complex $CKh^{Lee}(L)$, so

$$\begin{aligned} H_\bullet\left(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} \mathbb{Q}[\alpha]/(\alpha = 0)\right) &\cong Kh(L) = E^1 \\ H_\bullet\left(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} \mathbb{Q}[\alpha]/(\alpha - 1 = 0)\right) &\cong Kh^{Lee}(L) = E^\infty \end{aligned}$$

Theorem 4.8. *Let R_k denote the truncated polynomial ring $\mathbb{Q}[\alpha]/(\alpha^k = 0)$. Then the sequence $H_\bullet\left(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k\right)_{k \geq 0}$ completely encodes the Lee spectral sequence.*

Proof. Take an arbitrary complex

$$CKh^\alpha(L) \simeq p_E(t,q)E^\alpha \oplus \bigoplus_i p_i(t,q)C_i^\alpha.$$

Then,

$$CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k \simeq p_E(t,q)R_k \oplus \bigoplus_i p_i(t,q)R_k^i,$$

for the complex $R_k^i := R_k \xrightarrow{\alpha^i} R_k$, since the tensor product distributes over the direct summands. To compute homology of $CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k$, we firstly compute

$$\begin{aligned} H_\bullet(R_k) &\cong R_k \\ H_\bullet(R_k \xrightarrow{\alpha^i} q^{4i}R_k) &\cong \begin{cases} q^{4(k-i)}R_i \oplus q^{4i}R_i & \text{for } i < k \\ R_k \oplus q^{4i}R_k & \text{for } i \geq k \end{cases} \end{aligned}$$

Thus, with grading shifts,

$$H_\bullet(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k) \cong p_E(t,q)R_k \oplus \bigoplus_{i<k} p_i(t,q)(q^{4(k-i)}R_i \oplus q^{4i}R_i) \oplus \bigoplus_{i \geq k} p_i(t,q)(R_k \oplus q^{4i}R_k).$$

This is not as terrible as it looks. For example, suppose $CKh^\alpha(L) = p_E E^\alpha \oplus p_1 C_1^\alpha \oplus p_2 C_2^\alpha$. Then for $k = 1, 2, \ldots$, $H_\bullet(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k)$ is isomorphic to

$$\begin{array}{rl|ccccc} k=1: & p_E R_1 & \oplus & p_1(R_1 \oplus q^4 R_1) & \oplus & p_2(R_1 \oplus q^8 R_1) \\ k=2: & p_E R_2 & \oplus & p_1(q^4 R_1 \oplus q^4 R_1) & \oplus & p_2(R_2 \oplus q^8 R_2) \\ k=3: & p_E R_3 & \oplus & p_1(q^8 R_1 \oplus q^4 R_1) & \oplus & p_2(q^4 R_2 \oplus q^8 R_2) \\ k=4: & p_E R_4 & \oplus & p_1(q^{12} R_1 \oplus q^4 R_1) & \oplus & p_2(q^8 R_2 \oplus q^8 R_2) \\ \vdots & \vdots & & \vdots & & \vdots \end{array}$$

By looking at the above example, we notice that for $C_n^\alpha \otimes_{\mathbb{Q}[\alpha]} R_k$ the q-gradings remain constant for $k \leq n$, and for $k > n$, the q-grading of the first summand starts increasing by $4(k-n)$ (we have drawn a red line that separates $k = n$ and $k = n+1$ in each column).

This corresponds directly to the spectral sequence as described in Section 4.3.2. Take a summand of the Lee complex

$$0 \to \mathbb{Q} \xrightarrow{1} q^{4n}\mathbb{Q} \to 0,$$

then the E^kth page of the spectral sequence is a bigraded $\mathbb{Q}[\alpha]$-module with nonzero entries $\mathbb{Q}$ in bidegree $(1, 4n)$ for $n < k$. For $k = 4n$, there is a differential between the two $\mathbb{Q}$s that is an isomorphism, and on the $k > 4n+1$st page these $\mathbb{Q}$s disappear. Tensoring each page of the spectral sequence by R_k (over $\mathbb{Q}[\alpha]$), we instead have a R_k in each entry where there was a $\mathbb{Q}$. For $k \leq 4n$, the R_k on the kth page of the spectral sequence are precisely in the homological and q-grading specified by $H_\bullet(C_n^\alpha \otimes_{\mathbb{Q}[\alpha]} R_k)$, and for $k > 4n$, the R_k disappear. The point where the R_k disappear corresponds to the point where the q-gradings in the first summand starts increasing — that is, at $H_\bullet(C_n^\alpha \otimes_{\mathbb{Q}[\alpha]} R_{n+1})$.

To summarise, let D be the 'stable' part of the triply graded module, i.e. the part before the q-gradings start increasing (everything above and to the left of the red line). Then

$$H_\bullet(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k) \cap D \cong E^k \otimes_{\mathbb{Q}[\alpha]} R_k$$

So the sequence

$$\big(H_\bullet(CKh^\alpha(L) \otimes_{\mathbb{Q}[\alpha]} R_k)\big)_{k \geq 0}$$

completely encodes the pages of the Lee spectral sequence. □

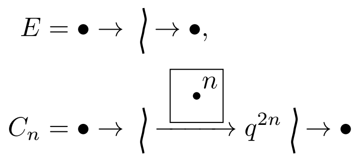
Thus, we can stay in the category of graded complexes of free $\mathbb{Q}[\alpha]$-modules and read off the spectral sequence. We already know that $CKh^\alpha(L)$ is a link invariant (up to homotopy) by Theorem 4.4, and since $CKh^\alpha(L)$ encodes the Lee spectral sequence, the isomorphism class of the pages of spectral sequence are link invariants, too. While it is not immediately obvious that something in $\mathsf{GKom}(\mathbb{Q}[\alpha] - \mathsf{Mod})$ is a useful invariant, Theorem 4.8 shows that the equivalence classes of two links $[L_1]$ and $[L_2]$, the equality $[L_1] = [L_2]$ is decidable by doing finitely much linear algebra, because finitely many of the $H_\bullet([L] \otimes R_k)$ determine the homotopy type of $[L]$.

4.4 Slice Genus and obtaining genus bounds from the s-invariant

An important result obtained from the Lee spectral sequence, first proved by Rasmussen [Ras10], is that the Lee spectral sequence allows us to compute the s-invariant, which gives lower bounds on the slice genus of knots. This will be the subject of the remainder of this thesis.

4.4.1 Some Notation

We briefly introduce some notation to be used heavily in the remainder of this thesis. We define E and C_n to be the complexes of $\mathbb{Q}[\alpha]$-modules obtained from $BN(T)$ for a (1,1)-tangle T



and E' and C'_n to be the complexes of $\mathbb{Q}[\alpha]$-modules

$$E' = \bullet \to \emptyset \to \bullet,$$
$$C'_n = \bullet \to \emptyset \xrightarrow{\alpha^n} q^{4n}\emptyset \to \bullet.$$

Thus the computation in Example 2.29 for $BN(3_1)$ shows that before we take the final trace, the cut open trefoil $c(3_1)$ has $BN(c(3_1)) \simeq q^2t^0E \oplus q^6t^2C_1$, where t keeps track of the homological height. If we take the trace, $BN(3_1) \simeq qE' \oplus q^3E' \oplus t^2q^5C'_1$.

The complexes E, C_n and E', C'_n are related by taking the trace and delooping:

$$tr(E) = \bigcirc \cong q^{-1}\emptyset \oplus q\emptyset \cong q^{-1}E' \oplus qE',$$
$$tr(C_n) = \bigcirc \xrightarrow{c_n} \bigcirc \cong q^{-1}\emptyset \oplus q\emptyset \xrightarrow{\begin{pmatrix} s^n & s^{n-1} \\ s^{n+1} & s^n \end{pmatrix}} q^{2n-1}\emptyset \oplus q^{2n+1}\emptyset,$$

where c_n is a cylinder with n dots on it, and s^n is a 2-sphere with n dots on it. Since $s^{2m+1} = \alpha^m$ (i.e. the sphere has an odd number of dots on it) and $s^{2m} = 0$ (i.e. even number of dots), we conclude that

$$s^n = \begin{cases} \alpha^{\frac{n-1}{2}} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Thus,

$$tr(C_n) \cong \begin{cases} q^{-1}C'_{\frac{n-1}{2}} \oplus qC'_{\frac{n-1}{2}} & \text{if } n \text{ odd,} \\ q^{-1}C'_{\frac{n}{2}} \oplus qC'_{\frac{n-2}{2}} & \text{if } n \text{ even.} \end{cases}$$

Remark 4.9. The E's are called 'pawn moves' (they always come in pairs and have a q-grading 2 apart) and C'_1 is called a 'knight's move' (it is homological height $+1$ and q-grading $+4$ apart, so it has bidegree (1,4)). It was until earlier this year that it was conjectured (the 'Knight's Move Conjecture') that the complex C'_n for $n \geq 2$ never appears in the complexes for tangles. However a C'_2 was found by Manolescu and Marengon earlier this year in [MM18] for a 38 crossing knot.

4.4.2 Defining the s-invariant

Theorem 4.10. *For a (1,1)-tangle T, the complex $BN(T)$ is isomorphic to a direct sum of complexes E and C_n.*

Proof. From a (1,1)-tangle T, we obtain a complex which will have at each vertex of the cube of resolutions a complete smoothing of T which will be a disjoint union of a single strand and circles. After delooping, we obtain a complex which for different homological heights in the complex will have a direct sum of single strands in different gradings. That is, $BN(T)$ is isomorphic to a complex such as

$$\begin{tikzcd} & & q^b \wr \arrow[rdd] & & & \\ & & \oplus & & & \\ \bullet \arrow[r] & q^a \wr \arrow[ruu] \arrow[r] \arrow[rdd] & q^c \wr \arrow[r] & q^e \wr \arrow[r] & q^g \wr \arrow[r] & \bullet \\ & & \oplus & \oplus & & \\ & & q^d \wr \arrow[r] & q^f \wr \arrow[ruu] & & \end{tikzcd}$$

By Theorem 2.20, each of the differentials in $BN(T)$ must have degree zero, so each differential is a matrix whose entries are scalars and monomials in $\bullet$ (otherwise the image of the differentials would be inhomogeneous). Explicitly, the degree zero maps are given by the following. Take a map $q^r \wr \to q^s \wr$. Such a cobordism will contain a sheet with possibly zero dots on it and a possibly empty set of unconnected components of closed surfaces. Thus,

$$\mathrm{Hom}(q^r \wr \to q^s \wr) \cong \mathbb{Q}[\alpha]\{\square, \boxed{\bullet}\} = \mathbb{Q}\{\square, \alpha\,\square, \alpha^2\,\square, \dots, \boxed{\bullet}, \alpha\,\boxed{\bullet}, \alpha^2\,\boxed{\bullet}, \dots\}.$$

The degree of the map is then given by

$$\deg(\alpha^p\,\square) = s - r - 4p$$
$$\deg(\alpha^p\,\boxed{\bullet}) = s - r - 2 - 4p$$

for some $p \in \mathbb{Z}_{\geq 0}$.

To find the degree zero map, we need $s - r - 4p = 0$ or $s - r - 4p - 2 = 0$. So,

- if $s \neq r \mod 2$, then the map with degree zero can only be the zero map (multiplication by zero);
- if $s < r$, then $s - r - 4p < 0$, so the map with degree zero is again the zero map;
- if $s > r$ and $s - k = 0 \mod 4$, then $\deg(\mathbb{Q}\{\alpha^{\frac{s-r}{4}} \square\}) = 0$;
- if $s > r$ and $s - k = 2 \mod 4$, then $\deg(\mathbb{Q}\{\alpha^{\frac{s-r-2}{4}} \boxed{\bullet}\}) = 0$.

So each differential has entries that are zero or are monomials in $\bullet$ in the above form. Now, by repeatedly using the Gaussian elimination of Lemma 4.2, we keep bubbling off C_n's until we are only left with E's at various homological heights. $\square$

Lemma 4.3. *The decomposition into a direct sum of E's and C_n's is unique.*

Proof. This is just a corollary of Lemma 4.2. $\square$

Fortunately, we know how many resulting E's appear in $BN(T)$ for a (1,1)-tangle by the following theorem first proved by Lee [Lee05] and proved 'locally' by Bar-Natan and Morrison [BNM06].

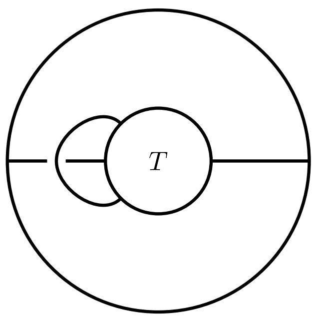
Theorem 4.11. *The number of E's is 2^{n-1} where n is the number of components of the 1-1 tangle.*

Thus, for a cut open knot, which is a connected two point tangle, there is a unique copy of E.

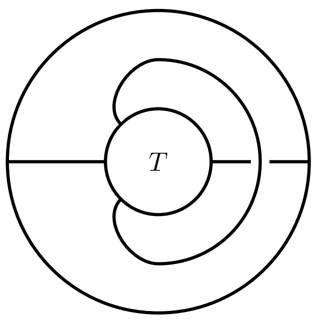
Definition 4.12. The s-invariant $s(K)$ of a knot K is the grading of the unique E of $BN(c(K))$, where $c(-)$ 'cuts open' a knot into a two point tangle.

Lemma 4.4. *The complex $BN(c(K))$ doesn't depend on where you cut.*

Proof. To see this, pick a basepoint t on the knot K (so we can think of K parametrised by time t, and give the knot orientation corresponding to time increasing) and denote the tangle cut at this point $c_t(K)$. Clearly, $BN(c_{t\pm\epsilon}(K))$ will not change for small enough ϵ, but it could change for a certain $t = t'$ where there is a crossing in the diagram for K. So we need to check that $BN(c_{t'+\epsilon}(K)) \cong B(c_{t'-\epsilon}(K))$. Cutting before the crossing gives



whereas cutting after the crossing gives



But $BN(-)$ is invariant under the Reidemeister moves by sweeping the strand over T, and finally performing two Reidemeister one moves, so $BN(-)$ of these two diagrams is homotopic equivalent (and have isomorphic homology). □

The above definition agrees with Rasmussen's definition of the s-invariant, which he defines as the average of the gradings of the E^α for Lee homology. Recall that Lee homology is given by

$$H_\bullet\left(CKh^\alpha(L)\otimes\mathbb{Q}[\alpha]/(\alpha-1=0)\right).$$

So for $q^{s(K)}E$, taking the trace $tr(q^{s(K)}E)=q^{s(K)-1}E'\oplus q^{s(K)+1}E'$, and thus

$$\begin{aligned}CKh^\alpha(K)&=\mathrm{Hom}_{\mathsf{BN}}(\emptyset, tr(q^{s(K)}E\oplus\bigoplus_i q^iC_{n_i}))\\&=q^{s(K)-1}\mathbb{Q}[\alpha]\oplus q^{s(K)+1}\mathbb{Q}[\alpha]\oplus\bigoplus_j q^jC_n^\alpha.\end{aligned}$$

Thus, by tensoring with $\mathbb{Q}[\alpha]/(\alpha-1=0)$ and taking homology,

$$Kh^{Lee}(K)\cong q^{s(K)-1}\mathbb{Q}\oplus q^{s(K)+1}\mathbb{Q},$$

since each $C_n^\alpha\otimes\mathbb{Q}[\alpha]/(\alpha-1=0)$ is contractible. Rasmussen defines $s(K)$ to be $\frac{a+b}{2}$ where a and b are the gradings of the copies of $\mathbb{Q}$ in Lee homology, i.e. $q^a\mathbb{Q}\oplus q^b\mathbb{Q}$. Here, we clearly have

$$\frac{s(K)-1+s(K)+1}{2}=s(K).$$

4.4.3 Slice Genus

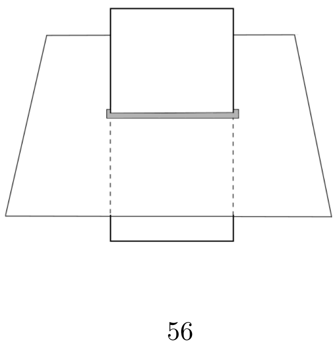
This section follows Cromwell's discussion in [Cro04].

Definition 4.13. The (smooth) slice genus or 4-ball genus of a link L is the minimum genus taken over all properly smoothly embedded orientable surfaces bounding L in the four-dimensional ball.

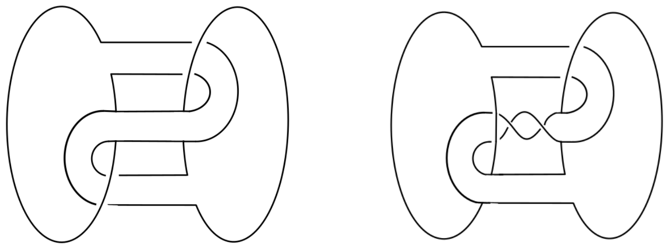
The properly embedded property means we think of L, which is the boundary of the surface bounding it, as a subset of the boundary of the four-ball, i.e. $L\subset\partial B^4=S^3$. The slice genus of a link L is denoted by $g_s(L)$ or $g_4(L)$.

To understand this, consider an example one dimension lower. Suppose two lines in $S^2=\partial B^3$ intersect transversely at a point. If part of one of the lines in a neighbourhood of the intersection is pushed into the interior of the ball B^3, then the intersection can be removed. Similarly, if two surfaces in $S^3=\partial B^4$ intersect transversely as in the figure below, then a neighbourhood of the intersection can be pushed into the 4-ball to remove the intersection.

Intersections of the form in the diagram are called *ribbon singularities*. An immersed surface in S^3 where all the intersections are ribbon singularities is called a ribbon surface. If we push all the intersections of a ribbon surface into a 4-ball, the surface can be made into a nonsingular embedded surface in $\mathbb{R}^4$.



For example consider two instances pictured below of the band sum of two unknots:



These knots can be spanned by disks connected by a band, and this strip must have ribbon singularities as it intersects each disk, thus giving us immersed disks bounding the knots. The ribbon surface is topologically a disk, so each knot bounds a disk properly embedded in B^4 and hence $g_s = 0$. We could construct more examples of ribbon surfaces of higher genus by adding more bands. Knots that are spanned by a disk with bands attached with ribbon singularities (i.e. knots bounded by a ribbon disk) in S^3 are called ribbon knots, and knots for which $g_s = 0$ are called *slice knots*. Indeed, every ribbon knot is slice.

Already it is clear that finding the slice genus of a knot is difficult, and we still do not know much about computing slice genus. For example, it is still unknown if every slice knot is actually ribbon. However, Rasmussen skilfully used the Lee spectral sequence to give us a lower bound on the slice genus of a knot. Moreover, finding a *lower* bound is the difficult inequality, and upper bounds are given by constructing explicit examples of surfaces.

4.4.4 Genus Bounds

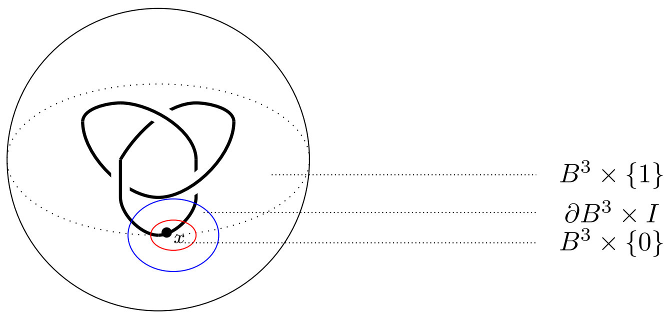
The following result was proved by Rasmussen [Ras10].

Theorem 4.14. *We have the following inequality*

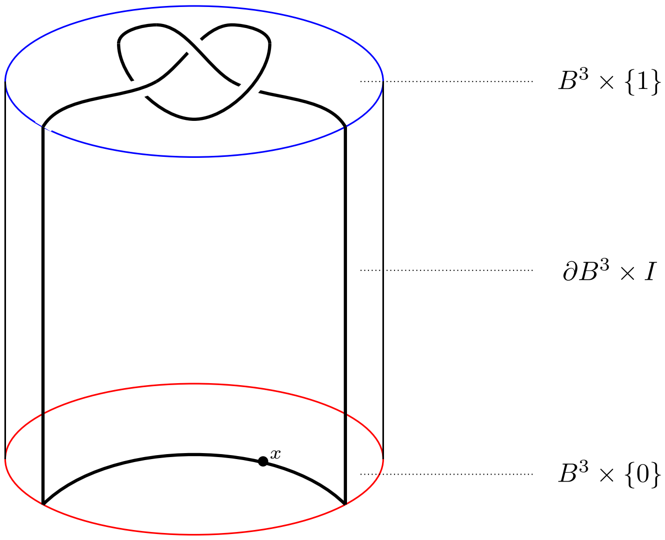
$$|s(K)| \leq 2g_4(K),$$

where $s(K)$ is the s-invariant for the knot K, and $g_4(K)$ is the slice (4-ball) genus for the knot K.

Proof. Take a surface Σ in B^4 bounding the knot $K \subset \partial B^4 = S^3$. Pick a basepoint $x \in K$ and parametrise B^4 as $B^3 \times I$ as follows:



which is homeomorphic to



where we compute the boundaries by the 'Leibniz rule':

$$\begin{aligned}\partial B^4 &= \partial(B^3 \times I) \\ &= ((\partial B^3) \times I) \cup (B^3 \times \partial I) \\ &= (S^2 \times I) \cup (B^3 \times \{0\}) \cup (B^3 \times \{1\}).\end{aligned}$$

As before, let $c(-)$ denote 'cut', so $c(U)$ is the cut unknot and $c(3_1)$ is the cut trefoil (which are in $B^3 \times \{0\}$ and $B^3 \times \{1\}$ in the above diagram, respectively). These are both $(1,1)$-tangles. The surface Σ induces a chain map $BN(c(U)) \to BN(c(3_1))$. By Theorem 4.10, $BN(c(K))$ is a direct sum of (grading shifted) Es and C_ns and there is exactly one copy of E, in grading $q^{s(K)}$.

Now, if Σ is a connected cobordism between two connected two point tangles $c(U)$ to $c(K)$, then the induced map $BN(\Sigma) : E \to q^{s(K)}E$ is nonzero. The proof for this is outside the scope of this thesis, but can be proved easily with the 'colour technology' introduced in [BNM06].

Recall that the degree of $BN(\Sigma)$ is given by the relative Euler characteristic. So in this case, since $c(U)$ and $c(K)$ are two point tangles, $\deg(BN(\Sigma)) = \chi(\Sigma) - \frac{1}{2}|B| = \chi(\Sigma) - 1$. The E for $c(U)$ is in q-grading 0, and by definition, the q-grading for the E for $c(K)$ is $s(K)$. So $BN(\Sigma)$ is some nonzero map

$$BN(\Sigma) : E \to q^{s(K)}E$$

of degree $\chi(\Sigma) - 1$. Since any map $E \to q^{s(K)}E$ is a (scalar multiple of a) sheet with $k \geq 0$ dots on it, and $\bullet$ has degree -2, $\chi(\Sigma) - 1 = s(K) - 2k$. Thus $\chi(\Sigma) = s(K) - 2k + 1$, and because $k \geq 0$,

$$\chi(\Sigma) \leq s(K) + 1.$$

Doing a similar computation for $BN(\Sigma) : q^{s(K)}E \to E$, we have

$$\chi(\Sigma) \leq -s(K) + 1.$$

Moreover, the Euler characteristic is given by $\chi(\Sigma) = -2g(\Sigma)$ (since $\chi = 2 - 2g - b$ where b is the the number of boundary components), so combining the previous bounds, we have

$$-2g_4(\Sigma) \leq -|s(K)|$$

and thus

$$2g_4(\Sigma) \geq |s(K)|.$$

and hence the s-invariant gives a lower bound for the slice genus. $\square$

For example, the s-invariant for the trefoil 3_1 is 2, and the slice genus $g_4(3_1) = 1$ (which can be found by looking at the Seifert surface of 3_1), so it does indeed satisfy $|s(3_1)| \leq 2g_4(K)$.

With a little more work, Theorem 4.14 then provides the following important corollaries, both found in [Ras10].

Corollary 4.1. *If K is a positive knot (that is, it admits a planar diagram with only positive crossings), $s(K)$ gives an equality $s(K) = 2g_4(K) = 2g(K)$, where g is the ordinary genus of the knot K.*

Another corollary is a combinatorial proof of the Milnor Conjecture, originally proved by Kronheimer and Mrowka using gauge theory in [KM93]:

Corollary 4.2. *(Milnor conjecture) The slice genus of the (p, q) torus knot is $(p-1)(q-1)/2$.*

4.4.5 Concluding Remarks

The Lee spectral sequence is a powerful tool: it contains the information of Khovanov homology, and also says something about 4-dimensional topology via the s-invariant. After some familiarisation, the equivariant Khovanov homology Kh^α is actually quite easy to work with, and finding the s-invariant from a knot is not so difficult by hand for knots with fewer crossings. Moreover, the pages of the Lee spectral sequence are encoded in this theory, which provides a way to bypass discussion of spectral sequences that are notoriously difficult. However, the comparison of Kh^α with the spectral sequence provides almost nothing new, and is perhaps a little underwhelming. This begs the questions:

- Is there a better way to view the Lee spectral sequence?
- How could we use this gadget to find more general versions of the s-invariant for arbitrary 4-manifolds?

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