Drinfeld centers for bimodule categories

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Except where otherwise indicated, this thesis is my own original work.

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To my mother Mellissa, and father Peter,
to whom I will be eternally grateful,
for everything.

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This has all been absolutely surreal, and I can't believe it's all over.

Abstract

We prove the folklore result that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a pivotal category $\mathcal{C}$ is contravariantly equivalent (as a braided monoidal category) to the category of representations of the annular category of $\mathcal{C}$, when $\mathcal{C}$ is finitely semisimple. This has only been briefly sketched in the literature previously.

Given bimodule categories ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ under mild hypotheses there are two general constructions we can make; first, we can form their Deligne product $\mathcal{M} \underset{\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{mop}}}{\boxtimes} \mathcal{N}$, which is a purely algebraic object. We can also consider representations of a "bimodule annular category", consisting of diagrams drawn in the annulus with equatorial boundaries labelled by ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$. We prove that the Deligne product and representations of the bimodule annular category are contravariantly equivalent in the finitely semisimple case. As corollaries we deduce characterisations of the bimodule Drinfeld center $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ and module functor categories $[{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}]$, each as representations of a special case of the bimodule annular category. To prove these equivalences we introduce a notion of "balanced tensor products" of module categories, which we show in particular gives a new model for the Deligne product $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ in the finitely semisimple case.

Finally, we use the balanced tensor product to define the notion of a bibalanced center $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ of an arbitrary bimodule category ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$, for which we establish a monoidal structure generalising the monoidal product in $\mathcal{Z}(\mathcal{C})$.

Contents

Chapter 0

Overview

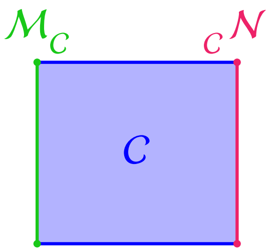
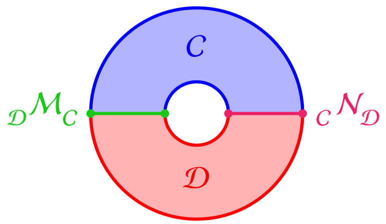
Bimodule categories are of great interest to both mathematicians studying tensor categories and physicists studying topological quantum computation. To a mathematician, one way to study a finite tensor category $\mathcal{C}$ is by calculating the equivalence classes of its invertible bimodules; this is the so called *Brauer-Picard group* $\mathrm{BrPic}(\mathcal{C})$ of $\mathcal{C}$. This group controls the group-graded extensions of $\mathcal{C}$ [11], and is equivalent to the category of *braided automorphisms* of the *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ [10]. For a fixed bimodule category $\mathcal{M}$, calculating the bimodule functors to another bimodule category $\mathcal{N}$ is another way to determine the structure of $\mathcal{M}$.

On the other hand, diagrammatic categories called *annular categories* play a key role in describing low energy excitations in some models of topological phases of matter [4]. In particular, *point defects* in the bulk of a topological phase are parameterised by representations of the annular category $\int_{S^1}\mathcal{C}$ for a pivotal category $\mathcal{C}$. Intersections of *domain walls* between topological phases are in turn parameterised by representations of the annular category for a collection of compatible bimodule categories. The promise of topological phases in quantum information theory is that they provide a way to perform quantum computation in a way which is robust to local noise [42, 14]. In these proposed devices, the quantum information is stored in excited states of the system. We would need to understand the explicit structure of these annular categories in order to determine the Hilbert spaces of excitations at domain walls and defects.

It is often extremely difficult to calculate directly with the algebraic objects we have just described [23, 32, 17, 2]. In this thesis we prove equivalence theorems which provide a connection between the mathematical and physical perspectives. As a key step along the way we define the *balanced tensor product* $\mathcal{M}\overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}}\mathcal{N}$ of a pair of module categories, and use it to define maps to both the algebraic and diagrammatic worlds. Assuming that all of our categories are *finitely semisimple*, we prove that all of these maps are equivalences.

The contents of this thesis are arranged as follows. In Chapter 1 we introduce basic definitions in the theory of $\Bbbk$-linear monoidal categories and their module categories. The key definition is the Drinfeld center $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ of a $(\mathcal{C},\mathcal{C})$-bimodule category, and the celebrated special case $\mathcal{Z}(\mathcal{C})$ for a monoidal category. In Chapter 2 we establish a folklore result—that there is an equivalence between the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and representations of the annular category $\int_{S^1}\mathcal{C}$—in much more detail than has ever been done previously. In Chapter 3 we define a generalised family of annular categories for bimodule categories, and connect them to our balanced tensor product $\mathcal{M}\overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}}\mathcal{N}$ and its sister construction, the category $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ of *bibalanced objects*. In Chapter 4 we define the *balanced center* $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ of a $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{M}$, and we relate its monoidal product to our earlier diagrammatic constructions.

The main equivalence theorems which we prove are summarised in the diagram below. The solid arrows depict functors which always exist when the categories they pass between make sense. Dotted morphisms indicate functors which only exist given semisimplicity hypotheses. When adjacent functors both exist and pass in opposite directions then they are part of a (possibly contravariant) equivalence. Equalities with a boxed label indicate the hypothesis required for them to exist. The first row is established in Chapters 1 and 2. The second and third rows are developed in Chapters 3 and 4.

"Physics" Rep (diagram category)		Algebraic "Centers"		Functor Categories
$\mathsf{Rep}\left(\int_{S^1} \mathcal{C}\right)$	$\rightleftarrows$	$\mathcal{Z}(\mathcal{C})$		
		$\boxed{\mathcal{M} = \mathcal{C}}$ $\Vert$		
		$\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$	$\rightleftarrows$	$[{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}]$
$\mathsf{Rep}\left(\right.$  $\left.\right)$	$\rightleftarrows$	$\mathcal{M} \underset{\mathcal{C}}{\overset{\text{bal}}{\otimes}} \mathcal{N}$	$\rightleftarrows$	$[{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$
		$\boxed{\mathcal{D} = \mathsf{Vec}}$ $\Vert$		
$\mathsf{Rep}\left(\right.$  $\left.\right)$	$\rightleftarrows$	$\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$	$\rightleftarrows$	$[{}_{\mathcal{C}}\mathcal{M}^*_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}]$
(Requires $\mathcal{C}, \mathcal{D}$ pivotal.)		$\boxed{\mathcal{N} = \mathcal{M}^*}$ $\Vert$		
		$\mathcal{Z}^{\text{bibal}}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}})$		

Chapter 1

Preliminaries

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote the set of morphisms from an object X of $\mathcal{C}$ to the object Y by $\mathcal{C}(X \to Y)$, and the category of functors from $\mathcal{C}$ to $\mathcal{D}$ by $[\mathcal{C} \to \mathcal{D}]$. The identity on X is denoted id_X. The relations $\cong$ and $\simeq$ denote isomorphism and equivalence respectively. Throughout $\Bbbk$ will be an algebraically closed field of characteristic 0. Let $I = [0,1]$ denote the unit interval.

1.1 Basic $\Bbbk$-linear category theory

Throughout we mostly adhere to the notation of [12]. A comprehensive resource is [15].

Definition 1.1.1. A category $\mathcal{C}$ is $\Bbbk$-*linear* if all of the following hold:

- For all $X, Y \in \mathcal{C}$ the homset $\mathcal{C}(X \to Y)$ is a $\Bbbk$-vector space (and not merely a set).
- The composition law $\circ : \mathcal{C}(Y \to Z) \times \mathcal{C}(X \to Y) \to \mathcal{C}(X \to Z)$ is a bilinear map.
- There is a *zero object* $0 \in \mathcal{C}$ with $\dim \mathcal{C}(X \to 0) = \dim \mathcal{C}(0 \to X) = 0$ for all $X \in \mathcal{C}$.
- The category $\mathcal{C}$ *has direct sums*: for all $X_1, X_2 \in \mathcal{C}$ there exists an object $Y \in \mathcal{C}$ together with morphisms $p_1 : Y \to X_1$, $p_2 : Y \to X_2$, $i_1 : X_1 \to Y$, $i_2 : X_2 \to Y$ such that $p_1 \circ i_1 = \mathrm{id}_{X_1}$, $p_2 \circ i_2 = \mathrm{id}_{X_2}$, and $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Y$. If such a Y exists it is unique up to unique isomorphism, and it is denoted $X_1 \oplus X_2$.

Thus there is a canonical *zero morphism* $0 \in \mathcal{C}(X \to Y)$. A functor $\mathcal{C} \to \mathcal{D}$ between $\Bbbk$-linear categories $\mathcal{C}$ and $\mathcal{D}$ is *linear* if $F : \mathcal{C}(X \to Y) \to \mathcal{D}(F(X) \to F(Y))$ is a linear map for all $X, Y \in \mathcal{C}$.

In this thesis unless otherwise specified all categories will be $\Bbbk$-linear and we will consider only the linear functors between them. Since linear functors preserve the equations defining a direct sum, all of our functors will preserve direct sums.

Definition 1.1.2. The *direct sum completion* of a category $\mathcal{C}$ is a new category $\mathrm{Mat}(\mathcal{C})$ with objects formal direct sums $X_1 \oplus \cdots \oplus X_n$ of objects X_i of $\mathcal{C}$. The morphisms $X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$ are $n \times m$ matrices with a morphism $X_i \to Y_j$ in their (i, j)-th entry. The composition law is matrix multiplication.

Definition 1.1.3. Given categories $\mathcal{C}$ and $\mathcal{D}$ we can form the category $\mathcal{C} \otimes^{\mathrm{pure}} \mathcal{D}$ with objects formal products $X \otimes Y$ (just pairs $(X, Y) \in \mathcal{C} \times \mathcal{D}$) and homsets $(\mathcal{C} \otimes^{\mathrm{pure}} \mathcal{D})(X \otimes Y \to Z \otimes W) := \mathcal{C}(X \to Z) \otimes \mathcal{D}(Y \to W)$.

The *naïve tensor product* of the categories $\mathcal{C}$ and $\mathcal{D}$ is the direct sum completion $\mathcal{C} \otimes \mathcal{D} := \mathrm{Mat}(\mathcal{C} \otimes^{\mathrm{pure}} \mathcal{D})$. For functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{F}$ there is a natural product $F \otimes G : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E} \otimes \mathcal{F}$ by acting separately on each factor (and extending to direct sums).

Definition 1.1.4. Let $f : X \to Y$ be an morphism in $\mathcal{C}$. Then an object $K \in \mathcal{C}$ together with $k : K \to X$ satisfying $f \circ k = 0$ is a *kernel* of f if for all other morphisms $k' : K' \to X$ with $f \circ k' = 0$ there exists a unique $g : K' \to K$ such that $k' = k \circ g$. Such a pair (K, k) is denoted by $\ker f$ (if it exists). In this case if $K = 0$ then f *is a monomorphism*.

The formally dual notion is called a *cokernel*. That is, a pair $(C, c : Y \to C)$ with $c \circ f = 0$ is a cokernel of f if for all other morphisms $c' : Y \to C'$ such that $c' \circ f = 0$ there exists $h : C \to C'$ such that $c' = h \circ c$. If it exists we denote it by $\mathrm{coker}(f) := (C, c)$, and if $C = 0$ then f *is an epimorphism*.

It is a standard exercise to verify that kernels and cokernels are each unique up to unique isomorphism [29, 15].

Definition 1.1.5. A category $\mathcal{C}$ is *abelian* if all morphisms $f : X \to Y$ fit into a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "i"] \arrow[rr, bend left, "f"] & I \arrow[r, "j"] & Y \arrow[r, "c"] & C \end{tikzcd}$$

such that $\ker(f) = (K, k)$, $\mathrm{coker}(f) = (C, c)$, $\ker(c) = (I, j)$, and $\mathrm{coker}(k) = (I, i)$. Such a diagram is called a *canonical decomposition of* f, and $\mathrm{im}(f) := I$ is called the *image of* f.

The category $\underline{\mathsf{Vec}}$ of $\Bbbk$-vector spaces and its full subcategory Vec of finite dimensional vector spaces are both examples of ($\Bbbk$-linear) abelian categories.

Definition 1.1.6. Let X be an object of an abelian category $\mathcal{C}$. A *subobject of* X is a monomorphism $\iota : Y \to X$ or by abuse, just Y. Similarly a *quotient of* X is an epimorphism $\pi : X \to Z$, or just Z. If $\iota : Y \to X$ is a subobject then the quotient is $X/Y := \mathrm{coker}(f)$.

The object X is *simple* if its only subobjects are 0 and itself, and is *semisimple* if it is a direct sum of simple objects. The category $\mathcal{C}$ is *semisimple* if all of its objects are semisimple, and is *finitely semisimple* if $\mathcal{C}$ has finitely many isomorphisms classes of simple objects.

Lemma 1.1.7 (Schur's lemma). *If $f : X \to Y$ is a morphism between simple objects in an abelian category $\mathcal{C}$, then f is an isomorphism or zero.*

In particular $\mathcal{C}(X \to X)$ is made a $\Bbbk$-algebra by the subspace spanned by id_X, so if X is simple then $\mathcal{C}(X \to X)$ is a division algebra. We assume that $\Bbbk$ is algebraically closed, and thus in this case whenever $\mathcal{C}(X \to X)$ is finite dimensional actually $\mathcal{C}(X \to X) \cong \Bbbk$.

Definition 1.1.8. An object X of an abelian category $\mathcal{C}$ *has finite length* if there exists a chain of monomorphisms

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X$$

with the successive quotients X_{i+1}/X_i all simple.

Definition 1.1.9. An abelian category $\mathcal{C}$ is *locally finite* if $\dim \mathcal{C}(X \to Y) < \infty$ for all $X, Y \in \mathcal{C}$, and all objects of $\mathcal{C}$ have finite length.

We assume that all of our abelian categories are locally finite.

Definition 1.1.10. A sequence of morphisms $\cdots \to X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \to \cdots$ is *exact* if we have $\mathrm{im}(f_i) = \ker(f_{i+1})$ for all i. A functor F is *left (respectively, right) exact* if for all (short) exact sequences $0 \to X \to Y \to Z \to 0$ the image sequence $0 \to F(X) \to F(Y) \to F(Z)$ (respectively, $F(X) \to F(Y) \to F(Z) \to 0$) is exact.

Definition 1.1.11. An object P of an abelian category $\mathcal{C}$ is *projective* if the hom-functor $\mathcal{C}(P \to -)$ is exact. An object X of $\mathcal{C}$ has a *projective cover* if there is an epimorphism $p : P \to X$ with P projective such that for all other epimorphisms $q : Q \to X$ with Q projective there exists an epimorphism $e : Q \to P$ such that $q = p \circ e$.

The category $\mathcal{C}$ has *enough projectives* if every simple object has a projective cover. If $\mathcal{C}$ is locally finite, has finitely many isomorphism classes of simple objects, and has enough projectives, then $\mathcal{C}$ is *finite*. Note that every finitely semisimple category is automatically finite.

Definition 1.1.12. If $\mathcal{C}$ and $\mathcal{D}$ are locally finite abelian categories then there is an abelian category $\mathcal{C} \boxtimes \mathcal{D}$ called *Deligne's tensor product* [7] and a functor $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ such that: for every abelian category $\mathcal{E}$ and functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is right exact in both variables there exists a unique right exact functor $\overline{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$ such that $F = \overline{F} \circ \boxtimes$.

Proposition 1.1.13. *When $\mathcal{C}$ and $\mathcal{D}$ are finitely semisimple the naïve tensor product $\mathcal{C} \otimes \mathcal{D}$ and the Deligne product $\mathcal{C} \boxtimes \mathcal{D}$ are equivalent.*

Proof. The hypotheses imply that $\mathcal{C} \otimes \mathcal{D}$ is finitely semisimple, and the universal property of Definition 1.1.12 may be checked directly in this case. □

1.2 Monoidal, rigid, pivotal, and braided categories

Definition 1.2.1. A *monoidal category* is a category $\mathcal{C}$ equipped with a (bi)functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *monoidal (or tensor) product*, an object $\mathbb{1} \in \mathcal{C}$ called the *unit*, and natural isomorphisms (with $X, Y, Z \in \mathcal{C}$)

$$\begin{aligned} \alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \\ l_X &: \mathbb{1} \otimes X \to X, \text{ and} \\ r_X &: X \otimes \mathbb{1} \to X \end{aligned}$$

called the *associator*, *left unitor*, and *right unitor* respectively, such that the diagrams

$$\begin{tikzcd}[column sep=small] & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\ ((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"] & & W \otimes (X \otimes (Y \otimes Z)) \\ (W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"'] \end{tikzcd}$$

(*associativity constraint*)

$$\begin{tikzcd} (X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\ & X \otimes Y & \end{tikzcd}$$

(*identity constraint*)

commute for all $W, X, Y, Z \in \mathcal{C}$.

The categories $\underline{\mathsf{Vec}}$ and Vec are both examples of monoidal categories under the tensor product of vector spaces. We defer more examples until the end of this section.

If $\mathcal{C}$ has the structure of a monoidal category with monoidal product $\otimes$ and associator α, then there is another monoidal product $\otimes^{\mathrm{mop}}$ given by $X \otimes^{\mathrm{mop}} Y := Y \otimes X$. We obtain another monoidal category $\mathcal{C}^{\mathrm{mop}}$ from the product $\otimes^{\mathrm{mop}}$ with the same unit as $\mathcal{C}$, the associator $\alpha^{\mathrm{mop}}_{X,Y,Z} := \alpha^{-1}_{Z,Y,X}$, and by swapping the roles of the unitors in $\mathcal{C}$.

Definition 1.2.2. The category $\mathcal{C}^{\mathrm{mop}}$ is called the *monoidal opposite* of $\mathcal{C}$.

Definition 1.2.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between monoidal categories. The functor F is *monoidal* if it is equipped with a natural isomorphism[1] (for $X, Y \in \mathcal{C}$)

$$J_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \to F(X \otimes^{\mathcal{C}} Y)$$

called the *tensorator* and an isomorphism $\iota : \mathbb{1}^{\mathcal{D}} \to F(\mathbb{1}^{\mathcal{C}})$ called the *identitor*, such that the diagrams

$$\begin{tikzcd}
(F(X) \otimes^{\mathcal{D}} F(Y)) \otimes^{\mathcal{D}} F(Z) \arrow[r, "\alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes^{\mathcal{D}} \mathrm{id}_{F(Z)}"] & F(X) \otimes^{\mathcal{D}} (F(Y) \otimes^{\mathcal{D}} F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes^{\mathcal{D}} J_{Y,Z}"] \\
F(X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{D}} Z \arrow[d, "J_{X \otimes^{\mathcal{C}} Y, Z}"] & F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) \arrow[d, "J_{X, Y \otimes^{\mathcal{C}} Z}"] \\
F((X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{C}} Z) \arrow[r, "F(\alpha^{\mathcal{C}}_{X,Y,Z})"] & F(X \otimes^{\mathcal{C}} (Y \otimes^{\mathcal{C}} Z))
\end{tikzcd}$$

(*associativity constraint*)

$$\begin{tikzcd}
\mathbb{1}^{\mathcal{D}} \otimes^{\mathcal{D}} F(X) \arrow[r, "l^{\mathcal{D}}_{F(X)}"] \arrow[d, "\iota \otimes \mathrm{id}_{F(X)}"] & F(X) \\
F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[r, "J_{\mathbb{1},X}"] & F(\mathbb{1}^{\mathcal{C}} \otimes^{\mathcal{C}} X) \arrow[u, "F(l^{\mathcal{C}}_X)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes^{\mathcal{D}} \mathbb{1}^{\mathcal{D}} \arrow[r, "r^{\mathcal{D}}_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes \iota"] & F(X) \\
F(X) \otimes^{\mathcal{D}} F(\mathbb{1}^{\mathcal{C}}) \arrow[r, "J_{X,\mathbb{1}}"] & F(X \otimes^{\mathcal{C}} \mathbb{1}^{\mathcal{C}}) \arrow[u, "F(r^{\mathcal{C}}_X)"]
\end{tikzcd}$$

(*left identity constraint*) (*right identity constraint*)

commute for all $X, Y, Z \in \mathcal{C}$.

We write $F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$ to indicate that F is a monoidal functor between monoidal categories.

Definition 1.2.4. Let $\eta : F \to G$ be a natural transformation of monoidal functors F and $G : \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$ with respective tensorators J and K. The natural transformation η is *monoidal* if the diagram

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "J_{X,Y}"] \arrow[d, "\eta_X \otimes \eta_Y"] & F(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
G(X) \otimes G(Y) \arrow[r, "K_{X,Y}"] & G(X \otimes Y)
\end{tikzcd}$$

is commutative for all $X, Y \in \mathcal{C}$.

Monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are *monoidally equivalent* (or simply $\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$) if there is an equivalence $(F, G, \eta, \varepsilon)$ of ordinary categories with F and G monoidal functors, and η and ε monoidal natural transformations. A monoidal category $\mathcal{C}$ is *strict* if its associator and left and right unitors are all identity morphisms.

Example 1.2.5. The category $\mathrm{End}(\mathcal{C})$ of endofunctors of any ordinary category $\mathcal{C}$ is a strict monoidal category under composition of functors, with unit the identity functor.

Theorem 1.2.6 (Mac Lane's Strictness Theorem)**.** *Every monoidal category $\mathcal{C}$ is monoidally equivalent to a strict monoidal category $\mathcal{C}^{\wr}$.*

Proof. This is Theorem XI.3.1 of [29]. $\square$

[1]We use a superscript to distinguish the monoidal products, associators, and so on of the categories in question.

Corollary 1.2.6.1 (Mac Lane's Coherence Theorem)**.** *Let C be a monoidal category. Every diagram with edges labelled by tensor products of α, l, r, and identity morphisms is commutative.*

Definition 1.2.7. An object X of a monoidal category has *left dual* X^* if there exist morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$ such that the composites

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$
$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes X^*} X^* \tag{1.2.1}$$

are both identities. A *right dual* of $X \in C$ is an object ${}^*X$ which has X as a left dual.

Duals on either side are unique up to unique isomorphism [12]. The name left dual comes from the fact that a left dual in the monoidal category $\mathrm{End}(C)$ (of Example 1.2.5) is exactly a left adjoint functor.

Definition 1.2.8. A monoidal category C is *rigid* if every object has a left and a right dual.

Given a morphism $f : X \to Y$ in a monoidal category C, rigidity of C gives a way to construct a morphism $f^* : Y^* \to X^*$ by forming a composite

$$Y^* \xrightarrow{r^{-1}_{Y^*}} Y^* \otimes \mathbb{1} \xrightarrow{Y^* \otimes \mathrm{coev}_X} Y^* \otimes X \otimes X^* \xrightarrow{Y^* \otimes f \otimes X^*} Y^* \otimes Y \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes X^*} \mathbb{1} \otimes X^* \xrightarrow{l_{X^*}} X^*. \tag{1.2.2}$$

Here we have written objects of C to mean the identity on that object, and we will often use this convention in the sequel. Of course there is also a dual morphism ${}^*f : {}^*Y \to {}^*X$ on the other side, and in fact we have the following[2].

Proposition 1.2.9. *The operations $(-)^*$ and ${}^*(-)$ define monoidal functors $C \xrightarrow{\otimes} C^{\mathrm{op},\mathrm{mop}}$. There are monoidal natural isomorphisms ${}^*(X^*) \cong X \cong ({}^*X)^*$.*

Corollary 1.2.9.1. *Every rigid monoidal category C is monoidally equivalent to its opposite, monoidal opposite category $C^{\mathrm{op},\mathrm{mop}}$.*

Proposition 1.2.10 (Frobenius reciprocity)**.** *In a rigid monoidal category C for all $X, Y, Z \in C$ there are canonical isomorphisms*

$$C(X^* \otimes Y \to Z) \cong C(Y \to X \otimes Z).$$

Proof. The map is just a slight generalisation of (1.2.2). □

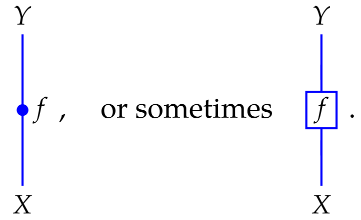
Definition 1.2.11. A monoidal natural isomorphism $\phi_X : X \to X^{**}$ is a *pivotal structure*. A rigid monoidal category C equipped with a pivotal structure is *pivotal* [16].[3] A monoidal functor $F : C \xrightarrow{\otimes} \mathcal{D}$ between pivotal categories is *pivotal* if $F(\phi^{C}_X) : F(X) \to F(X^{**}) \cong F(X)^{**}$ is just $\phi^{\mathcal{D}}_X$.

The form of (1.2.2) makes it clear that the approach of describing duals using diagrams of arrows labelled by morphisms will quickly become unmanageable. Fortunately there is a *rectangular string diagram calculus* in monoidal categories, which is particularly powerful in the rigid and pivotal case, and which greatly simplifies depictions of composites of the above kind.

[2] This is a standard exercise, see Remark 2.10.3 of [12].

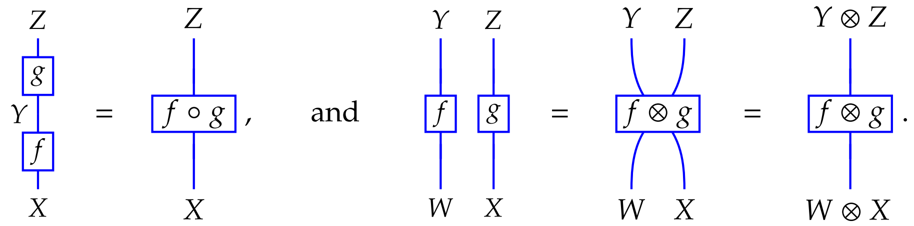
[3] Note that a canonical natural isomorphism ${}^*X \to X^*$ is readily obtained from any such natural isomorphism ϕ_X. Pivotal structures are not unique.

Definition 1.2.12. The *string diagram* associated to a morphism $f : X \to Y$ in a monoidal category $\mathcal{C}$ is the diagram



Here strings are labelled by objects of $\mathcal{C}$ and the point is labelled by f. We sometimes draw a box over a point containing its label so that there can be no confusion as to what is being labelled.

We declare that vertical juxtaposition of morphisms is composition, and that horizontal concatenation gives the tensor product. That is, there are equalities of diagrams

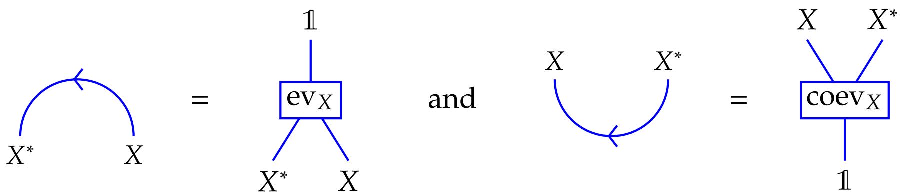


(1.2.3)

Let us always suppose that string diagrams are drawn inside a rectangle $\Sigma_R := I \times I$, with some number of strings meeting the top and bottom of the rectangle. Given two diagrams D and D', if strings meet the top edge of D in the same places where strings meet the bottom edge of D', and the labels of the respective strings agree, then we can glue the diagrams together and obtain a single diagram $D' \circ D$.

In fact we can be more flexible. We allow diagrams to be modified by isotopy rel boundary in the rectangle, subject to the requirement that all of the strings in a diagram always travel up the page and are never horizontal. By permitting local applications of the *local replacement* rules (1.2.3) inside diagrams, for any string diagram we can compute a morphism in $\mathcal{C}$ by reducing its contents to a single labelled point (the resulting label will be some series of composites and tensor products in $\mathcal{C}$). If from left to right the bottom edge of our original diagram meets strings labelled by $X_1, \dots, X_n$ and the top edge meets strings labelled by $Y_1, \dots, Y_m$, then after reduction we obtain a morphism $f : X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_m$.[4] It is well known[5] that the morphism f so obtained is unique.

Now suppose that $\mathcal{C}$ is rigid. Heretofore diagrams with horizontal-passing strings have been unacceptable, but we now declare

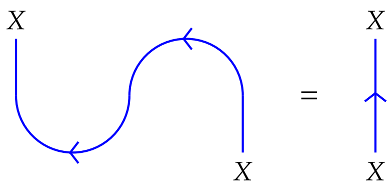


i.e. that left-oriented caps and cups are the string diagrams for the evaluation (ev_X) and coevaluation (coev_X) morphisms respectively. That is, we have dropped the labelled points ev_X

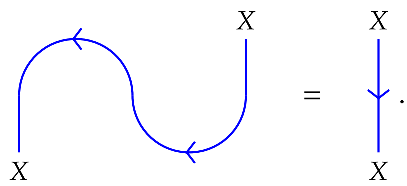
[4] If $\mathcal{C}$ is not strict then we must choose a way to associate these tensor products in order to obtain an actual morphism of $\mathcal{C}$. By the coherence theorem Corollary 1.2.6.1 all ways of doing this are consistent, and so in practice there will be no ambiguity.

[5] For instance, see Theorem 3.1 of the comprehensive review [38] and subsequent references.

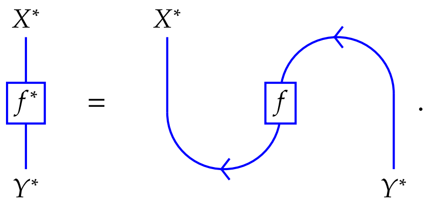
and coev_X, and have omitted the strings corresponding to the unit $\mathbb{1}$. So that string labelling remains consistent it is also convenient to now impose that all strings be oriented, with strings travelling down the page corresponding to the left dual of their label. This allows us to label the cup and cap strings above simply by X. For example, in this language the identities (1.2.1) correspond to formal equalities (note that identity morphisms can be represented by empty strings)



and



Nonetheless in practice we are often able to omit the string orientations without hindering their inference. Consequently we are now able to permit isotopies in which strings pass down the page as well as up, so long as strings never pass horizontally to the right[6] (since our cups only pass to the left, an ambiguity would arise). It is again well known [38] that rectangular string diagrams in rigid categories unambiguously correspond to morphisms in the category. For example, in this language the dual morphism f^* in (1.2.2) of $f : X \to Y$ is just the upside-down flip



We can interpret the structure of a pivotal category as lifting this final restriction and enabling a theory of arbitrary diagrams with arbitrary isotopy rel boundary—which in fact can be drawn in any 2-manifold—and we address this further in Section 1.4.

Definition 1.2.13. A monoidal category C is *braided* if it is equipped with a natural (in both variables) isomorphism

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

called the *braiding*, such that the diagrams

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "X \otimes b_{Y,Z}"] \arrow[d, "\alpha^{-1}_{X,Y,Z}"] & X \otimes (Z \otimes Y) \arrow[d, "\alpha^{-1}_{X,Z,Y}"] \\
(X \otimes Y) \otimes Z \arrow[d, "b_{X \otimes Y, Z}"] & (X \otimes Z) \otimes Y \arrow[d, "b_{X,Z} \otimes \mathrm{id}_Y"] \\
Z \otimes (X \otimes Y) \arrow[r, "\alpha^{-1}_{Z,X,Y}"] & (Z \otimes X) \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "b_{X,Y} \otimes Z"] \arrow[d, "\alpha_{X,Y,Z}"] & (Y \otimes X) \otimes Z \arrow[d, "\alpha_{Y,X,Z}"] \\
X \otimes (Y \otimes Z) \arrow[d, "b_{X, Y \otimes Z}"] & Y \otimes (X \otimes Z) \arrow[d, "\mathrm{id}_Y \otimes b_{X,Z}"] \\
(Y \otimes Z) \otimes X \arrow[r, "\alpha_{Y,Z,X}"] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

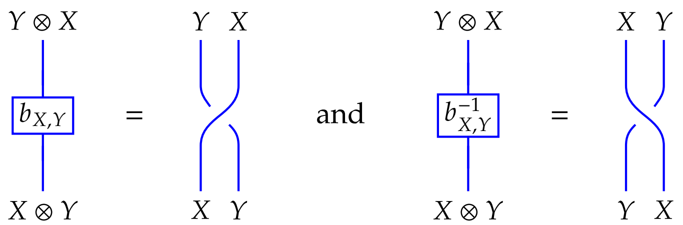
(*left hexagon identity*) (*right hexagon identity*)

are commutative for all $X, Y, Z \in C$.

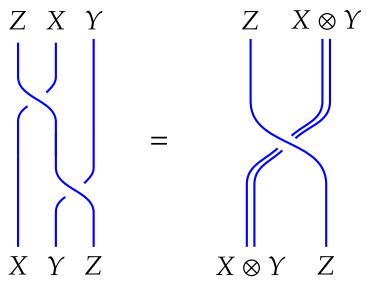
A braided monoidal category is *symmetric* if $b_{Y,X} \circ b_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in C$.

As the name suggests, string diagrams in braided monoidal categories admit a particularly elegant representation of the braiding. We simply declare

[6] Other authors replace labelled points with labelled coupons (balls of some finite radius), and state this condition as “coupons don’t rotate” [21, 38].



so that the braiding composed with its inverse can be untangled by pulling the strings under one another. With this interpretation, the left hexagon identity asserts that



and the right hexagon identity is the analogous fact for the opposite crossing orientation.

Definition 1.2.14. The *braided opposite (or reverse)* C^{bop} of a braided monoidal category C is the same underlying monoidal category as C, equipped with the new braiding $b^{\text{bop}}_{X,Y} := b^{-1}_{Y,X}$.

Definition 1.2.15. Let $F : C \xrightarrow{\otimes} \mathcal{D}$ be a monoidal functor between braided monoidal categories C and $\mathcal{D}$, with respective braidings b and c. Then F is *braided* if the diagram

$$\begin{tikzcd} F(X \otimes Y) \arrow[r, "F(b_{X,Y})"] & F(Y \otimes X) \\ F(X) \otimes F(Y) \arrow[u, "J_{X,Y}"] \arrow[r, "c_{F(X),F(Y)}"] & F(Y) \otimes F(X) \arrow[u, "J_{Y,X}"] \end{tikzcd} \tag{1.2.4}$$

is commutative for all $X, Y \in C$.

Note that in contrast to the notion of a monoidal functor, being a braided monoidal functor merely imposes a condition (as opposed to requiring some additional data, e.g. of a tensorator).

If C and $\mathcal{D}$ are braided monoidal categories then they are *braided equivalent* if there is a monoidal equivalence $(F, G, \eta, \varepsilon)$ with F and G braided. The following lemma eases the burden in producing a monoidal or braided monoidal equivalence.

Proposition 1.2.16. *Let $F : C \to \mathcal{D}$ be a monoidal functor between monoidal categories which is part of an equivalence when considered as a functor between ordinary categories. Then F is part of a monoidal equivalence. If F is braided then F is part of a braided equivalence.*

Proof. Every equivalence $(F, G, \eta, \varepsilon)$ can always be made an adjoint equivalence by replacing ε only [29]. If F is monoidal then its tensorator can be transported to G over the adjunction, in which case η and ε become monoidal natural transformations (see Remark 2.4.10 of [12]). One then checks directly that if F is braided then G (with its new tensorator) is braided too. □

Definition 1.2.17. A locally finite $\Bbbk$-linear abelian rigid monoidal category C is *tensor* if $C(\mathbb{1} \to \mathbb{1}) \cong \Bbbk$. A tensor category C is *fusion* if C is finitely semisimple.

Remark 1.2.18. Monoidal categories are a categorification[7] of the ordinary notion of an monoid, lifting algebraic equalities up to coherence isomorphisms; upon taking isomorphism classes the

[7] This is a overloaded term in the literature, and we mean it in a particular narrow sense.

tensor product $\otimes$ descends to a monoid operation with unit $\mathbb{1}$. Since our categories always have direct sums $\oplus$ as well, taking the so-called *Grothendieck group* we actually obtain ring.

A braided monoidal category then categorifies a commutative ring, since any braiding at all descends to an equality asserting commutativity of $\otimes$. In the next sections we will see categorifications of modules for a ring, and of their centers.

Example 1.2.19. The categories $\underline{\mathsf{Vec}}$ and Vec, along with the category $\mathsf{Rep}\,G$ of finite dimensional representations of a finite group G are all symmetric monoidal categories under their usual tensor product operation. The center construction of Section 1.5 will give a way to construct braided monoidal categories from arbitrary monoidal ones. The categories Vec and $\mathsf{Rep}\,G$ are pivotal and fusion with respective sets of simple objects the one-dimensional vector spaces (i.e. isomorphic to $\Bbbk$ thought of as a $\Bbbk$-vector space—the tensor unit), and the finite dimensional irreducible representations of G. More exotic examples of fusion categories include finite dimensional representations of quantum groups such as $\mathsf{Rep}\,U_q\mathfrak{sl}_2$ for q a root of unity (after taking the quotient by the so-called "negligible ideal") [27], and the even parts $\mathcal{EH}1$ and $\mathcal{EH}2$ of the Extended Haagerup subfactor [3], but defining these far exceeds the scope of this thesis—though by different orders of magnitude.

1.3 Module categories for a monoidal category

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{M}$ be categories with $\mathcal{C}$ monoidal. Then $\mathcal{M}$ is a left $\mathcal{C}$-module category if it is equipped with an *action* functor

$$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$$

and natural isomorphisms (with $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$)

$$\begin{aligned} m_{X,Y,M} &: (X \otimes Y) \triangleright M \to X \triangleright (Y \triangleright M), \text{ and} \\ l_M &: \mathbb{1} \triangleright M \to M \end{aligned}$$

called the *module associator* and *unitor* respectively, such that the diagrams

$$\begin{tikzcd} & (X \otimes Y) \triangleright (Z \triangleright M) \arrow[dr, "m_{X,Y,Z \triangleright M}"] & \\ ((X \otimes Y) \otimes Z) \triangleright M \arrow[ur, "m_{X \otimes Y,Z,M}"] \arrow[d, "m_{X,Y,Z} \triangleright M"'] & & X \triangleright (Y \triangleright (Z \triangleright M)) \\ (X \otimes (Y \otimes Z)) \triangleright M \arrow[rr, "m_{X,Y \otimes Z,M}"] & & X \triangleright ((Y \otimes Z) \triangleright M) \arrow[u, "X \triangleright m_{Y,Z,M}"'] \end{tikzcd}$$

(*associativity constraint*)

$$\begin{tikzcd} (X \otimes \mathbb{1}) \triangleright M \arrow[rr, "m_{X,\mathbb{1},M}"] \arrow[dr, "r_X \triangleright M"'] & & X \triangleright (\mathbb{1} \triangleright M) \arrow[dl, "X \triangleright l_M"] \\ & X \triangleright M & \end{tikzcd}$$

(*identity constraint*)

commute for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$.

There is of course the dual notion of a right $\mathcal{C}$-module category, which is concisely defined as a left $\mathcal{C}^{\mathrm{mop}}$-module category. We will often write $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ for its action, and similarly use $n_{M,X,Y}$ and r_M for its module associator and unitor isomorphisms.

Definition 1.3.2. A category $\mathcal{M}$ which is simultaneously a left $\mathcal{C}$- and right $\mathcal{D}$-module category with respective module associators m and n is a $(\mathcal{C}, \mathcal{D})$-*bimodule category* if it is equipped with an additional natural isomorphism (with $X \in \mathcal{C}$, $M \in \mathcal{M}$, and $Y \in \mathcal{D}$)

$$s_{X,M,Y} : (X \triangleright M) \triangleleft Y \to X \triangleright (M \triangleleft Y)$$

called the *bimodule (or middle) associator*, such that the diagrams (with $X, Y \in \mathcal{C}$ and $Z, W \in \mathcal{D}$)

$$\begin{tikzcd}
& (W \otimes X) \triangleright (M \triangleleft Y) \arrow[dr, "m_{W,X,M \triangleleft Y}"] & \\
((W \otimes X) \triangleright M) \triangleleft Y \arrow[ur, "s_{W \otimes X,M,Y}"] \arrow[d, "m_{W,X,M} \triangleleft Y"] & & W \triangleright (X \triangleright (M \triangleleft Y)) \\
(W \triangleright (X \triangleright M)) \triangleleft Y \arrow[rr, "s_{W,X \triangleright M,Y}"] & & W \triangleright ((X \triangleright M) \triangleleft Y) \arrow[u, "W \triangleright s_{X,M,Y}"']
\end{tikzcd}$$

(*left middle constraint*)

$$\begin{tikzcd}
& (X \triangleright M) \triangleleft (Y \otimes Z) \arrow[dl, "s_{X,M,Y \otimes Z}"'] \arrow[dr, "n_{X \triangleright M,Y,Z}"] & \\
X \triangleright (M \triangleleft (Y \otimes Z)) \arrow[d, "X \triangleright n_{M,Y,Z}"] & & X \triangleright (M \triangleleft (Y \otimes Z)) \\
X \triangleright ((M \triangleleft Y) \triangleleft Z) \arrow[rr, "s_{X,M \triangleleft Y,Z}"] & & (X \triangleright (M \triangleleft Y)) \triangleleft Z \arrow[u, "s_{X,M,Y} \triangleleft Z"']
\end{tikzcd}$$

(*right middle constraint*)

are commutative for all $W, X \in \mathcal{C}$, $M \in \mathcal{M}$, and $Y, Z \in \mathcal{D}$.

We sometimes write ${}_{\mathcal{C}}\mathcal{M}$ and $\mathcal{M}_{\mathcal{D}}$ to emphasise that $\mathcal{M}$ has a left $\mathcal{C}$- or right $\mathcal{D}$-module category structure, and similarly use ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ when $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category.

Example 1.3.3. Every monoidal category $\mathcal{C}$ is a $(\mathcal{C}, \mathcal{C})$-bimodule category over itself, with the left and right action given by tensor product on each side. Both of the associativity isomorphisms are just the associator α of $\mathcal{C}$, and similarly for the unitors.

Proposition 1.3.4. *Every monoidal functor $F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$ canonically turns a left $\mathcal{D}$-module category ${}_{\mathcal{D}}\mathcal{M}$ into a left $\mathcal{C}$-module category ${}_{F}\mathcal{M}$. The analogous statement for right $\mathcal{D}$-module and bimodule categories holds as well.*

Many common examples of module categories arise from a monoidal functor in this way (e.g. see Section 7.4 of [12]). For example $\mathsf{Rep}\, H$ is made a $(\mathsf{Rep}\, G, \mathsf{Rep}\, G)$-bimodule category for G a finite group and H a subgroup by the restriction functor Res^G_H.

Definition 1.3.5. Every left module category ${}_{\mathcal{C}}\mathcal{M}$ can be considered a right module category $\mathcal{M}_{\mathcal{C}^{\mathrm{mop}}}$ called its *flip* [8]. Similarly one can check directly that every bimodule category ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ corresponds to module categories $\mathcal{M}_{\mathcal{C}^{\mathrm{mop}} \otimes \mathcal{D}}$ (the *right flip*) and ${}_{\mathcal{C} \otimes \mathcal{D}^{\mathrm{mop}}}\mathcal{M}$ (the *left flip*). Note here $\mathcal{C}^{\mathrm{mop}} \otimes \mathcal{D}$ and $\mathcal{C} \otimes \mathcal{D}^{\mathrm{mop}}$ are made monoidal categories by using the tensor product in each factor and extending over direct sums.

Definition 1.3.6. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between left $\mathcal{C}$-module categories. Then F is a $\mathcal{C}$*-module functor* if it is equipped with a natural isomorphism (with $X \in \mathcal{C}$ and $M \in \mathcal{M}$)

$$c_{X,M} : F(X \triangleright M) \to X \triangleright F(M)$$

called the *modulator*, such that the diagrams

$$\begin{tikzcd}
& (X \otimes Y) \triangleright^{\mathcal{N}} F(M) \arrow[dr, "m^{\mathcal{N}}_{X,Y,F(M)}"] & \\
F((X \otimes Y) \triangleright^{\mathcal{M}} M) \arrow[ur, "c_{X \otimes Y,M}"] \arrow[d, "F(m^{\mathcal{M}}_{X \otimes Y,M})"] & & X \triangleright^{\mathcal{N}} (Y \triangleright^{\mathcal{N}} F(M)) \\
F(X \triangleright^{\mathcal{M}} (Y \triangleright^{\mathcal{M}} M)) \arrow[rr, "c_{X,Y \triangleright^{\mathcal{M}} M}"] & & X \triangleright^{\mathcal{N}} F(Y \triangleright^{\mathcal{M}} M) \arrow[u, "X \triangleright^{\mathcal{N}} c_{Y,M}"']
\end{tikzcd}$$

(*associativity constraint*)

$$\begin{tikzcd}
F(\mathbb{1} \triangleright^{\mathcal{M}} M) \arrow[rr, "c_{\mathbb{1},M}"] \arrow[dr, "F(l^{\mathcal{M}}_M)"'] & & \mathbb{1} \triangleright^{\mathcal{N}} F(M) \arrow[dl, "l^{\mathcal{N}}_{F(M)}"] \\
& F(M) &
\end{tikzcd}$$

(*identity constraint*)

commute for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

Of course there is also the formally dual notion of a right $\mathcal{D}$-module functor obtained by setting $\mathcal{C} := \mathcal{D}^{\mathrm{mop}}$ above, with a modulator we typically denote by $d_{M,X}$. From now on we suppress the superscript $\mathcal{C}$, $\mathcal{D}$, $\mathcal{M}$, and $\mathcal{N}$ annotations when no ambiguity can arise. Similarly, when it eases readability we drop an explicit tensor product $X \otimes Y$ in favour of the concatenation XY.

Definition 1.3.7. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between $(\mathcal{C}, \mathcal{D})$-bimodule categories $\mathcal{M}$ and $\mathcal{N}$ with respective bimodule associators s and t. Suppose that F is simultaneously a left $\mathcal{C}$- and right $\mathcal{D}$-module functor with respective modulators c and d. Then F is a $(\mathcal{C}, \mathcal{D})$*-bimodule functor* [19] if the diagram

$$\begin{tikzcd}
F((X \triangleright M) \triangleleft Y) \arrow[r, "d_{X\triangleright M,Y}"] \arrow[d, "F(s_{X,M,Y})"] & F(X \triangleright M) \triangleleft Y \arrow[d, "c_{X,M}\triangleleft Y"] \\
F(X \triangleright (M \triangleleft Y)) \arrow[d, "c_{X,M\triangleleft Y}"] & (X \triangleright F(M)) \triangleleft Y \arrow[d, "t_{X,F(M),Y}"] \\
X \triangleright F(M \triangleleft Y) \arrow[r, "X\triangleright d_{M,Y}"] & X \triangleright (F(M) \triangleleft Y)
\end{tikzcd} \tag{1.3.1}$$

commutes for all $X \in \mathcal{C}$, $M \in \mathcal{M}$, and $Y \in \mathcal{D}$.

Definition 1.3.8. Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{C}$-module categories, and let F and $G : \mathcal{M} \to \mathcal{N}$ be a pair of left $\mathcal{C}$-module functors. Then a natural transformation $\eta : F \to G$ is a *morphism of left* $\mathcal{C}$*-module functors* if the diagram

$$\begin{tikzcd}
F(X \triangleright M) \arrow[r, "\eta_{X\triangleright M}"] \arrow[d, "c_{X,M}"] & G(X \triangleright M) \arrow[d, "d_{X,M}"] \\
X \triangleright F(M) \arrow[r, "X\triangleright\eta_M"] & X \triangleright G(M)
\end{tikzcd}$$

commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$. The formally dual notion for right $\mathcal{C}$-module functors is clear.

A *morphism of* $(\mathcal{C}, \mathcal{D})$*-bimodule functors* is just a natural transformation which is simultaneously a morphism of left $\mathcal{C}$- and right $\mathcal{D}$-module functors.

If $\mathcal{M}$ and $\mathcal{N}$ are (bi)module categories then they are *equivalent as (bi)module categories* if there is an equivalence $(F, G, \eta, \varepsilon)$ with F and G morphisms of (bi)module functors. We write $[{}_{\mathcal{C}}\mathcal{M} \to {}_{\mathcal{C}}\mathcal{N}]$ for the category of left $\mathcal{C}$-module functors, and so on for right $\mathcal{D}$-module and $(\mathcal{C}, \mathcal{D})$-bimodule functors. There is similarly a strictness and coherence theorem[8] for module categories (and thus also bimodule categories by taking flips).

Theorem 1.3.9. *Every left $\mathcal{C}$-module category $\mathcal{M}$ is equivalent to a left $\mathcal{C}$-module category where the module associator and unit isomorphisms are the identity.*

Proposition 1.3.10. *Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. There are canonical isomorphisms (with $X \in \mathcal{C}$, and $M, N \in \mathcal{M}$)*

$$\mathcal{M}(X^* \triangleright M \to N) \cong \mathcal{M}(M \to X \triangleright N).$$

Proof. The proof proceeds in a fashion completely analogous to the proof of the same fact Proposition 1.2.10 for monoidal categories. □

[8] See Remark 7.2.4 of [12].

Definition 1.3.11. Let $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ be module categories. A functor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{E}$ is $\mathcal{C}$-*balanced* if[9] it is equipped with a natural (in each variable) isomorphism $a_{M,X,N} : F(M \triangleleft X, N) \to F(M, X \triangleright N)$ such that the diagram

$$\begin{tikzcd}
 & F(M, X \triangleright (Y \triangleright N)) & \\
F(M, (X \otimes Y) \triangleright N) \arrow[ur, "{(M, m_{X,Y,N})}"] & & F(M \triangleleft X, Y \triangleright N) \arrow[ul, "a_{M,X,Y \triangleright N}"'] \\
F(M \triangleleft (X \otimes Y), N) \arrow[u, "a_{M,X\otimes Y,N}"'] \arrow[rr, "{(n_{M,X,Y}, N)}"] & & F((M \triangleleft X) \triangleleft Y, N) \arrow[u, "a_{M \triangleleft X,Y,N}"]
\end{tikzcd}$$

is commutative for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, and $N \in \mathcal{N}$.

If $\mathcal{C}$ is a finite tensor category there is a *Deligne product* $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ *of module categories* [9] and a functor $\underset{\mathcal{C}}{\boxtimes} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$, which satisfies the following universal property (analogous to that of Definition 1.1.12): for every finite abelian category $\mathcal{E}$ and right exact in both variables $\mathcal{C}$-balanced bifunctor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{E}$ there exists a unique right exact functor $\overline{F} : \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N} \to \mathcal{E}$ such that $F = \overline{F} \circ \underset{\mathcal{C}}{\boxtimes}$. When $\mathcal{M}$ and $\mathcal{N}$ are respectively $(\mathcal{D}, \mathcal{C})$- and $(\mathcal{C}, \mathcal{E})$-bimodule categories then $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ is naturally a $(\mathcal{D}, \mathcal{E})$-bimodule category.

1.4 Formal diagrams of morphisms

In Section 1.2 we introduced string diagrams as a relatively informal but consistent way to express compositions of morphisms in monoidal categories, and especially pivotal categories. Every argument via (valid) string diagram manipulations can be translated into a chain of corresponding algebraic equalities upon inspection, and this alone is a very compelling argument for their use.

Nonetheless we stress that for any pivotal category $\mathcal{C}$ there is a completely formal theory of string diagrams in any 2-manifold Σ with boundary.[10] The basic definition is the following.

Definition 1.4.1 (String diagram in a 2-manifold)**.** An *unlabelled string diagram* $\widetilde{\Sigma}$ *of shape* Σ is a stratification

$$\emptyset = \Sigma_{-1} \hookrightarrow \Sigma_0 \hookrightarrow \Sigma_1 \hookrightarrow \Sigma_2 = \Sigma$$

subject to the following conditions.

1. (Finiteness): Each $\Delta_k := \Sigma_k \setminus \Sigma_{k-1}$ itself an oriented k-manifold with finitely many connected components. We call the connected components of Δ_1 the *strings*.

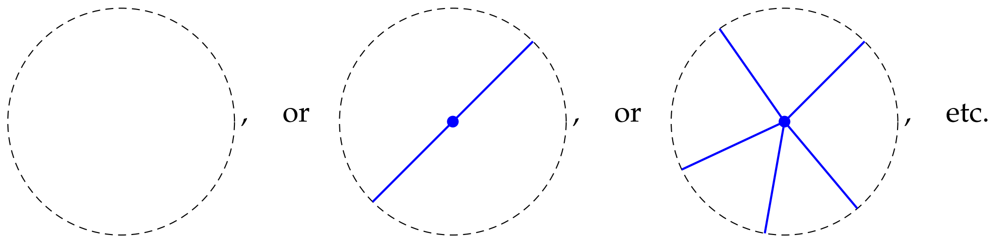
2. (Transversality): The Σ_1-stratum meets the boundary $\partial\Sigma$ at all points transversely.

3. (Regularity): Each $x \in \Sigma \setminus \partial\Sigma$ has a disk neighbourhood $U \subset \Sigma$ which is a cone with

[9] This is Definition 3.1 of [11].

[10] There are a number of approaches using a number of different models—both using local relations [39] and arbitrary isotopy [26, 4, 40]. The resource [38] is a comprehensive review of graphical languages is various settings (including the pivotal case).

respect to the stratification; i.e. locally U is of the form



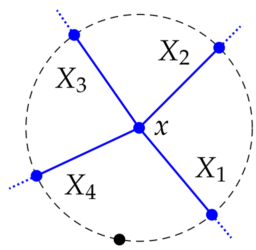
with the blue line segments depicting $U \cap \Sigma_1$. This prohibits, for example, strings terminating with an open end in the interior of Σ.

The conditions 1–3 are motivated by our intent to label points $x \in \Sigma_0 \setminus \partial\Sigma$ by morphisms of a monoidal category; morphisms can only be between tensor products of finitely many simple objects, and we prohibit various topological pathology. Of course there is also the notion of an isotopy between unlabelled diagrams of shape Σ, being an isotopy of Σ which takes one stratification to another though stratifications all satisfying 1–3.

Now we want to label an unlabelled string diagram $\widetilde{\Sigma}$ as in Section 1.2. Thus fix a pivotal category C. Beginning with the strings themselves, suppose we have made an assignment of an object of C to each connected component of Δ_1. For each $x \in \Sigma_0 \setminus \partial\Sigma$ regularity (condition 3) implies that there is a disk neighbourhood U_x of x in Σ which is homeomorphic to a cone on $\partial U_x \cong S^1$. The finitely many points p_L at which ∂U_x intersects strings $L \subset \Delta_1$ each then acquire a label with an object of C coming from the label of L. Namely, if L is entering U_x at p_L (with respect to the orientation of L) then we assign the label of L to p_L, and if L is leaving at p_L then we assign the left dual of the label of L.

Definition 1.4.2. Let Ω be a 1-manifold. A C-*decorated* Ω is a labelling of finitely points of Ω with an object of C. We let $\int_\Omega C$ denote the set of all such objects.[11]

Thus an assignment of an object of C to every string $L \subset \Delta_1$ gives a C-decorated S^1 denoted B_x for each point $x \in \Sigma_0 \setminus \partial\Sigma$. In order to label x, distinguish a point $\bullet \in B_x$ which is not labelled and which we call the *dot*. Reading anticlockwise around B_x starting from the dot we can build a tensor product of objects of C from the labels we encounter. For example, we would decode the C-decorated S^1 (drawn below with respect to a local neighbourhood of the string diagram from which it was obtained)



as the tensor product $P_x = ((((\mathbb{1} \otimes X_1) \otimes X_2) \otimes X_3) \otimes X_4)$. Note by convention we always begin the product with $\mathbb{1}$. A *label* for B_x is then a morphism $f_x : P_x \to \mathbb{1}$ in C.

Definition 1.4.3. A C-*labelling* of an unlabelled string diagram $\widetilde{\Sigma}$ of shape Σ is an assignment of an object of C to every string in $\widetilde{\Sigma}$, and of a morphism in C to every $x \in \Sigma_0 \setminus \partial\Sigma$ which respects the string labelling in the way in which we have just described.

We denote the collection of all C-labelled string diagrams of shape Σ by $\underline{\Sigma}(C)$.

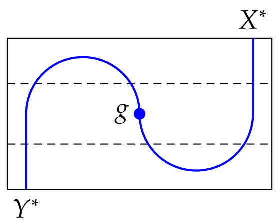
Of course there is a natural notion of isotopy of C-labelled diagrams.

[11] We will mostly be interested in the case of $\Omega = S^1$ in Chapter 2.

Theorem 1.4.4. *There is a map* **eval** *sending rectangular string diagrams (i.e. in $\underline{\Sigma}_{\mathrm{R}}(C)$) which do not meet their vertical boundary to ordinary morphisms in C.*

Proof idea. We do not prove the theorem, but we at least explain the idea.[12] Being embedded in a rectangle, diagrams $D \in \underline{\Sigma}_{\mathrm{R}}(C)$ have strings meeting the top and bottom faces and associating objects of C to the points of intersection. A tensor product P of objects of C can then be read across the bottom of the rectangle from left-to-right, and a tensor product Q for the top is obtained by left-to-right reading similarly (by convention we include $\mathbb{1}$ in these products so that they are never empty).

We aim to produce a morphism $P \to Q$ in C by interpreting diagrams as bottom-to-top composites as we informally did in in Section 1.2, by cutting the diagram into horizontal strips which can each be interpreted as tensor products of morphisms in C, and then composing the strips. It is at this stage that we convert caps and cups into evaluation and coevaluation morphisms. In order to decode a point $x \in \Sigma_0 \setminus \partial\Sigma$ labelled by a morphism $f_x : (\cdots(\mathbb{1} \otimes X_1) \otimes \cdots) \otimes X_n \to \mathbb{1}$ in C we use rigidity to migrate those objects X_i coming from strings leaving x and rising up the page to the target object of f_x. This process is schematically depicted below (with $g : X \to Y$ a morphism in C, and suppressing associator isomorphisms for readability).



$$\longmapsto \quad (\mathrm{ev}_Y \otimes X^*) \circ (Y^* \otimes g \otimes X^*) \circ (Y^* \otimes \mathrm{coev}_X)$$

Some ambiguity can arise if a string enters a labelled point x horizontally, since we must perform a perturbation of the string in order to decode the point as a morphism—and there are two possible choices. One sees that pivotality of C exactly means that the result of this process is independent of the choice. □

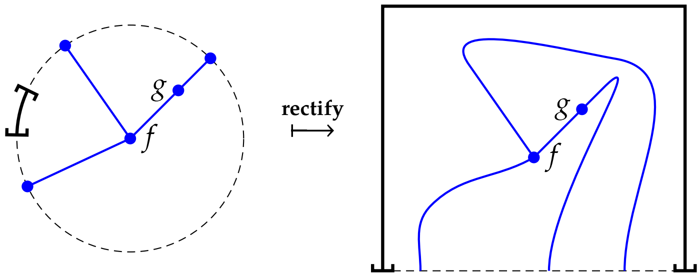
When Σ is a product $\Omega \times I$ with a 1-manifold Ω, diagrams in $\underline{\Sigma}(C)$ have two natural Ω-boundary components, a *bottom* $\Omega \times \{0\}$ and a *top* $\Omega \times \{1\}$, each C-decorated by the strings incident on them. Thus for every pair A and B of C-decorated copies of Ω we have a set $\underline{\Sigma}(A \to B)$ of string diagrams with bottom A and top B. There is also a natural composition law $\underline{\Sigma}(B \to C) \times \underline{\Sigma}(A \to B) \to \underline{\Sigma}(A \to C)$ by stacking and gluing along the common boundary B.

Definition 1.4.5. There is a diagram category $\int_\Omega C$ for every 1-manifold Ω and pivotal category C. The morphisms $\int_\Omega C(A \to B)$ are a quotient of the free $\mathbb{k}$-vector space on $\underline{\Sigma}(A \to B)$ by isotopy (making the composition associative on-the-nose), and by a pair of relations called *local replacement* and *linearity*, which we describe below. The identity $A \to A$ is just the labelled string diagram formed from the product $A \times I$ using the labels of A.

The local replacement relation imposed on $\underline{\Sigma}(A \to B)$ asserts that a disk in any string diagram may be substituted for any other disk which represents the same morphism in C. In order to decide if two diagrams in disks (with the same C-decorated boundary) represent the same morphism, we *rectify* them into rectangles which can be evaluated by Theorem 1.4.4. This is done by inflating an unlabelled closed interval in their boundary into the top three sides of a rectangle, with the bottom side formed from the remainder of the boundary. The result are **eval**uable rectangles. An example of a rectified string diagram is depicted below, with the

[12]The algorithm is explained in detail in [40] using the formalism of Penrose diagrams [24].

chosen unlabelled interval shown in black.



Since the sets $\mathcal{C}(X \to Y)$ are vector spaces, there is a notion of a linear relation in the free vector space on string diagrams $\underline{\Sigma}(A \to B)$. Namely, a formal scalar multiple $\lambda \cdot D$ of a diagram is declared equal to any diagram formed by multiplying a morphism label in $\mathcal{D}$ by λ. Moreover if a diagram D has a morphism label which is a sum $f + g$ of morphisms, then we declare D to be equal to the formal sum $D_f + D_g$ of the diagrams D_f and D_g respectively obtained by replacing the label $f + g$ with f or g.

Thus we obtain vector space homsets $\int_\Omega \mathcal{C}(A \to B)$. The category $\int_\Omega \mathcal{C}$ also has a zero object by labelling an arbitrary point in each connected component of Σ with the zero object of $\mathcal{C}$. In Chapter 2 we will see that when Ω is connected $\int_\Omega \mathcal{C}$ also has direct sums, so is often a $\Bbbk$-linear category. When $\Omega = I$ there is not only a category $\int_I \mathcal{C}$, but horizontal concatenation of diagrams (which we saw informally above) gives a strict monoidal product. We have the following famous result (originally due to Joyal and Street [26], see [38] for a summary of related results) on the structure of pivotal categories.

Theorem 1.4.6. *Every pivotal category $\mathcal{C}$ is pivotally equivalent to the diagram category $\int_I \mathcal{C}$.*

Proof. It is sufficient to verify that **eval** descends to a map from $\int_I \mathcal{C}(A \to B)$. This is done in [40, 26]. □

1.5 The Drinfeld center of a $(\mathcal{C}, \mathcal{C})$-bimodule category

Definition 1.5.1. Whenever $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{C})$-bimodule category we can form the *Drinfeld center*[13], a new category where

- its objects are pairs (M, β) for $M \in \mathcal{M}$ and a natural isomorphism $\beta_X : M \triangleleft X \to X \triangleright M$ (with $X \in \mathcal{C}$) called a *half-braiding*, such that for all $X, Y \in \mathcal{D}$ the diagram

$$\begin{tikzcd} M \triangleleft (X \otimes Y) \arrow[r, "n_{M,X,Y}"] \arrow[d, "\beta_{X\otimes Y}"] & (M \triangleleft X) \triangleleft Y \arrow[d, "\beta_X \triangleleft Y"] \\ (X \otimes Y) \triangleright M \arrow[d, "m_{X,Y,M}"] & (X \triangleright M) \triangleleft Y \arrow[d, "s_{X,M,Y}"] \\ X \triangleright (Y \triangleright M) & X \triangleright (M \triangleleft Y) \arrow[l, "X \triangleright \beta_Y"] \end{tikzcd} \tag{1.5.1}$$

  is commutative, and

[13] This notion of Drinfeld center of a bimodule category ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ was first given in [18]. The original notion for a monoidal category $\mathcal{C}$ is much older and due to unpublished work of Drinfeld [12], and appeared in [30, 25].

- its morphisms $(M,\beta) \to (N,\gamma)$ are maps $f : M \to N$ in $\mathcal{M}$ such that for every $X \in \mathcal{C}$ the diagram

$$\begin{tikzcd} M \triangleleft X \arrow[r, "\beta_X"] \arrow[d, "f \triangleleft X"] & X \triangleright M \arrow[d, "X \triangleright f"] \\ N \triangleleft X \arrow[r, "\gamma_X"] & X \triangleright N \end{tikzcd} \tag{1.5.2}$$

  is commutative.

Morphisms compose because squares of the form (1.5.2) juxtapose, and the identity map for any $(M,\beta) \in \mathcal{Z}(\mathcal{M})$ has underlying morphism the identity on M in $\mathcal{M}$.

It is easy to see that $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ has direct sums, since the module action functors $\triangleright$ and $\triangleleft$ must necessarily preserve them. Thus $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ is a $\Bbbk$-linear category.

Example 1.5.2 (Drinfeld center of a monoidal category)**.** Every monoidal category $\mathcal{C}$ is itself naturally a $(\mathcal{C},\mathcal{C})$-bimodule category (Example 1.3.3), and hence we obtain the Drinfeld center $\mathcal{Z}(\mathcal{C}) := \mathcal{Z}({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}})$ of a monoidal category. The category $\mathcal{Z}(\mathcal{C})$ originally appeared in [30, 25], and has found many applications. One reason is that when $\mathcal{C}$ is fusion the category $\mathcal{Z}(\mathcal{C})$ is a so-called *modular tensor category* [33], which is an object that is extremely overconstrained by Galois theory and the representation theory of $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$. Thus $\mathcal{Z}(\mathcal{C})$ can be used to control $\mathcal{C}$ in many cases.

Example 1.5.3 (Drinfeld center of a functor)**.** Every monoidal functor $F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$ naturally equips a $(\mathcal{D},\mathcal{D})$-bimodule category ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}$ with the structure ${}_{F}\mathcal{M}_{F}$ of a $(\mathcal{C},\mathcal{C})$-bimodule category as in Proposition 1.3.4, of which we can also take the center $\mathcal{Z}({}_{F}\mathcal{M}_{F})$. As a special case, such an F equips $\mathcal{D}$ itself with a $(\mathcal{C},\mathcal{C})$-bimodule category structure. Thus in particular we can talk of the center $\mathcal{Z}(F) := \mathcal{Z}({}_{F}\mathcal{D}_{F})$ of a monoidal functor F.[14]

Proposition 1.5.4 (Bowtie product)**.** *Let $F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$ be a monoidal functor and let ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}$ be a $(\mathcal{D},\mathcal{D})$-bimodule category. Then there are left and right action functors*

$$\ltimes : \mathcal{Z}(F) \times \mathcal{Z}({}_{F}\mathcal{M}_{F}) \to \mathcal{Z}({}_{F}\mathcal{M}_{F}) \text{ and}$$
$$\rtimes : \mathcal{Z}({}_{F}\mathcal{M}_{F}) \times \mathcal{Z}(F) \to \mathcal{Z}({}_{F}\mathcal{M}_{F}).$$

Proof. We set $(Y,\beta) \ltimes (M,\gamma) := (Y \triangleright M, \alpha)$ with each component α_X for $X \in \mathcal{C}$ defined by the commutative diagram

$$\begin{tikzcd} (Y \triangleright M) \triangleleft F(X) \arrow[r, "s_{Y,M,X}"] \arrow[d, dotted, "\alpha_X"] & Y \triangleright (M \triangleleft F(X)) \arrow[r, "Y \triangleright \gamma_X"] & Y \triangleright (F(X) \triangleright M) \\ F(X) \triangleright (Y \triangleright M) & (F(X) \otimes Y) \triangleright M \arrow[l, "m_{F(X),Y,M}"'] & (Y \otimes F(X)) \triangleright M. \arrow[l, "\beta_X \triangleright M"'] \arrow[u, "m_{Y,F(X),M}"'] \end{tikzcd} \tag{1.5.3}$$

These components are obviously natural, and checking that they obey (1.5.1) amounts to filling a large square with the associativity constraint diagrams for s and m together with the half-braiding diagrams for β and γ. For $f : (Y_1,\beta_1) \to (Y_2,\beta_2)$ and $g : (M_1,\gamma_1) \to (M_2,\gamma_2)$ we set $f \ltimes g := f \triangleright g$. The fact that $f \triangleright g$ is a morphism in $\mathcal{Z}({}_{F}\mathcal{M}_{F})$ is readily checked by using the

[14] We calculate a toy example in the next section.

fact that f and g are themselves morphisms in Drinfeld centers. The action $(M,\gamma) \rtimes (Y,\beta)$ has underlying object $M \triangleleft Y$ for which a half-braiding is constructed completely analogously. □

Proposition 1.5.5. *The categories* $\mathcal{Z}(\mathcal{C})$ *and* $\mathcal{Z}(F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D})$ *are monoidal. More generally, the left and right actions of Proposition 1.5.4 equip* $\mathcal{Z}({}_F\mathcal{M}_F)$ *with a* $(\mathcal{Z}(F),\mathcal{Z}(F))$*-bimodule category structure.*

Proof. All of the claims follow from the same computation (note that $\mathcal{Z}(\mathcal{C})$ is the center of the identity functor on $\mathcal{C}$). For $(X,\beta),(Y,\gamma) \in \mathcal{Z}(F)$ and $(M,\delta) \in \mathcal{Z}({}_F\mathcal{M}_F)$ the half-braiding identity (1.5.1) forces the left associator isomorphism to be a morphism

$$m_{X,Y,M} : ((X,\beta) \otimes (Y,\gamma)) \ltimes (M,\delta) \to (X,\beta) \ltimes ((Y,\gamma) \ltimes (M,\delta))$$

in $\mathcal{Z}({}_F\mathcal{M}_F)$. Consequently $m_{X,Y,M}$ is the desired left associator for $\ltimes$. The unit object in $\mathcal{Z}(F)$ is $\mathbb{1}^{\mathcal{Z}(F)} := (\mathbb{1}^{\mathcal{D}}, r^{\mathcal{D}-1}_{F(-)} \circ l^{\mathcal{D}}_{F(-)})$, and the unitors are just those in ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}$. The fact that the unitors are morphisms in $\mathcal{Z}({}_F\mathcal{M}_F)$ now follows directly from the module category axioms. The situation for $\rtimes$ is completely dual. □

Proposition 1.5.6. *The category* $\mathcal{Z}(\mathcal{C})$ *is braided.*

Proof. We set $b_{(X,\beta),(Y,\gamma)} := \beta_Y$ for each $(X,\beta),(Y,\gamma) \in \mathcal{Z}(\mathcal{C})$. The fact that each component of b is a morphism in $\mathcal{Z}(\mathcal{C})$ follows from the half-braiding axiom for β combined with naturality of β. Naturality of b in the second variable amounts to naturality of β, and naturality of b in the first variable is asserted by the morphism compatibility square (1.5.2). One of the hexagon identities for β holds by the definition (1.5.1) of a half-braiding, and the other follows from the definition of the braiding on the tensor product $(X,\beta) \otimes (Y,\gamma)$. □

Proposition 1.5.7. *If* $\mathcal{C}$ *is rigid or pivotal then so is* $\mathcal{Z}(\mathcal{C})$*. In general* $\mathcal{Z}(F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D})$ *is rigid when* $\mathcal{C}$ *and* $\mathcal{D}$ *both are, and is pivotal when in* F *is a pivotal functor between pivotal categories.*

Proof. The left dual of $(X,\beta) \in \mathcal{Z}(F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D})$ is (X^*,γ) with $\gamma_Y = \beta^*_{{}^*Y}$, since monoidal functors preserve duals.

If F is pivotal it is sufficient to verify that each component of the pivotal structure on $\mathcal{D}$ is a morphism in $\mathcal{Z}(F)$. This slightly subtle; the proof of Proposition 2.3 in [21] of the $F = \mathrm{id}_{\mathcal{C}}$ case directly upgrades to the case of nontrivial F. □

Theorem 1.5.8. *If* $\mathcal{C}$ *and* $\mathcal{M}$ *are finitely semisimple then so is* $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$*.*

Proof. A standard proof in the $\mathcal{M} = \mathcal{C}$ case is in [33], and [18, 11] each contain the general claim. □

Proposition 1.5.9. *Let* ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}$ *be a* $(\mathcal{D},\mathcal{D})$*-bimodule category. Every monoidal functor* $F : \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$ *induces canonical image and restriction functors*

$$F_* : \mathcal{Z}(\mathcal{C}) = \mathcal{Z}({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}) \to \mathcal{Z}({}_F\mathcal{D}_F) = \mathcal{Z}(F) \quad \textit{and} \quad F^* : \mathcal{Z}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}) \to \mathcal{Z}({}_F\mathcal{M}_F).$$

Proof. Given $(X,\beta) \in \mathcal{Z}(\mathcal{C})$ we let $F_*(X,\beta)$ be the object $(F(X),\alpha)$ of $\mathcal{Z}(F)$ with each component α_Y defined by the composite

$$F(X) \otimes F(Y) \xrightarrow{J_{X,Y}} F(X \otimes Y) \xrightarrow{F(\beta_Y)} F(Y \otimes X) \xrightarrow{J^{-1}_{Y,X}} F(Y) \otimes F(X).$$

Naturality of α is immediate from the naturality of J and β. On morphisms $f : (X,\beta) \to (Y,\gamma)$ we set $F_*(f) = F(f)$, which yields a morphism in $\mathcal{Z}(F)$ by applying F to (1.5.2). This data obviously assembles into a functor.

To define F^* just note that given $(M,\gamma) \in \mathcal{Z}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}})$ we obtain an object $(M, \gamma_{F(-)})$ of $\mathcal{Z}({}_F\mathcal{M}_F)$ by whiskering—there is nothing to check. Setting $F^*(f) = f$ on morphisms yields the desired functor. $\square$

1.6 Toy example: $\mathcal{Z}$ of $\mathrm{Res}^G_H : \mathsf{Rep}\, G \to \mathsf{Rep}\, H$ for G cyclic

We begin with essentially the simplest possible example. Given any ($\Bbbk$-linear) monoidal category C there is an associated canonical monoidal functor $K_C : \mathsf{Vec} \to C$, defined by picking-out the tensor unit. We simply send n direct sum copies of the unit $\mathbb{1}^{\mathsf{Vec}}$ (from a skeleton of Vec) to the same number of direct sum copies of $\mathbb{1}^C$, and the extension of the definition of K_C to morphisms by preserving identities is clear as well. Monoidality of K_C is obtained just using the unitors in Vec and C. Indeed, it is easy to see that any monoidal functor $\mathsf{Vec} \to C$ must be naturally isomorphic to K_C.

Lemma 1.6.1. *Let $F : C \xrightarrow{\otimes} \mathcal{D}$ be a monoidal functor and let $(X, \beta) \in \mathcal{Z}(F)$. Then when $Y = \bigoplus_i Y_i$ is any direct sum of objects of C, the component $\beta_Y : X \otimes F(Y) \to F(Y) \otimes X$ is completely determined by the components β_{Y_i}. In particular, when C is semisimple any such natural transformation β is completely determined by its components at the simple objects of $\mathcal{D}$.*

Proof. If $\iota : Y_i \to Y$ is an inclusion, naturality of β yields that $\beta_Y \circ (X \otimes F(\iota)) = (F(\iota) \otimes X) \circ \beta_{Y_i}$. As we allow Y_i to vary over all direct summands, we obtain β_Y as a direct sum of the maps $(\iota \otimes X) \circ \beta_{Y_i}$, as desired. $\square$

This is a manifestation of the general phenomenon that natural transformations in semisimple categories are completely determined by their components on the simple objects. Conversely, a natural transformation can be defined by specifying its components on the simple objects exactly when those components pairwise satisfy every naturality square which can be drawn for them.

Proposition 1.6.2. *There is a monoidal equivalence $C \overset{\otimes}{\simeq} \mathcal{Z}(K_C)$.*

Proof. Given $(X, \beta) \in \mathcal{Z}(K_C)$ an object, by Lemma 1.6.1 every component of β is determined by $\beta_{\mathbb{1}} : X \otimes K_C(\mathbb{1}) \to K_C(\mathbb{1}) \otimes X$. Equivalently, using the unitors of C this is the same data an automorphism $\widetilde{\beta_{\mathbb{1}}} : X \to X$. But now suppressing the tensor unit in C the half-braiding hexagon (1.5.1) for (X, β) asserts[15] that $\widetilde{\beta_{\mathbb{1}}}^2 = \widetilde{\beta_{\mathbb{1}}}$. Thus $\widetilde{\beta_{\mathbb{1}}} = \mathrm{id}_X$ and the half-braiding β is unique. Any morphism in C commutes with the identities, so C and $\mathcal{Z}(K_C)$ are equivalent categories. The forgetful functor $U : \mathcal{Z}(K_C) \to C$ is monoidal, so the equivalence is as well by Proposition 1.2.16. $\square$

Henceforth in this section let $G = C_n$ be a cyclic group with generator g, let $H \subseteq G$ be a subgroup of order m, and let $R = \mathrm{Res}^G_H : \mathsf{Rep}\, G \to \mathsf{Rep}\, H$ be the restriction functor. Fixing a primitive nth root of unity ζ, in this situation G has n distinct irreducible representations τ^n_k (for $0 \le k < n$), where in τ^n_k the element $g^l \in G$ acts as multiplication by ζ^{kl}.

Lemma 1.6.3. *Every object (π, β) of $\mathcal{Z}(R)$ has the natural isomorphism β completely determined by the component $\beta_{\tau^n_1} : \pi \otimes \tau^n_1 \to \tau^n_1 \otimes \pi$, or equivalently, an endomorphism L of π for which $L^n = \mathrm{id}$.*

Proof. By the argument of Lemma 1.6.1 exploiting the naturality of β, we find that the components of β are determined by the n components corresponding to the n 1-dimensional irreducible representations of G (the simple objects of $\mathsf{Rep}\, G$). Given any such component $\beta_{\tau^n_k} : \pi \otimes \tau^n_k \to$

[15] This is a well-known consequence (Remark 2.2 of [36]) of the half-braiding axiom, which follows from naturality of β and the fact that $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$. We will see a variation of this in the proof of Proposition 3.2.3.

$\tau_k^n \otimes \pi$ for $0 \leq k < n$, the fact that $\mathsf{Rep}\, H$ is a symmetric monoidal category allows us to commute the tensor product in the target, and upon taking duals we equivalently obtain a family of components $\widetilde{\beta_{\tau_k^n}} : \pi \to \pi \otimes (\tau_k^{n*} \otimes \tau_k^n)$.

Critically, the representations τ_k^n are all 1-dimensional, so we can post-compose with the evaluation isomorphism $\tau_k^{n*} \otimes \tau_k^n \to \mathbb{1}$ and view each morphism $\widetilde{\beta_{\tau_k^n}}$ as the equivalent data of an endomorphism $L_k : \pi \to \pi$.

Applying the symmetric braiding in $\mathsf{Rep}\, H$ and then the manipulation involving duals which we have just described to the monoidality constraint imposed on β, we conclude that monoidality is equivalent to the requirement that the map $G \to \mathrm{End}(\pi)$ sending $g^k \in G$ to the endomorphism L_k be a group homomorphism. Explicitly, we have $L_{k+l} = L_k L_l$ (with $k + l$ taken modulo n).

Hence, all of β is completely determined by the endomorphism $L = L_1$ of π for which $L^n = \mathrm{id}$, or equivalently, its component $\beta_{\tau_1^n}$. □

Lemma 1.6.4. *Every object of $\mathcal{Z}(R)$ decomposes as a direct sum of objects (π, β) with*

- *π irreducible (hence 1-dimensional),*
- *the component $\beta_{\tau_1^n}$ equal to an nth root of unity times the braiding in $\mathsf{Rep}\, H$, and*
- *any choice of nth root of unity suffices to produce a corresponding object of $\mathcal{Z}(R)$.*

Proof. The recipe of Lemma 1.6.3 by which we can reconstruct all of β given $L \in \mathrm{End}(\pi)$ shows that if π is a direct sum and if in addition L splits as a direct sum of maps between summands of π, then the entire object (π, β) is itself a direct sum.

However by Schur's lemma π splits as a direct sum of multiples of the irreducible representations of H, and hence we can always diagonalise the resulting endomorphisms of these multiples since each satisfies $L^n = \mathrm{id}$ (each then has n distinct eigenvalues). As a consequence, every object of $\mathcal{Z}(R)$ is a direct sum of objects (π, β) with π 1-dimensional and β equivalent to the data of an endomorphism L of π of order n, hence a scalar and nth root of unity.

The recipe by which we can reconstruct all of β can now be applied beginning with any nth root of unity, since all of the relevant naturality squares are then trivial. □

Theorem 1.6.5. *The tensor category $\mathcal{Z}(R)$ is finitely semisimple with its set of simple objects in correspondence with pairs (k, l) with $0 \leq k < m$ and $0 \leq l < n$. The correspondence maps (k, l) to the object $(\tau_k^m, \kappa_{l,-}^n)$, where $\kappa_{l,-}^n$ is that natural isomorphism associated to ζ^l (for ζ an nth root of unity) by Lemma 1.6.4. Moreover, we have the formula*

$$(\tau_k^m, \kappa_{l,-}^n) \otimes (\tau_{k'}^m, \kappa_{l',-}^n) \cong (\tau_{k+k'}^m, \kappa_{l+l',-}^n)$$

when we interpret $k + k'$ modulo m and $l + l'$ modulo n.

Hence in particular $\mathcal{Z}(R)$ has $nm = |G||H|$ simple objects.

Proof. We just need to show that $X = (\tau_k^m, \kappa_{l,-}^n)$ and $Y = (\tau_{k'}^m, \kappa_{l',-}^n)$ are isomorphic only when $k = k'$ and $l = l'$. Since any isomorphism $f : X \to Y$ in $\mathcal{Z}(R)$ forgets to an ordinary morphism $f : \tau_k^m \to \tau_{k'}^m$ in $\mathsf{Rep}\, H$, the requirement that $k = k'$ is immediate from Schur's lemma. Now remembering the constraint which a morphism in $\mathcal{Z}(R)$ must satisfy, we must have a commuting square

$$\begin{tikzcd}
\tau^m_k \otimes \mathbb{1} \arrow[r, "\kappa^n_{l,-}"] \arrow[d, "f\otimes\mathbb{1}"] & \mathbb{1}\otimes\tau^m_k \arrow[d, "\mathbb{1}\otimes f"] \\
\tau^m_k \otimes \mathbb{1} \arrow[r, "\kappa^n_{l',-}"] & \mathbb{1}\otimes\tau^m_k
\end{tikzcd} .$$

Just thinking of this square as a commutative diagram of linear maps of 1-dimensional vector spaces, we see directly that either $f = 0$ or $\kappa^n_{l,-} = \kappa^n_{l',-}$. The former is prohibited if f is an isomorphism, and the latter holds precisely when $l = l'$, exactly as desired.

In order to recover the tensor product rule we begin by recalling that $\tau^m_k \otimes \tau^m_{k'} \cong \tau^m_{k+k'}$ just by definition of the irreducible representations τ^m_k of H which we have been using. The extension of this fact to the rule

$$X \otimes Y = (\tau^m_k, \kappa^n_{l,-}) \otimes (\tau^m_{k'}, \kappa^n_{l',-}) \cong (\tau^m_{k+k'}, \kappa^n_{l+l',-})$$

is then not any more difficult; once we forget the morphisms $\kappa^n_{l,-}$ and $\kappa^n_{l',-}$ down to linear maps, one-dimensionality of all of the irreducible representations yields that the half-braiding of $X \otimes Y$ corresponds to to $\zeta^l \zeta^{l'} = \zeta^{l+l'}$, as desired. □

Proposition 1.6.6. *The simple objects of $\mathcal{Z}(R)$ are all realised as the image of simple objects of $\mathcal{Z}(\mathsf{Rep}\, G)$ under the map $F_* : \mathcal{Z}(\mathsf{Rep}\, G) \to \mathcal{Z}(R)$. The simple objects of $\mathcal{Z}(R)$ in the image of $F^* : \mathcal{Z}(\mathsf{Rep}\, H) \to \mathcal{Z}(R)$ are all those of the form $(\tau^n_k, \kappa^n_{l,-})$ with l a multiple of $[G : H]$.*

Proof. Using the correspondence of Theorem 1.6.5 between simple objects of $\mathcal{Z}(\mathsf{Rep}\, G)$ and pairs of bounded integers, the object $(\tau^n_k, \kappa^n_{l,-})$ is mapped via F_* to the object $(\tau^m_{k \bmod m}, \kappa^n_{l,-})$. Hence pairs (k, l) and (k', l') give the same simple object under F_* if and only if $k \equiv k' \bmod m$.

Similarly, F_* fixes the underlying $\mathsf{Rep}\, H$-object of any simple object $(\tau^m_k, \kappa^m_{l,-}) \in \mathcal{Z}(\mathsf{Rep}\, H)$, and the half-braiding becomes[16] $\kappa^n_{[G:H]l,-}$. □

Remark 1.6.7. In general it should be expected that computing $\mathcal{Z}(\mathrm{Res}^G_H)$ for G a finite group and $H \subseteq G$ any subgroup is a difficult problem. Indeed, by taking $H = G$ in Theorem 1.6.5 we described the ordinary center $\mathcal{Z}(\mathsf{Rep}\, G)$ when G is cyclic. Already the structure of $\mathcal{Z}(\mathsf{Rep}\, G)$ when G is not cyclic is complicated [5, 20].

1.7 The bimodule functor category $[{}_{C}C_{C} \to {}_{C}\mathcal{M}_{C}]$

Just as a $\Bbbk$-linear monoidal category C categorifies the notion of a ring, its center $\mathcal{Z}(C)$ categorifies the ordinary center of a ring. The following proposition gives a precise formulation of this latter claim, since the center of a ring R is isomorphic to its ring of bimodule endomorphisms. Our Proposition 1.7.1 was first proved for the $\mathcal{M} = C$ case in [35] as Proposition 2.5, and was extended to the general case in [18] (see Remark 2.2 thereof).

Proposition 1.7.1. *The categories $\mathcal{Z}({}_{C}\mathcal{M}_{C})$ and $[{}_{C}C_{C} \to {}_{C}\mathcal{M}_{C}]$ are equivalent.*

[16] The fact that there is any change here is perhaps a slightly subtle point. Although the underlying linear map of the half-braiding does not change at all, by our definitions $\kappa^m_{l,-}$ should multiply by the lth-power of a fixed mth root of unity, while $\kappa^n_{l,-}$ should multiply by the lth power of an nth root of unity. Thus l must change to $[G : H]l$ to accommodate the fact that the associated linear map is just multiplication by the same underlying constant.

Proof. Let $F : {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ be a bimodule functor with left modulator c and right modulator d. We canonically associate an object $M := F(\mathbb{1})$ of $\mathcal{M}$ to F, and moreover have a natural isomorphism (for $X \in \mathcal{C}$)

$$\tau_X : F(X) \xrightarrow{F(r_X^{\mathcal{C}-1})} F(X \otimes \mathbb{1}) \xrightarrow{c_{X,\mathbb{1}}} X \triangleright F(\mathbb{1}) = X \triangleright M.$$

Note that the functor $- \triangleright M : \mathcal{C} \to \mathcal{M}$ is made a left $\mathcal{C}$-module functor by the left module associator m of $\mathcal{M}$. We see that the natural isomorphism τ_X is an isomorphism of $\mathcal{C}$-module functors by combining naturality and the modulator associativity constraint for c with the identity constraint for m. Remembering the right $\mathcal{C}$-module functor structure on F, the isomorphism τ_X canonically equips the functor $- \triangleright M$ with the structure of a right $\mathcal{C}$-module functor.

Thus it suffices to prove the claim for the full subcategory of $[{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}]$ consisting of those functors F which are left action on a fixed object of $M \in \mathcal{M}$ with the left modulator induced by m and an arbitrary right modulator d. For such F we have a natural isomorphism

$$\beta_X : X \triangleright M \xrightarrow{F(l_X^{\mathcal{C}-1})} (\mathbb{1} \otimes X) \triangleright M \xrightarrow{d_{\mathbb{1},X}} (\mathbb{1} \triangleright M) \triangleleft X \xrightarrow{F(l_M^{\mathcal{M}})} M \triangleleft X.$$

We obtain an object $(M, \beta) \in \mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$, by filling in (1.5.1) with the bimodule functor condition (1.3.1) to obtain a commutative diagram. On the other hand d is completely determined by β_X since the bimodule actions in $\mathcal{C}$ are both just tensor product in $\mathcal{C}$.

The constraint for a morphism $\eta : (- \triangleright M) \to (- \triangleright N)$ of left $\mathcal{C}$-module functors exactly says that every component η_X is determined by the component $\eta_{\mathbb{1}} : \mathbb{1} \triangleright M \to \mathbb{1} \triangleright N$, i.e exactly a morphism $f : M \to N$. If η is a morphism of $(\mathcal{C}, \mathcal{C})$-bimodule functors then corresponding constraint for the right module structure precisely asserts that the morphism constraint (1.5.2) for the Drinfeld center commutes. The equivalence of $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ and $[{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}]$ now follows immediately. □

Proposition 1.7.2. *When $\mathcal{M} = \mathcal{C}$ the equivalence between $\mathcal{Z}(\mathcal{C})$ and $[{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}]$ is monoidal.*

Proof. Recall from Example 1.2.5 that the monoidal structure in $[{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}]$ is given by composition of functors. By Proposition 1.2.16 it is sufficient to produce a tensorator for the functor $\mathcal{Z}({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}) \to [{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}]$. From the explicit description of this functor above we see that the associator of $\mathcal{C}$ suffices for this purpose. □

Representations of the annular category

In this chapter we prove that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a pivotal category $\mathcal{C}$ is contravariantly equivalent as a braided monoidal category to the representation category of the annular category of $\mathcal{C}$ (and we define the latter object). Our results are folklore[1] in the TQFT (topological quantum field theory) community, and we check them in much greater detail than has ever been done previously.

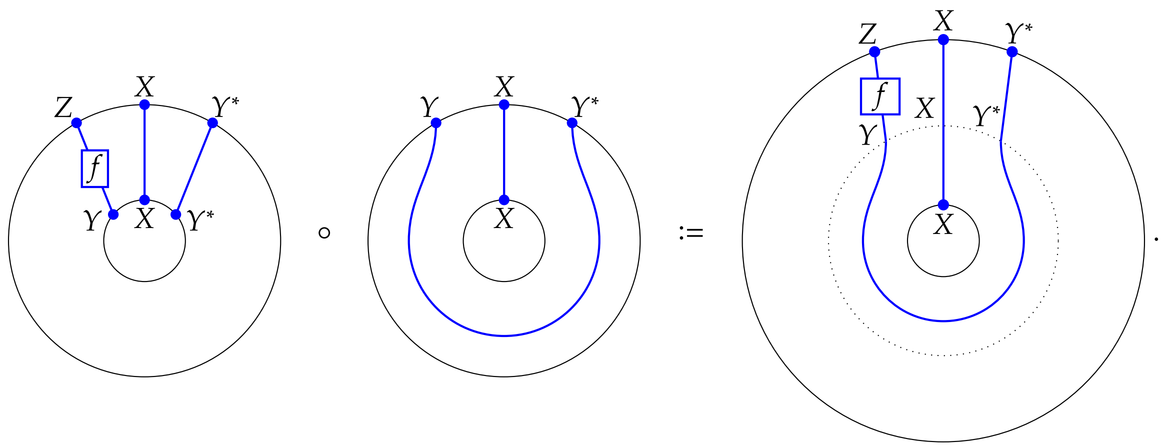
Definition 2.0.1. The *annular category* of a pivotal category $\mathcal{C}$ is the category of diagrams $\int_{S^1} \mathcal{C}$ (as in Section 1.4).

Recall that the objects of $\int_{S^1} \mathcal{C}$ are $\mathcal{C}$-*decorated* copies of S^1, with finitely many points of S^1 labelled by an object of $\mathcal{C}$, for example as depicted in Figure 2.1a with $X, Y, Z, W, U \in \mathcal{C}$. Moreover, we will think of the morphisms from some object ω to some other object τ as (equivalence classes of) string diagrams drawn in an annulus which have inner boundary ω and outer boundary τ. The identity $\omega \to \omega$ consists of a finite number of radial arcs, for example as depicted in Figure 2.1b.

(a) (b)

Figure 2.1: Objects and identities in $\int_{S^1} \mathcal{C}$.

A pair of morphisms $f : \omega \to \tau$ and $g : \tau \to \kappa$ are composed by lining up the outer annulus of f with the inner annulus of g and "fusing" the result along the boundary; for example



[1] The only reference we can find is Paragraph 5.2.30 of Kevin Walker's TQFT notes [41].

This operation is obviously associative, and thus we obtain a category. We will see in Section 2.2 that $\int_{S^1} \mathcal{C}$ is $\Bbbk$-linear, but $\int_{S^1} \mathcal{C}$ does not have a natural monoidal structure.

The category $\int_{S^1} \mathcal{C}$ is closely related to a certain ordinary algebraic gadget called the *tube algebra* of $\mathcal{C}$ [34]. When $\mathcal{C}$ is a finitely semisimple pivotal category the tube algebra is defined to be the direct sum

$$\mathrm{Tube}(\mathcal{C}) = \bigoplus_{X,Y,W} \mathcal{M}(X \otimes W^* \to W \otimes Y)$$

with X, Y, W all ranging over a set $\mathrm{Irr}(\mathcal{C})$ of representatives of isomorphism classes of the simple objects of $\mathcal{C}$. For $\zeta \in \mathcal{M}(X_1 \otimes W_1^* \to W_1 \otimes Y_1)$ and $\xi \in \mathcal{M}(X_2 \otimes W_2^* \to W_2 \otimes Y_2)$ the product $\zeta \cdot \xi$ is defined to be zero if $Y_1 \neq X_2$, and otherwise is a certain sum over a basis for the homsets $\mathcal{C}(W_2 \otimes W_1 \to Z)$ with $Z \in \mathrm{Irr}(\mathcal{C})$. We use a similar construction in the proof of Proposition 2.3.3.

The algebra $\mathrm{Tube}(\mathcal{C})$ can be thought of as a one-object category, and upon taking the direct sum completion and *Karoubi envelope* we obtain a category $\mathrm{Mat}(\mathrm{Kar}(\mathrm{Tube}(\mathcal{C})))$ which is equivalent to $\int_{S^1} \mathcal{C}$. This equivalence has antecedents in the subfactor literature, but it is difficult to translate this earlier work because the description is given in terms of endomorphisms of factors, rather than fusion categories. The upshot is that since $\mathrm{Tube}(\mathcal{C})$ is a finite dimensional algebra, explicit computations are tractable with computer assistance [22, 23]. The results of this chapter then give corresponding implications for $\mathcal{Z}(\mathcal{C})$.

2.1 Basic properties of representation categories

The theory of linear representations of classical algebraic gadgets can be realised as a special case of a particular categorical construction known as taking the "representation category".

Definition 2.1.1. For any ordinary category $\mathcal{C}$, its *(linear) representation category* $\mathsf{Rep}\,\mathcal{C}$ is the functor category $[\mathcal{C} \to \mathsf{Vec}]$ (also known as the category of $\mathcal{C}^{\mathrm{op}}$ presheaves). That is, the objects of $\mathsf{Rep}\,\mathcal{C}$ are functors $\mathcal{C} \to \mathsf{Vec}$, and the morphisms or *intertwiners* between these functors are just ordinary natural transformations between them.

For example, let G be a finite group. Each such G is equivalent to the data of a (not $\Bbbk$-linear) category $\mathbf{G}$ with a single object $\bullet$ and a morphism for each element of G—the composition is just multiplication in G. It is easy to see that a (finite dimensional) linear representation of G is precisely the same data as a functor $R : \mathbf{G} \to \mathsf{Vec}$; namely elements of G act on the vector space $R(\bullet)$ by their image under R. The formal equivalence of all of this data is encoded by the equivalence (actually, isomorphism) of the categories $\mathsf{Rep}\,\mathbf{G}$ and the category of finite dimensional G-representations $\mathsf{Rep}\,G$.

When $\mathcal{C}$ is $\Bbbk$-linear we naturally require that $\mathsf{Rep}\,\mathcal{C}$ consist of only the linear functors from $\mathcal{C}$ to Vec and their natural transformations. In the setting of a finite group G as before there is an analogous category $\mathbf{G}_\Bbbk$ with a single object, now with its endomorphisms the group algebra $\Bbbk[G]$. It is easy to see that in this case $R \in \mathsf{Rep}\,\mathbf{G}_\Bbbk$ is exactly the data of a vector space $V \in \mathsf{Vec}$ together with an algebra homomorphism $\Bbbk[G] \to \mathrm{End}(V)$, i.e. again just a linear representation of G, with an analogous equivalence result involving $\mathsf{Rep}\,\mathbf{G}_\Bbbk$ holding.

Proposition 2.1.2. *The functor category $[\mathcal{C} \to \mathcal{D}]$ is $\Bbbk$-linear whenever $\mathcal{D}$ is.*

Proof. Sums and scalar multiples of natural transformations of functors $\mathcal{C} \to \mathcal{D}$ are themselves natural transformations, and so $[\mathcal{C} \to \mathcal{D}]$ has vector space homsets (with a bilinear composition). There is always a zero functor (mapping all objects and morphisms to the zero object or morphism), and one now checks directly that we can similarly take direct sums "pointwise in $\mathcal{D}$". □

Definition 2.1.3. Let C be a $\Bbbk$-linear[2] category. Then C *has all copowers* (or alternatively, *is tensored over* Vec) if for all $X \in C$ the functor $C(X \to -) : C \to \mathsf{Vec}$ has a left-adjoint functor $- \odot X : \mathsf{Vec} \to C$. That is, for all $U \in \mathsf{Vec}$ and $X, Y \in C$ there is a natural isomorphism (in Vec)

$$C(U \odot X \to Y) \xrightarrow{\sim} \mathsf{Vec}(U \to C(X \to Y)). \tag{2.1.1}$$

If such a functor $- \odot X : \mathsf{Vec} \to C$ exists it is determined up to isomorphism by the adjunction. We will often omit "$\odot$" when no ambiguity is left.

Proposition 2.1.4. *Every $\Bbbk$-linear category C is tensored over* Vec*.*

Proof. We will just define the functor $- \odot - : \mathsf{Vec} \times C \to C$ using the skeleton of Vec consisting of direct sum copies of the unit $\mathbb{1}_\Bbbk \in \mathsf{Vec}$. In this case for any $X \in C$ we just set $k\mathbb{1}_\Bbbk \odot X = kX$ (with kY the k-fold direct sum of an object Y with itself). Now given $L : k\mathbb{1}_\Bbbk \to l\mathbb{1}_\Bbbk$ and $f : X \to Y \in C$, we need to define $L \odot f : kX \to lY$. Let $\iota_{Z,i} : Z \to nZ$ denote the inclusion of a single $Z \in C$ summand into the ith factor of some number of direct sum copies of Z (we infer n from context), and similarly let $\pi_{Z,i} : nZ \to Z$ be the projection onto the ith summand. The 1-dimensional endomorphisms of $\mathbb{1}_\Bbbk$ which arise as the composites $L_{i,j} = \pi_{\mathbb{1}_\Bbbk,j} \circ L \circ \iota_{\mathbb{1}_\Bbbk,i}$ may now just be interpreted as the matrix entries of L written with respect to the standard basis of $k\mathbb{1}_\Bbbk$. Finally we set $\pi_{Y,j} \circ (L \odot f) \circ \iota_{X,i} = L_{i,j} f$. The functor $\odot$ then extends to all of $\mathsf{Vec} \times C$ (up to natural isomorphism). It is easy to see that the adjunction (2.1.1) is then satisfied. □

Proposition 2.1.5. *If C is $\Bbbk$-linear and finitely semisimple, then every $F : C \to \mathsf{Vec}$ in $\mathsf{Rep}\, C$ is explicitly representable by*

$$\bigoplus_{X_i \in \mathrm{Irr}(C)} F(X_i) X_i.$$

Proof. Given $F : C \to \mathsf{Vec}$ as above, consider the new functor $G : C \to \mathsf{Vec}$ defined by

$$G(X) = C\left(\bigoplus_{X_i \in \mathrm{Irr}(C)} F(X_i) X_i \to X \right).$$

(As usual we will have $G(f : X \to Y)$ just be the linear map given by post-composition with f.) For any $X_j \in \mathrm{Irr}(C)$ we then just have

$$\begin{aligned} G(X_j) &= C\left(\bigoplus_{X_i \in \mathcal{O}(C)} F(X_i) X_i \to X_j \right) \\ &\cong C\left(F(X_j) X_j \to X_j \right) \\ &\cong F(X_j), \end{aligned}$$

a composite of natural isomorphisms, as desired. □

Given an ordinary (not necessarily $\Bbbk$-linear) category C and object $X \in C$, let h^X be the hom-functor $C(X \to -) : C \to \mathsf{Set}$ to the category of sets. That is, on objects $Y \in C$ we will have $h^X(Y) = C(X \to Y)$, and on morphisms $f : Y \to Z$ the induced map $h^X(f) : C(X \to Y) \to C(X \to Z)$ will just be post-composition with f. In this situation, we have the following famous result.

[2] In fact, C need only be *enriched over $\Bbbk$-vector spaces* [28]. The notion of C having copowers then naturally extends to enrichment over any *closed* monoidal category $\mathcal{D}$.

Theorem 2.1.6 (Yoneda lemma)**.** *For any functor $F : \mathcal{C} \to \mathsf{Set}$ there is an isomorphism*

$$[\mathcal{C} \to \mathsf{Set}](h^X \to F) \cong F(X),$$

where it is understood that in the statement of this theorem it is asserted that the left-hand side is always a set.

Proof. Naturality of any $\eta \in [\mathcal{C} \to \mathsf{Set}](h^X, F)$ implies that the diagram

$$\begin{tikzcd} h^X(X) \arrow[r, "\eta_X"] \arrow[d, "h^X(f)"'] & F(X) \arrow[d, "F(f)"] \\ h^X(Y) \arrow[r, "\eta_Y"] & F(Y) \end{tikzcd}$$

commutes for every $f : X \to Y$ in $\mathcal{C}$. Since $\mathrm{id}_X \in h^X(X) = \mathcal{C}(X \to X)$, it follows that

$$\eta_Y(f) = \eta_Y(f \circ \mathrm{id}_X) = \eta_Y(h^X(f)(\mathrm{id}_X)) = F(f)(\eta_X(\mathrm{id}_X)),$$

and this implies that every component η_Y is completely determined by $\eta_X(\mathrm{id}_X) \in F(X)$.

Conversely, given $x \in F(X)$, the equation $\eta_Y(f : X \to Y) = F(f)(x)$ at least gives a well-defined family of functions $\eta_Y : h^X(Y) \to F(Y)$. In fact they assemble into a natural transformation $h^X \to F$ just by the unwrapping of the definition of naturality which we have just performed. These two constructions are obviously mutually inverse, and so the claim follows. □

The case of $F = h^Y$ for some $Y \in \mathcal{C}$ above is especially important. There the Yoneda lemma says explicitly that there is an isomorphism

$$[\mathcal{C}, \mathsf{Set}](h^X \to h^Y) \cong \mathcal{C}(Y \to X),$$

and the proof gives the formula

$$\eta_Z(f) = h^Y(f)(\eta_X(\mathrm{id}_X)) = f \circ \eta_X(\mathrm{id}_X)$$

for any $\eta : h^X \to h^Y$ and $f : X \to Z$. Thus we obtain the following.

Corollary 2.1.6.1. *Given a morphism of representable functors $\eta : h^X \to h^Y$, every component of η arises just as post-composition with the same morphism $f : Y \to X$, and any such morphism $Y \to X$ suffices to produce an entire natural transformation $h^X \to h^Y$.*

Moreover, there is a fully faithful functor $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathsf{Set}]$ (the Yoneda embedding) defined by sending X to h^X and a morphism $f : Y \to X$ to the natural transformation $\eta : h^X \to h^Y$ represented by f.

It is clear that Set can safely be replaced with Vec above, and we will use the corresponding results so obtained without further comment.

2.2 Objects of $\mathsf{Rep} \int_{S^1} \mathcal{C}$ for $\mathcal{C}$ semisimple

The purpose of this section is to prove the following proposition, and hence to understand representations of $\int_{S^1} \mathcal{C}$ when $\mathcal{C}$ is a finitely semisimple pivotal category. To simplify notation, we define $\mathsf{Rep}^{\mathrm{op}}\, \mathcal{D} = (\mathsf{Rep}\, \mathcal{D})^{\mathrm{op}}$ for any category $\mathcal{D}$.

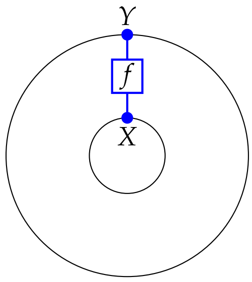
Proposition 2.2.1. *When C is finitely semisimple there is a faithful functor $F : \mathsf{Rep}^{\mathrm{op}} \int_{S^1} C \to \mathcal{Z}(C)$.*

We begin with a general construction which holds regardless of semisimplicity of C.

Proposition 2.2.2. *There is an essentially surjective functor $J : C \to \int_{S^1} C$.*

Proof. Such an "inclusion" functor $J_\Omega : C \to \int_\Omega C$ arises for each choice of closed interval in a 1-manifold Ω; this proposition is a special case, but the idea in general is exactly the same. It will also be easy to see that J_Ω is essentially surjective and up to natural isomorphism independent of the choice of open interval whenever Ω is connected.

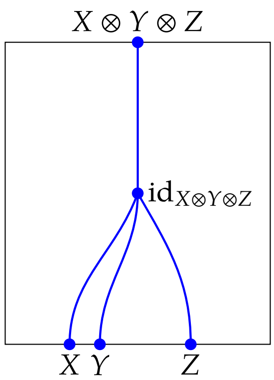
After fixing an open interval $K \subseteq \Omega$ observe that $K \times I$ is homeomorphic to $\Sigma_{\mathbb{R}}$ of Section 1.4, and hence (as in Theorem 1.4.4) each $f : X \to Y$ in C is canonically represented by a diagram in $K \times I$. Since $K \times I \subseteq \Omega \times I$ is an inclusion which respects boundaries we obtain a diagram in $\Sigma_\Omega = \Omega \times I$ associated to f. The associated diagrammatic picture is straightforward; given $f : X \to Y$, we simply draw the annular diagram



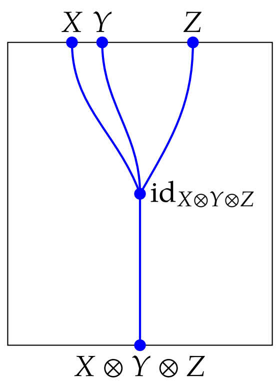
.

The bottom and top boundaries of the resulting labelled diagram are fixed for all morphisms $X \to Y$, and it is clear that the construction is functorial when diagrams are taken modulo isotopy. Hence this data assembles into a functor J.

Essential surjectivity is also not difficult to establish. We first show the claim for $\Omega = I$, in which case for any $\omega \in \int_I C$ we have an isomorphism



with inverse



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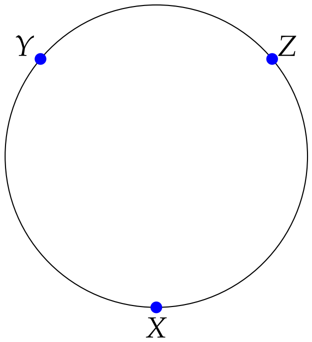
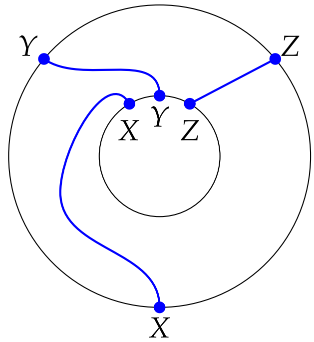
In the general case so long as Ω is connected there is an isotopy of Ω taking the distinguished points of any $S \in \int_\Omega C$ into the open interval K. From such an isotopy we obtain a morphism from S to another copy S' of Ω with all of its distinguished points in K, and this morphism is an isomorphism since the isotopy from which it is built is invertible. The diagrammatic perspective is that given $S \in \int_{S^1} C$ of the form



we always have an isomorphism



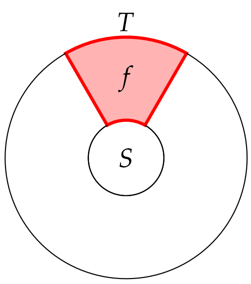
(2.2.1)

which "bunches up" the distinguished objects inside a small ball in S^1. Again so long as Ω is connected there is an isotopy of Ω taking any open interval $K \subseteq S^1$ into any other, and consequently the functor J constructed for a fixed K is naturally isomorphic to any other. $\square$

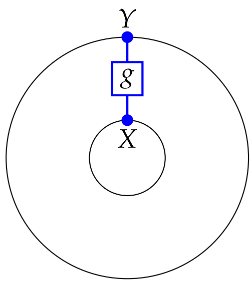
In what follows we sometimes refer to objects of C in place of objects of $\int_{S^1} C$, suppressing a choice of distinguished point on S^1 which actually carries the label. Up to equivalence, we can assume that C is strict as well by Theorem 1.2.6. Thus similarly when no confusion can arise we write an unassociated tensor product of objects of C (e.g. XY^*Z) for a copy of S^1 with a point labelled by each object in the tensor product as we pass clockwise. Also note that since C has direct sums and J is a linear functor, for all $X, Y \in C$ the image $J(X \oplus Y)$ is a direct sum of $J(X)$ and $J(Y)$—one simply checks that the conditions defining a direct sum are preserved by any linear functor. Since J is essentially surjective this implies that $\int_{S^1} C$ has direct sums as well, and thus is a $\Bbbk$-linear category.

Henceforth assume that the pivotal category C is also finitely semisimple, and suppose that we have some representation $V : \int_{S^1} C \to \mathsf{Vec}$. We will now try to understand V and in particular pin-down the data which V encodes.

The first observation we should make is that the inclusion functor $J : C \to \int_{S^1} C$ allows us to forget V down to a representation $V \circ J$ of C only. The hypothesis that C is finitely semisimple permits application of Proposition 2.1.5, and hence produces an object $X \in C$ so that $V \circ J$ is naturally isomorphic to the hom-functor $C(X \to -)$. This means that we know completely how V acts on morphisms in the image of J, i.e. morphisms $f : S \to T$ in $\int_{S^1} C$ which can be drawn entirely in the red shaded region below:

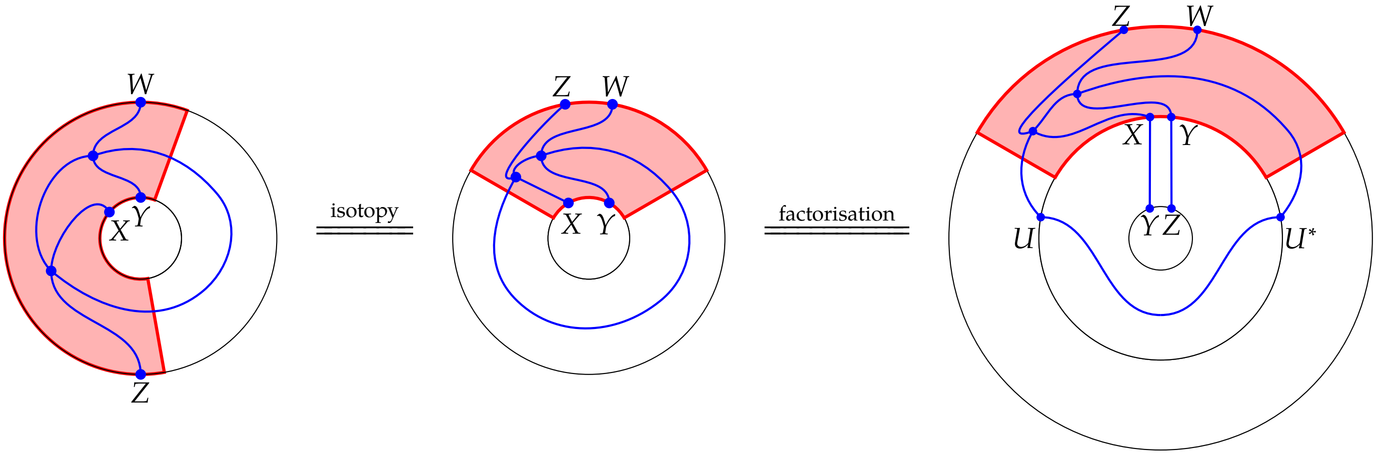


, e.g.



with $g \in C(X \to Y)$.

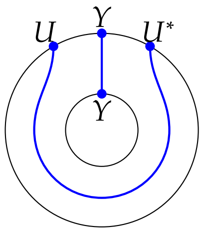
Namely, $V(J(f : Y \to Z))$ is just the map $C(X \to Y) \to C(X \to Z)$ given by post-composition with f. This already pins V down completely on a large class of morphisms in $\int_{S^1} C$. The key observation is that any morphism in $\int_{S^1} C$ can be isotoped so that "everything interesting appears in the image of J". Diagrammatically, we can always perform an isotopy taking a potentially complicated morphism (with some particular labels on its interior vertices) to a composite as shown below.



Of course in practice the object U might have to be some tensor product of objects of C which actually appear as labels in the original morphism $Y \to Z$. Diagrammatically this corresponds to bunching up multiple strings which pass around the back of the annulus (outside the shaded red region above) into a single string. Thus we have established the following lemma.

Lemma 2.2.3 (Annular factorisation). *Every morphism $Y \to Z$ factors as a composite of a morphism $Y \to UYU^*$ with a morphism $UYU^* \to Z$ for some $U \in C$, where:*

- *the first morphism $r_{Y,U} : Y \to UYU^*$ is precisely of the form*



, *and*

- *the second morphism $UYU^* \to Z$ is in the image of J.*

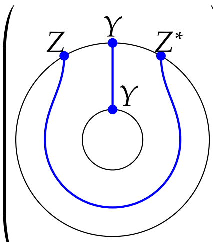
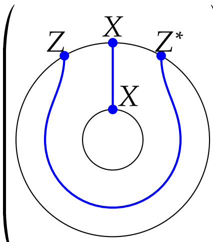
Note that such a factorisation is clearly non-canonical.

As a consequence of this factorisation, our representation V is completely determined by the representing object X together with knowledge of $V(r_{Y,Z})$ for every $Y, Z \in C$. The following lemma reduces this information further.

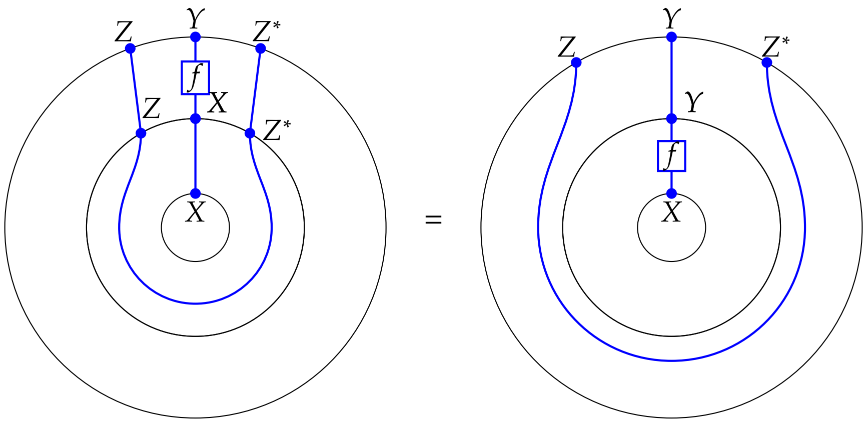
Lemma 2.2.4. *Let $V : \int_{S^1} C \to$ Vec be any representation which when restricted to a representation of C is represented by $X \in C$. Then given any morphism $f : X \to Y$ and object $Z \in C$ we have*

$$V(r_{Y,Z})(f) = ZfZ^* \circ V(r_{X,Z})(\mathrm{id}_X).$$

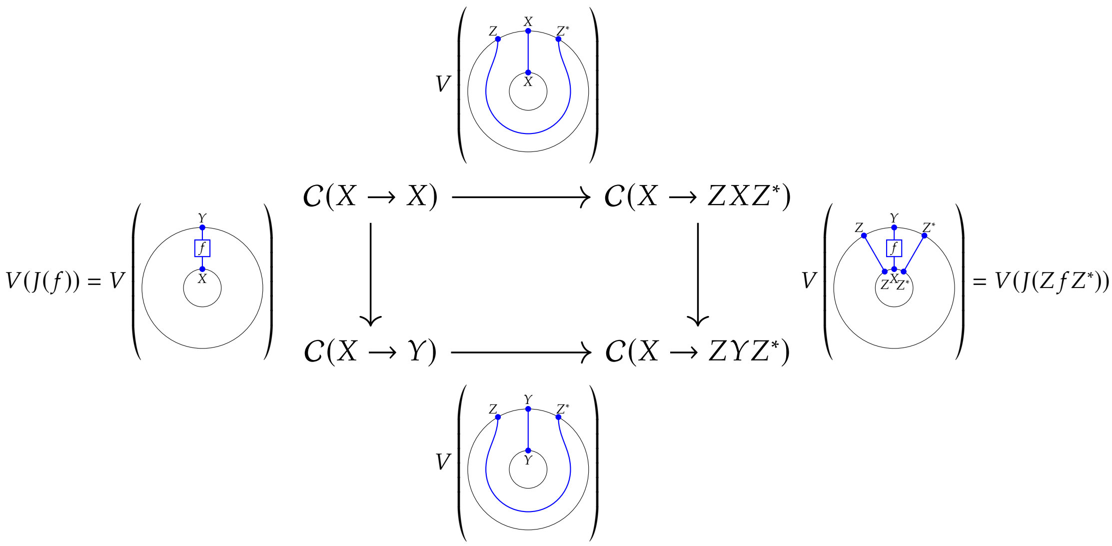
Diagrammatically,

$V \Bigg($  $\Bigg)(f) = ZfZ^* \circ V \Bigg($  $\Bigg)(\mathrm{id}_X).$

Proof. By an evident planar isotopy, we have the following equality of two composites of morphisms in $\int_{S^1} \mathcal{C}$.



Functoriality of $V : \int_{S^1} \mathcal{C} \to \mathsf{Vec}$ then yields a commutative diagram



Since the vertical morphisms each come from a morphism in $\int_{S^1} \mathcal{C}$ which lies in the image of $J : \mathcal{C} \to \int_{S^1} \mathcal{C}$, representability of V implies that the left-side morphism $V(J(f))$ is merely post-composition with f, and likewise for the right-side morphism.

Chasing the identity $\mathrm{id}_X : \mathcal{C}(X \to X)$ around this diagram then gives

$$\begin{tikzcd}
f \circ \mathrm{id}_X = f \arrow[d, mapsto] & \mathrm{id}_X \arrow[l, mapsto] \arrow[r, mapsto] & V(r_{X,Z})(\mathrm{id}_X) \arrow[d, mapsto] \\
V(r_{Y,Z})(f) & = & ZfZ^* \circ V(r_{X,Z})(\mathrm{id}_X)
\end{tikzcd},$$

precisely as desired. □

Lemma 2.2.4 is the key fact allowing us to calculate with arbitrary representations of $\int_{S^1} \mathcal{C}$.

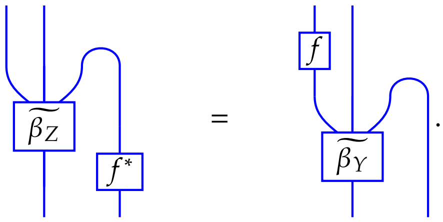
As a corollary, we deduce that $V : \int_{S^1} \mathcal{C} \to \mathsf{Vec}$ is completely determined by its $\mathcal{C}$-representing object X, along with a morphism $\widetilde{\beta_Y} : X \to YXY^*$ for every $Y \in \mathcal{C}$, each $\widetilde{\beta_Y}$ being obtained from evaluating $V(r_{X,Y}) : \mathcal{C}(X \to X) \to V(X \to YXY^*)$ at id_X. Now observe that each morphism $\widetilde{\beta_Y}$ is canonically adjoint (via Frobenius reciprocity) to a morphism $\beta_Y : XY \to YX$.

Proposition 2.2.5. *The morphisms $\beta_Y : XY \to YX$ thus constructed assemble into a natural transformation.*

Proof. Let $f : Y \to Z$ be any morphism in $\mathcal{C}$. We need to verify commutativity of the square

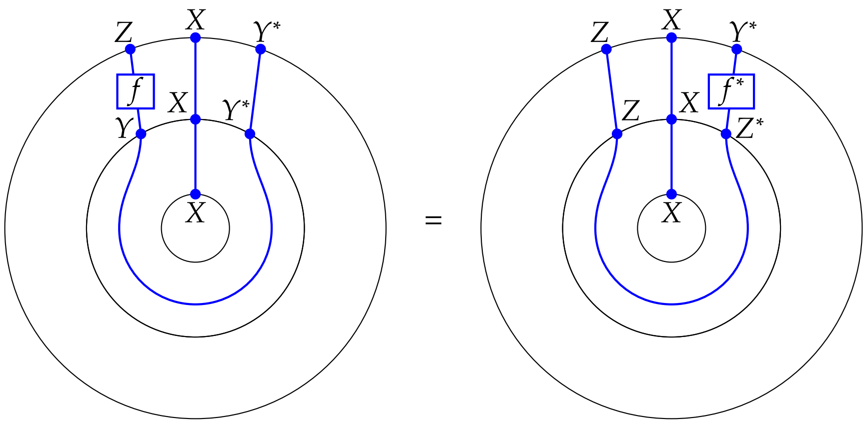
$$\begin{tikzcd} XY \arrow[r, "\beta_Y"] \arrow[d, "Xf"] & YX \arrow[d, "fX"] \\ XZ \arrow[r, "\beta_Z"] & ZX \end{tikzcd} .$$

Replacing both components β with their definitions in terms of the morphisms $\widetilde{\beta}$, this square is equivalent to the equality of string diagrams



(2.2.2)

Now, there is a planar isotopy between the following two composites of morphisms obtained just by sliding the box labelled "f" around the annulus.



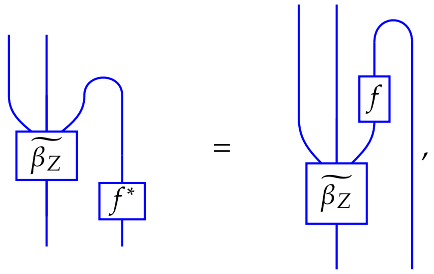
Thus we obtain another commuting diagram

$$\begin{tikzcd} \mathcal{C}(X \to X) \arrow[r, "V(r_{X,Y})"] \arrow[d, "V(r_{X,Z})"'] & \mathcal{C}(X \to YXY^*) \arrow[d, "(V \circ J)(fXY^*)"] \\ \mathcal{C}(X \to ZXZ^*) \arrow[r, "(V \circ J)(ZXf^*)"] & \mathcal{C}(X \to ZXY^*) \end{tikzcd} ,$$

and chasing the identity we find

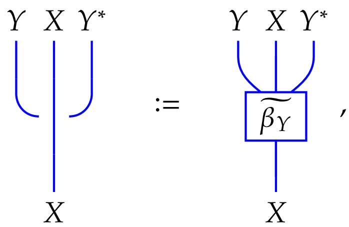
$$\begin{tikzcd} V(r_{X,Z})(\mathrm{id}_X) = \widetilde{\beta_Z} \arrow[d, mapsto] & \mathrm{id}_X \arrow[l, mapsto] \arrow[r, mapsto] & V(r_{X,Y})(\mathrm{id}_X) = \widetilde{\beta_Y} \arrow[d, mapsto] \\ ZXf^* \circ \widetilde{\beta_Z} & = & fXY^* \circ \widetilde{\beta_Y} \end{tikzcd} .$$

The right-hand side of this last equation is verbatim the right-hand side of (2.2.2), so writing $ZXf^* \circ \widetilde{\beta_Z}$ as a string diagram it suffices to check that

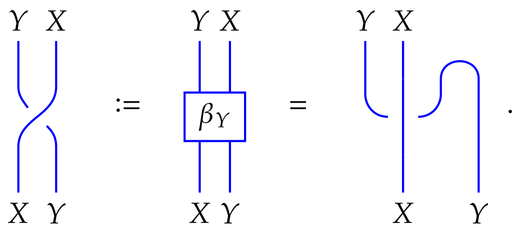


and thus we are done by the evident isotopy, or concretely, by an elementary property of duals. □

It will be convenient for us to introduce the notation

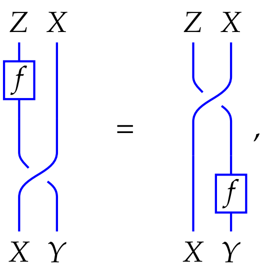


where it is meant that a single string joining Y and Y^* passes underneath the vertical X to X string. Similarly, we will define

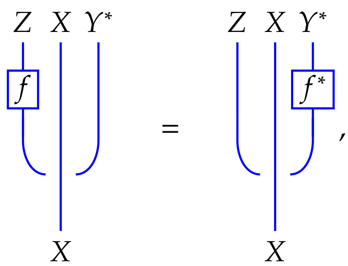


(2.2.3)

At least for now, we must be careful when manipulating this notation within our diagrammatic calculus. In particular, we must ensure that we do not perform moves which hold topologically, but which we have not established as algebraic properties of the underlying morphisms. For instance, it is the *content* of Proposition 2.2.5 that for any morphism $f : Y \to Z$ in $\mathcal{C}$, there is an equality of diagrams



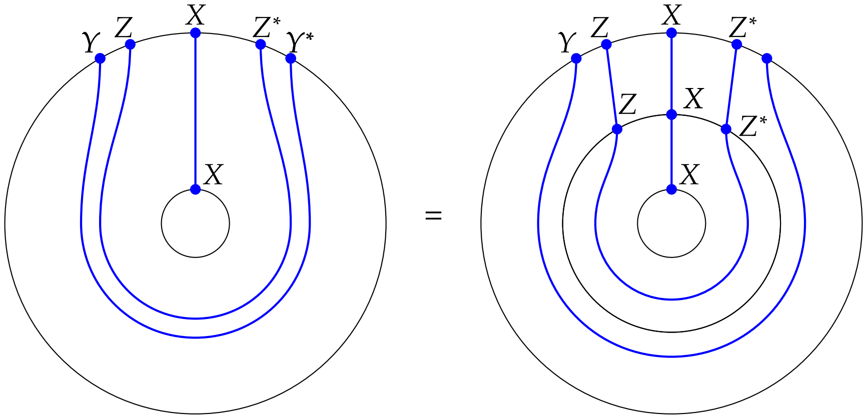
or equivalently



and hence the box labelled "f" may in either case be pulled-through the crossing under the X-to-X string. The following proposition establishes the similarly topologically intuitive property that a tensor product can be braided under an X-to-X string all in one go, or each object in the tensor product can be braided separately.

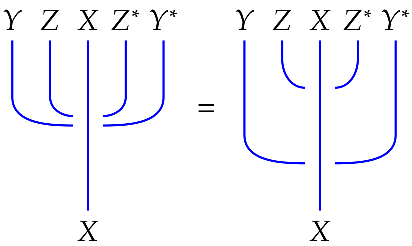
Proposition 2.2.6. *The natural transformation $\beta_Y : XY \to YX$ obeys the hexagon identity describing a half-braiding for X.*

Proof. There is again a planar isotopy witnessing an equality



,

and in turn functoriality of V gives rise to an equality $V(r_{X,YZ}) = V(r_{ZXZ^*,Y}) \circ V(r_{X,Z})$. Evaluating both sides at id_X gives $\widetilde{\beta_{YZ}} = V(r_{ZXZ^*,Y})(\widetilde{\beta_Z})$, and by Lemma 2.2.4 this further simplifies to $\widetilde{\beta_{YZ}} = Y\widetilde{\beta_Z}Y^* \circ \widetilde{\beta_Y}$. By unwrapping the definition of the components of β we are now done, since[3] this last equation establishes an equality of the diagrams



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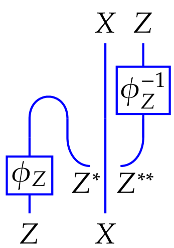
∎

Proposition 2.2.7. *Each component $\beta_Y : XY \to YX$ is an isomorphism.*

Proof. Fixing any $Y \in C$, we construct an explicit inverse $g : ZX \to XZ$ of $\beta_Z : XZ \to ZX$ by the composite

g :=


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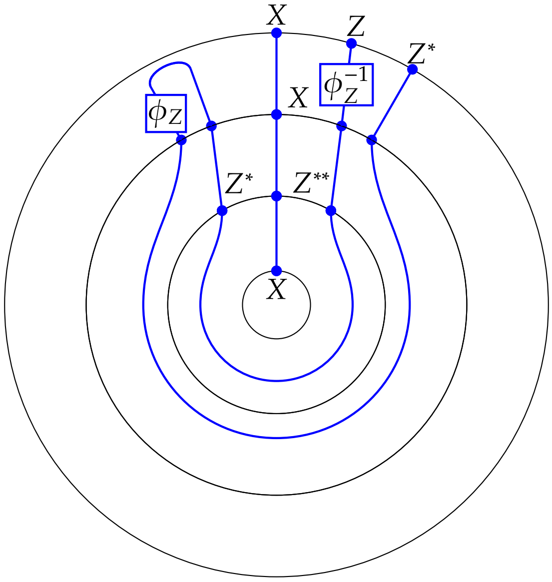
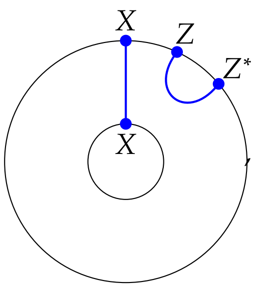
[3] All one must do to obtain the conventional half-braiding relation is to bend the two rightmost strings in each picture back down to the bottom using the evaluation map—the desired relation between the components of β is then directly recovered.

Drawing the diagram of composites of morphisms in $\int_{S^1} C$



$$,\tag{2.2.4}$$

we see that we have an isotopy to the much simpler morphism



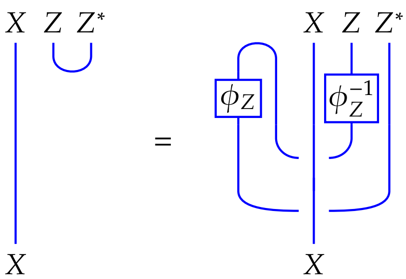
,

where we have annihilated the boxes labelled ϕ_Z and ϕ_Z^{-1} by bringing them together.

This isotopy thus asserts an equality of morphisms (again using Lemma 2.2.4)

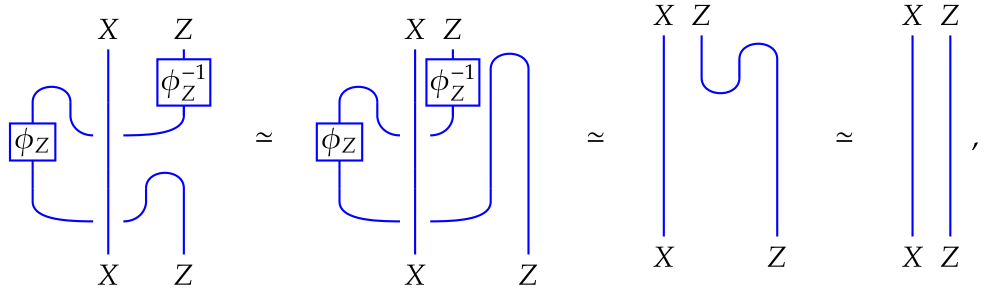
$$\begin{aligned}
X\,\mathrm{coev}_Z &= (V \circ J)(X\,\mathrm{coev}_Z)(\mathrm{id}_X)\\
&= \Big((V \circ J)((\mathrm{ev}_{Z^*} \circ \phi_Z Z^*)X\phi_Z^{-1}Z^*) \circ V(r_{Z^*XZ^{**},Z}) \circ V(r_{X,Z^*})\Big)\,(\mathrm{id}_X)\\
&= \Big((V \circ J)((\mathrm{ev}_{Z^*} \circ \phi_Z Z^*)X\phi_Z^{-1}Z^*) \circ V(r_{Z^*XZ^{**},Z})\Big)\,(\widetilde{\beta_{Z^*}})\\
&= (V \circ J)((\mathrm{ev}_{Z^*} \circ \phi_Z Z^*)X\phi_Z^{-1}Z^*)(Z\widetilde{\beta_{Z^*}}Z^* \circ \widetilde{\beta_Z})\\
&= (\mathrm{ev}_{Z^*} \circ \phi_Z Z^*)X\phi_Z^{-1}Z^* \circ Z\widetilde{\beta_{Z^*}}Z^* \circ \widetilde{\beta_Z},
\end{aligned}$$

which in terms of an equality of string diagrams says

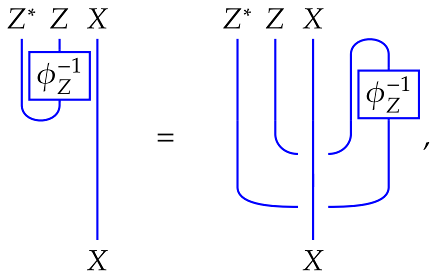


$$.\tag{2.2.5}$$

Now, the composite $g \circ \beta_Z$ may now be manipulated using (2.2.5) to obtain



i.e. that g is a post-inverse of β_Z. By a slight modification of (2.2.4), attempting to mirror the diagram over the vertical and adjusting the pivotal isomorphisms accordingly, we obtain the relation



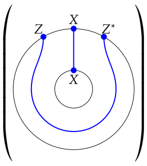
from which we derive that $\beta_Z \circ g = \mathrm{id}_{ZX}$ in the same way as well. Hence β_Z and g are mutually inverse, as desired. $\square$

We now collect together the observations which we have made thus far.

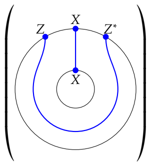
Proof of Proposition 2.2.1. For objects we have essentially described the necessary construction already; first, given a representation $V : \mathsf{Rep} \int_{S^1} \mathcal{C} \to \mathsf{Vec}$, restrict V to $\mathcal{C}$ and determine a representing object $X \in \mathcal{C}$. Then use Propositions 2.2.5, 2.2.6, and 2.2.7 to obtain a half-braiding $\beta_Y : XY \to YX$. Thus we have an object $F(V) = (X, \beta)$ of $\mathcal{Z}(\mathcal{C})$.

Suppose that now that in addition to $V : \int_{S^1} \mathcal{C} \to \mathsf{Vec}$ we have fixed a second representation W, and an intertwiner $\eta : V \to W$. Once again regard both V and W as representations of $\mathcal{C}$ using the inclusion functor J and select representing objects X and Y for V and W respectively. Then since η restricts to a morphism of representable functors $\mathcal{C} \to \mathsf{Vec}$, Corollary 2.1.6.1 (a direct corollary of the proof of the Yoneda lemma) yields that η is completely determined by the value $f : Y \to X$ of the component $\eta_X : \mathcal{C}(X \to X) \to \mathcal{C}(Y \to X)$ on id_X. Setting $F(\eta) = f$, functoriality of the result and faithfulness of this map is part of the content of the Corollary. It just remains to verify that the morphism $f = \eta_X(\mathrm{id}_X) : Y \to X$ so extracted from the intertwiner η is a actually a morphism in $\mathcal{Z}(\mathcal{C})$.

Let the morphisms $\widetilde{\beta_Z} : X \to ZXZ^*$ and $\widetilde{\gamma_Z} : Y \to ZYZ^*$ be those obtained by evaluating $V(r_{X,Z})$ and $W(r_{Y,Z})$ at id_X and id_Y respectively, and similarly let β_Z and γ_Z be obtained in the usual way (2.2.3) by Frobenius reciprocity. Naturality of η gives commutativity of the square

V 

$$\begin{tikzcd} \mathcal{C}(X \to X) \arrow[r, "\eta_X"] \arrow[d] & \mathcal{C}(Y \to X) \arrow[d] \\ \mathcal{C}(X \to ZXZ^*) \arrow[r, "\eta_{ZXZ^*}"] & \mathcal{C}(Y \to ZXZ^*) \end{tikzcd} \tag{2.2.6}$$

W 

and hence chasing the identity on X we obtain an equality

$$\eta_{ZXZ^*}(\widetilde{\beta_Z}) = \eta_{ZXZ^*}(V(r_{X,Z})(\mathrm{id}_X)) = W(r_{X,Z})(\eta_X(\mathrm{id}_X)) = W(r_{X,Z})(f).$$

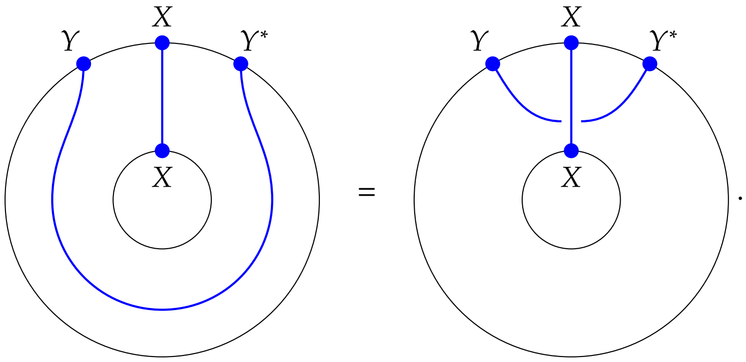
The right-hand side evaluates to $W(r_{X,Z})(f) = ZfZ^* \circ \widetilde{\gamma_Z}$ by Lemma 2.2.4, while the Yoneda lemma Corollary 2.1.6.1 says that η_{ZXZ^*} is pre-composition with $f = \eta_X(\mathrm{id}_X)$. Hence $\widetilde{\beta_Z} \circ f = ZfZ^* \circ \widetilde{\gamma_Z}$, which by Frobenius reciprocity exactly says that f defines a morphism $(X, \beta) \to (Y, \gamma)$ in $\mathcal{Z}(\mathcal{C})$. □

2.3 The relationship between $\mathcal{Z}(\mathcal{C})$ and Rep $\int_{S^1} \mathcal{C}$

Without requiring finite semisimplicity of $\mathcal{C}$ we still have a direct, though perhaps slightly creative, means of obtaining representations of $\int_{S^1} \mathcal{C}$ from objects of $\mathcal{Z}(\mathcal{C})$.

Proposition 2.3.1. *There is a functor $G : \mathcal{Z}(\mathcal{C}) \to \mathsf{Rep}^{\mathrm{op}} \int_{S^1} \mathcal{C}$.*

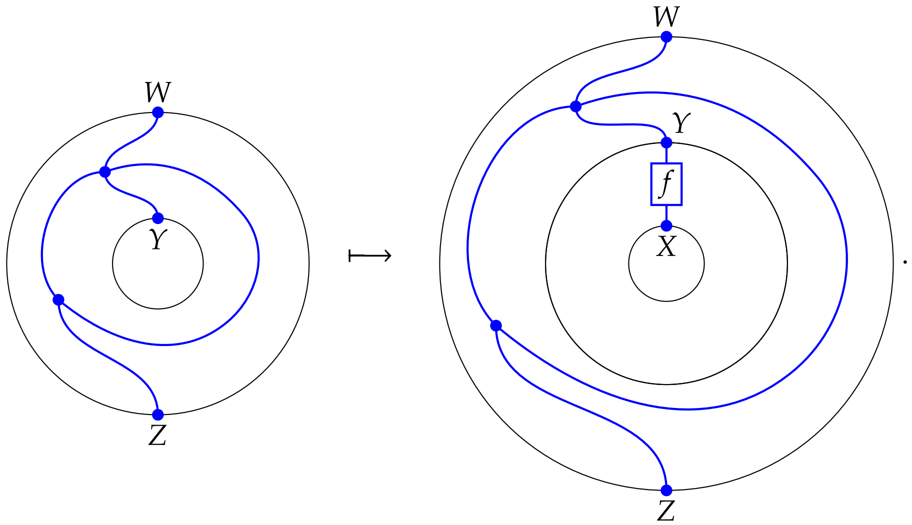
Proof. We construct G by applying the adage "use every piece of data exactly once". Fix an object (X, β) of $\mathcal{Z}(\mathcal{C})$. To each object S of $\int_{S^1} \mathcal{C}$ we will assign a quotient of the vector space $(\int_{S^1} \mathcal{C})(J(X) \to S)$ by an equivalence relation which we now describe. Namely, we identify diagrams drawn in the annulus where an edge has been "pulled through" the puncture using the half-braiding β. Diagrammatically, in any annular neighbourhood of the puncture we declare



This partially defines a representation $V : \int_{S^1} \mathcal{C} \to \mathsf{Vec}$ by specifying how V acts on the objects.

In order to extend V to an actual functor, given a morphism $f : S \to T$ in $\int_{S^1} \mathcal{C}$ we produce a linear map $V(S) \to V(T)$ just by gluing an annular diagram $v \in V(S)$ inside the annular diagram defined by f using the embedding Proposition 2.2.2, and interpreting the result in $V(T)$. It is then immediate from our construction that V sends the identity to the identity (inserting the identity diagram certainly gives the identity linear map), and that the functor so obtained is actually functorial.

This only defines the functor G on the objects of $\mathcal{Z}(\mathcal{C})$. Now given a morphism $f : (X, \beta) \to (Y, \gamma)$, we need to specify the components $\eta_T : G(Y, \gamma)(T) \to G(X, \beta)(T) \in \mathsf{Vec}$ of a natural transformation in $\mathsf{Rep}\int_{S^1} C$ (where T varies over the objects of $\int_{S^1} C$). We do this just given a diagram $v \in G(Y,\gamma)$ in the annulus with inner boundary $J(Y)$ and outer boundary T by gluing the outer boundary of the annular diagram corresponding to $J(f)$ along the inner boundary of the annular diagram v; diagrammatically



This certainly defines a linear map $\left(\int_{S^1} C\right)(J(Y) \to T) \to \left(\int_{S^1} C\right)(J(X) \to T)$, but a priori it need not descend to a linear map $V(Y)(T) \to V(X)(T)$ in the quotient. In diagrammatic terms, a string in v may be pulled over the annular hole using γ, then $J(f)$ may be glued on the inside, or $J(f)$ may be glued first and then β may be used to pull the string over the annular hole. Fortunately, the fact that $f : Y \to X$ is a morphism in $\mathcal{Z}(C)$ precisely asserts that the two diagrams so obtained are equal after a local replacement.

Naturality of the components η_T thus constructed amounts to demanding that the operation of gluing any morphism $g : T \to S$ along the outer boundary of any $v \in G(Y,\gamma)$, and the operation of gluing the morphism $J(f) : J(X) \to J(Y)$ along the inner boundary of that same $v \in G(Y,\gamma)$ commute. This is a manifest property of our construction, and thus the proof is complete. □

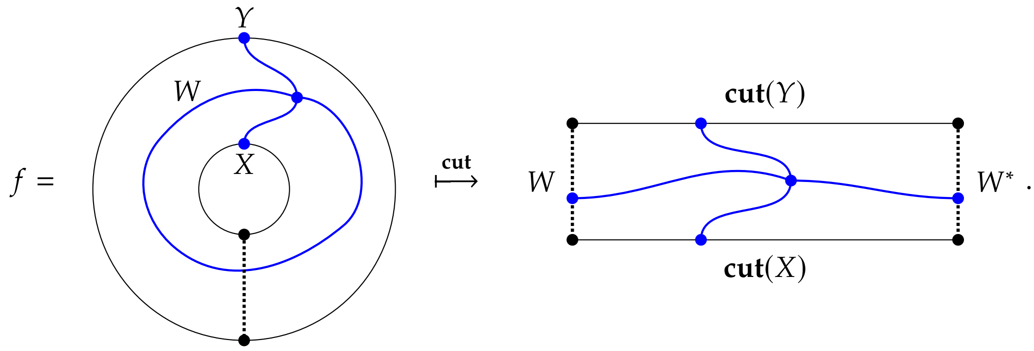
Remark 2.3.2. We will shortly see that (as expected) when $G(Y,\gamma)$ is restricted to a representation of C, the resulting functor is represented by Y. Thus algebraically each map $\eta_T = G(f : (X,\beta) \to (Y,\gamma))$ has a particularly simple description; every $g \in G(Y,\gamma)(T)$ is sent to $g \circ J(f)$—just pre-composition with $J(f)$. Theorem 2.1.6 (the Yoneda lemma) says that this rule automatically defines a natural transformation $G(Y,\gamma) \to G(X,\beta)$ of representations of C.

Note however that this does not make our explicit argument below redundant, since we desire a natural transformation of representations of all of $\int_{S^1} C$.

Proposition 2.3.3. *If* $\mathrm{End}(\mathbb{1}) \cong \mathbb{k}$ *then the functor* J *is faithful.*

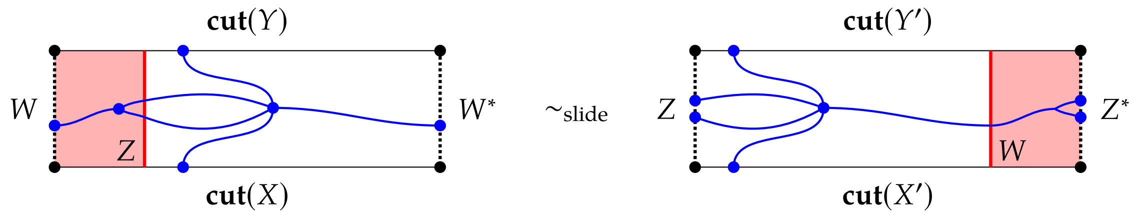
Proof. Fix objects $X, Y \in \int_{S^1} C$. In order to reduce to the case of Theorem 1.4.6, we define an operation **cut** on morphisms $f : X \to Y$ induced by cutting $S^1 \times I$ along a radial arc determined by a distinguished point $\bullet \in S^1$. For instance, when X and Y are themselves in the image of J

we would have



Note that in order for this to make sense we must fix a point • which does not coincide with a labelled point of X or Y. As shown above, for each $f : X \to Y$ the map **cut** produces an object $W \in \int_I C$ along with a rectangular string diagram with bottom side labelled by $\mathbf{cut}(X)$, top side labelled by $\mathbf{cut}(Y)$, and the left and right sides labelled by W and W^* respectively (here the dual W^* has the same distinguished points of W with the left duals of the original labels being taken). Of course it might be the case that no string in diagram f intersected the cutting line, in which case we take $W = \mathbb{1}$. We can then form the free vector space $V_{X,Y,W}$ on rectangular string diagrams of this kind. On each such vector space we impose a relation $\sim_{\text{dia}}$ which identifies isotopic rectangular diagrams, and diagrams obtained by local replacement or linearity relations (in the sense of Section 1.4) in the interior of each rectangle. The result is a quotient $Q_{X,Y,W} = V_{X,Y,W}/\sim_{\text{dia}}$, for which we consequently assemble the direct sum $Q_{X,Y} = \bigoplus_{W \in \int_I C} Q_{X,Y,W}$. We assert that after imposing a second relation $\sim_{\text{slide}}$ on $Q_{X,Y}$, the quotient $H_{X,Y} = (\bigoplus_W V_{X,Y,W}/\sim_{\text{dia}})/\sim_{\text{slide}}$ exactly recovers the homset $\left(\int_{S^1} C\right)(X \to Y)$.

The second relation $\sim_{\text{slide}}$ is generated by sliding moves of the form depicted below (with the induced C-decorations on the red lines of the left and right diagrams corresponding to the objects Z and W of $\int_{S^1} C$ respectively).



(2.3.1)

The relation $\sim_{\text{slide}}$ exists to reflect that fact that our rectangular string diagrams should be thought of annuli which have been cut up. Critically, the relation $\sim_{\text{dia}}$ does not mix the summands $V_{X,Y,W}$ (by definition), while the relation $\sim_{\text{slide}}$ does. Since it is always possible to factor an ambient isotopy into a finite composite of ambient isotopies which modify a single ε-neighbourhood at each stage (for instance, see Lemma B.0.1 of [31]), we see that the quotient construction $H_{X,Y}$ exactly recovers $\left(\int_{S^1} C\right)(X \to Y)$.

It now suffices to show that the inclusion of any ordinary morphism $g : U \to V$ in C into the $V_{X,Y,\mathbb{1}}$ summand of $Q_{X,Y}$ as a rectangular diagram, and then passing to the quotient $H_{X,Y}$, is an injective map. It is the content of Theorem 1.4.6 that this map is injective after taking the first quotient $\sim_{\text{dia}}$, but before taking the second. But now observe that the relation $\sim_{\text{slide}}$ is transitively closed; a chain of sliding moves between any two rectangular diagrams can always be accomplished by a single move. This means that it in turn suffices to verify that if two classes of diagrams D and D' in $V_{X,Y,\mathbb{1}}/\sim_{\text{dia}}$ are related by a single sliding move then they are actually the same class. If there was such a sliding move, then the (right) interior boundary of the red region of (2.3.1) would also be labelled by just the unit $\mathbb{1}$. But now by the hypothesis $\mathrm{End}(\mathbb{1}) \cong \Bbbk$, the diagram inscribed within the red region must be (possibly a scalar multiple of) an empty diagram. A pair of representatives of D and D' are therefore isotopic, and hence $D = D'$. This completes the proof. □

So that Proposition 2.3.3 applies, we will assume that $\mathrm{End}(\mathbb{1}) \cong \Bbbk$ from now on.

We are almost ready to prove the main theorem of this section. The following lone and essentially knot-theoretic obstacle remains to be overcome.

Theorem 2.3.4. *Let $(X, \beta) \in \mathcal{Z}(C)$. Then for each $Z \in C$ the inclusion $C(X \to Z) \to G(X, \beta)(Z)$ (via J, and then passing to the quotient) is a linear isomorphism.*

Moreover, under this isomorphism the natural transformation $G(f)$ for any $f : (X, \beta) \to (Y, \gamma)$ just becomes precomposition with f on every homset.

The proof is deferred until the end of this section. Note that it is certainly the case that for each representative v of an equivalence class in $G(X, \beta)(Z)$ we can construct a morphism in $C(X \to Z)$. Indeed, after isotoping v and locally replacing disks to obtain the factorised form of Lemma 2.2.3 as we have done now many times, we need only pull a single string across the annular puncture using the quotient relation defining $G(X, \beta)(Z)$. This yields a morphism in the image of the inclusion J, hence in $C(X \to Z)$ (and thus establishes that J is surjective in the quotient). The second part of Theorem 2.3.4 is also then immediately clear. The factorisation of Lemma 2.2.3 being highly non-canonical, the only question is whether after arbitrary isotopy and local replacement and subsequent factorisation, the same morphism in $C(X \to Z)$ is always obtained; that is, whether J has nontrivial kernel in the quotient. This is the content of the deferred proof, and as expected it uses that the half-braidings in Drinfeld centers are natural, satisfy the half-braiding hexagon, and are isomorphisms (corresponding respectively to Propositions 2.2.5, 2.2.6, and 2.2.7) all in a critical way. Note that by Theorem 2.3.4 the functor G is faithful, since we can recover $f : X \to Y$ by evaluating the component η_Y at id_Y; the result will be the image of f under the faithful functor J.

Theorem 2.3.5. *Let C be finitely semisimple so that the functor F from Proposition 2.2.1 exists, and suppose $\mathrm{End}(\mathbb{1}) \cong \Bbbk$. Then F and G together give an equivalence of the categories $\mathcal{Z}(C)$ and $\mathsf{Rep}^{\mathrm{op}} \int_{S^1} C$.*

Proof. We will actually show that we can slightly modify F and G into functors respectively from and to a category $\mathcal{D}^{\mathrm{op}}$ equivalent to $\mathsf{Rep}^{\mathrm{op}} \int_{S^1} C$, so that F and G witness an isomorphism between $\mathcal{D}^{\mathrm{op}}$ and $\mathcal{Z}(C)$; the claim will then follow immediately.

The point of this technical manoeuvring is to avoid making non-canonical choices. Namely, our definition of the functor $F : \mathsf{Rep}^{\mathrm{op}} \int_{S^1} C \to \mathcal{Z}(C)$ is slightly inconvenient, requiring an arbitrary choice[4] of representing object $X \in C$ for every representation $V : \int_{S^1} C \to \mathsf{Vec}$. This is equivalent to fixing an isomorphism from each such representation V to another representation $V' : \int_{S^1} C \to \mathsf{Vec}$ and object $X \in C$ so that V' restricts on C to the literal hom-functor $C(X \to -)$. That is, $V'(Y)$ is equal to $C(X \to Y)$ on-the-nose, and $V'(J(f : Y \to Z))$ is literally post-composition with f. Obviously, the choice of such a representation V' and the isomorphism thereto is non-canonical.

In order to not have to deal directly with these isomorphisms, observe that the category $\mathsf{Rep} \int_{S^1} C$ is equivalent to a category $\mathcal{D}$ consisting of the functors $\int_{S^1} C \to \mathsf{Vec}$ which restrict on C to hom-functors, and thinking[5] of F as a functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{Z}(C)$ in this situation there is a canonical choice $F(V) = (X, \beta)$ of both the object X and natural isomorphism β. Similarly,

[4]Really, to each such family of choices is associated a particular functor $F : \mathsf{Rep}^{\mathrm{op}} \int_{S^1} C \to \mathcal{Z}(C)$.

[5]Of course, technically we are actually now defining a new functor—though very similar—with the same name F.

via Theorem 2.3.4 we can construct a functor $G : \mathcal{Z}(C) \to \mathcal{D}^{\mathrm{op}}$ such that for each $Z \in C$ the vector space $G(X,\beta)(Z)$ is literally the homset $C(X \to Z)$, and so that $G(f : (X,\beta) \to (Y,\gamma))$ is just precomposition with f.

This all ensures that starting with an object $(X,\beta) \in \mathcal{Z}(C)$, for $(Y,\gamma) := FG(X,\beta)$ we have $Y = X$ and $\gamma_Z = G(X,\beta)(\mathrm{id}_X) = \beta_Z$ on-the-nose. Similarly, let $V : \int_{S^1} C \to \mathsf{Vec}$ be a representation restricting to a hom-functor represented by $X \in C$. Then we have $F(V) = (X,\beta)$ with β some half-braiding for X, and for any $Y \in C$ we have $GF(V)(Y) = G(X,\beta)(Y) = C(X \to Y) = V(Y)$ an equality.

Finally, for any $f : Y \to Z$ we must check that $GF(V)(f) = V(f)$, and this is a non-trivial calculation. First, apply Lemma 2.2.3 in order to factor f as $f = J(g) \circ r_{Y,W}$ for some morphism $g : WYW^* \to Z$. Define $W = GF(V)$ to simplify the notation. Now by definition, $W(f) = G(X,\beta)(f)$ sends a morphism $h : X \to Y$ to $f \circ J(h) = J(g) \circ r_{Y,W} \circ J(h)$; composition here is being performed in $\int_{S^1} C$ using representatives and then descending to the quotient, and we continue this convention in the sequel. The isotopy of the proof of Lemma 2.2.4 then yields an equality

$$W(f)(h) = J(g) \circ J(WhW^*) \circ r_{X,W}.$$

We can then write $r_{X,W} = J(\widetilde{\beta_W})$ using quotient defining $W(X)$, and so

$$W(f)(h) = J(g \circ WhW^* \circ \widetilde{\beta_W}).$$

Now unfolding the definition of $\widetilde{\beta_W}$, applying Lemma 2.2.4, and using functoriality of V we obtain

$$\begin{aligned} g \circ WhW^* \circ \widetilde{\beta_W} &= g \circ \big(WhW^* \circ V(r_{X,W})(\mathrm{id}_X)\big) \\ &= g \circ V(r_{Y,W})(h) \\ &= V(J(g))\,\big(V(r_{Y,W})(h)\big) \\ &= V(J(g) \circ r_{Y,W})(h) \\ &= V(f)(h). \end{aligned} \tag{2.3.2}$$

Hence $W(f) = V(f)$ identically, as desired.

It remains to show that F and G are mutually inverse on morphisms. First fix some $f : (X,\beta) \to (Y,\gamma)$ in $\mathcal{Z}(C)$. Then unwinding definitions we have $FG(f) = G(f)_Y(\mathrm{id}_Y)$, and we observed above that $G(f)_Y$ is just pre-composition with f. Consequently, $FG(f) = f$. Now fix some natural transformation $\eta : V \to W$ of representations of $\int_{S^1} C$ which restrict on C to hom-functors represented by X and Y respectively. Then for any $Z \in C$ and $g : X \to Z$ we can compute $GF(\eta)$ componentwise by

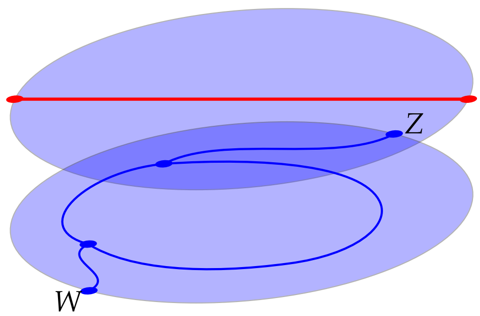
$$(GF(\eta))_Z(g) = G(\eta_X(\mathrm{id}_X))_Z(g) = g \circ \eta_X(\mathrm{id}_X) = \eta_Z(g)$$

with the last equality following from the Yoneda lemma. This shows that $GF(\eta) = \eta$ and completes the proof. □

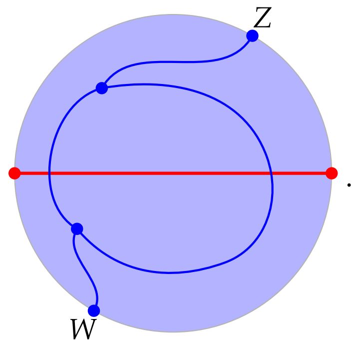
In general $\mathcal{Z}(C)$ and $\mathsf{Rep}^{\mathrm{op}} \int_{S^1} C$ need not be the same. For example, when C is not finitely semisimple there may be exotic representations of $\int_{S^1} C$ which are not representable, and of course when C is not pivotal the annular category does not even make sense.

In the previous section we determined basic properties of objects of $\mathsf{Rep} \int_{S^1} C$, at least when C is finitely semisimple. Theorem 2.3.5 together with the results of the next section show that those properties are all of the important ones; the functors F and G then explain in extremely concrete terms how to pass between the two viewpoints of the same data.

Finally, we tie up the loose-end. We will deduce Theorem 2.3.4 by appeal to the following knot-theoretic result. We do not prove it carefully, but we at least explain the general method by which results of this kind are obtained [37]. Consider two identical disks embedded in 3-space, one overlaid above the other so that they coincide when viewed from above. On the "upper disk" we will inscribe a fixed, distinguished line, and on the bottom disk we will allow any unlabelled diagram with labels in C to be drawn. A sample configuration to have in mind is depicted below.



or, from above



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As depicted, when viewed from above we obtain a ⦶-*diagram*; a knot diagram with crossings wherever the distinguished line passes over the labelled diagram inscribed below. We will prohibit ⦶-diagrams with "singular" points, with a diagram string intersecting the distinguished line tangentially, or a labelled vertex lying exactly on the distinguished line. It is clear that isotopies of labelled string diagrams in the disk give rise to a sensible notion of isotopy of ⦶-diagrams when viewed from above (though the induced isotopy of ⦶-diagrams might pass through singular points).

Theorem 2.3.6. *In the geometrical situation just described suppose there is a function ψ defined on the set of ⦶-diagrams (obtained by viewing the two disks from above). Then ψ is an invariant of the underlying labelled string diagram exactly if ψ is invariant under isotopy of ⦶-diagrams, and local applications of*

1. *the Reidemeister 2 move, pulling loops under and through each side of the distinguished line, and*

2. *moving a labelled vertex across under the distinguished line.*

Proof sketch. This is a Reidemeister type theorem, and thus may be established via the following general programme. First, we extend the space of diagrams to include those which can innocently arise out of viewing an isotopy of knots as an isotopy of ⦶-diagrams, but nonetheless ψ does not take values on. We do this by labelling each unacceptable point of a knot diagram which we disallowed (ψ does not accept) a *singular point* with some multiplicity. In our case by definition we cannot have a string meeting the distinguished line tangentially, and we cannot have a vertex hidden under the distinguished line. On their own we declare both such points to have multiplicity 1. It can also happen that both of these possibilities simultaneously occur at the same point, and we declare in this situation to have a singular point of multiplicity 2. (In our case, since we only allow ambient isotopy to begin with, we do not need to consider notions of higher multiplicity crossings as one would when defining a typical knot invariant, for example.)

Next we consider a filtration

$$F_0 \hookrightarrow F_1 \hookrightarrow F_2 \hookrightarrow \cdots$$

of the infinite-dimensional manifold F of all ⦶-diagrams, with the subset F_i consisting of those ⦶-diagrams which have sums of multiplicities of singular points at most i. Then we establish separately that

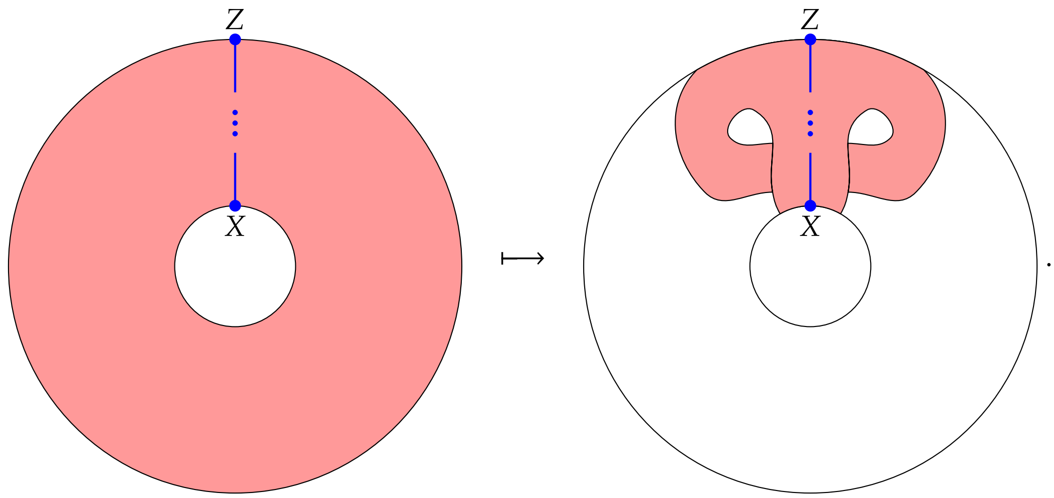
a) There exists some $k \geq 0$ such that any string diagram isotopy rel boundary inducing an isotopy of $\mathbb{O}$-diagrams may be modified to give an isotopy through only F_k. That is, if any singular points of multiplicity greater than k arise, we can arrange their creation can be avoided by modifying the isotopy.

b) The function ψ is invariant under all possible ways of resolving all of all kinds of multiplicity $\leq k$ singularities which can occur. The resulting diagrammatic equations are the associated *Reidemeister moves* for domain in which we are working.

In our case it is sufficient to set $k = 1$. To see this, observe the following. First, since only finitely many singular points can arise during the entire isotopy, we can pick a ε small enough that whenever multiple singularities do simultaneously occur, ε-balls about all of the singular points never intersect. Recall once again that every ambient isotopy of the disk can be refined into a finite composite of ambient isotopies where only ε-balls are modified at each stage [31], and we do this. Consequently we dispatch with any diagram of singular multiplicity greater than one, so long as that diagram is singular due to multiple multiplicity 1 singular points present on the distinguished line simultaneously. Finally, it remains to deal with the case of a single multiplicity 2 singularity, but this is straightforward; we just slide a labelled vertex under the distinguished line before we resolve the Reidemeister 2-type crossing by pulling the remaining loop through.

Hence we obtain the desired isotopy though diagrams in F_1 only. The value of ψ is constant on isotopic diagrams by assumption, and the other hypotheses of the theorem assert that ψ is invariant as the isotopy passes through any (multiplicity one) singular diagram. The claim follows. $\square$

Proof of Theorem 2.3.4. By the discussion following the statement of the theorem we just need to show that the inclusion $C(X \to Z) \to G(X, \beta)(Z)$ induced by J is injective in the quotient. Let η be an isotopy between diagrams f and g, each representing a vector in $G(X, \beta)(Z)$. Now observe that we can embed the annulus into itself via the *mushroom-ification* map

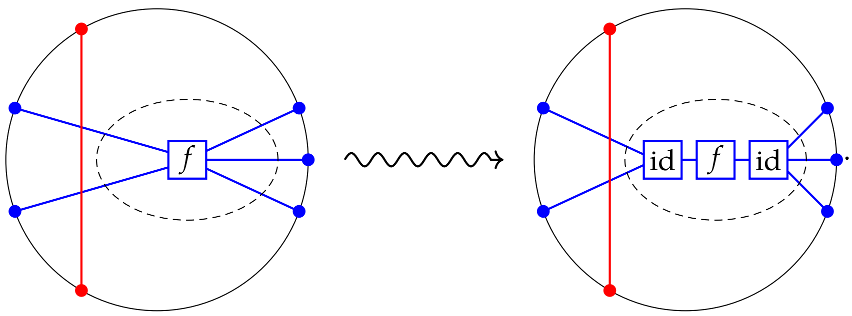


When we perform this re-embedding, we are careful to preserve a small neighbourhood of the inner distinguished point labelled by X, in order that the annulus crosses under itself only under the single string originating at X.

The resulting re-embeddings of f and g evidently factor through the inclusion of a rectangle into the annulus, and we denote the associated diagrams in the rectangle by $\widetilde{f}$ and $\widetilde{g}$ respectively. For now we consider $\widetilde{f}$ and $\widetilde{g}$ only topologically, since they may have intersecting strings and hence not make sense as actual labelled string diagrams. Nonetheless it is easy to then see that the isotopy η gives rise to an isotopy $\widetilde{\eta}$ between $\widetilde{f}$ and $\widetilde{g}$.

The point of all of this is that we can think of the string connecting to X on the boundary as a distinguished line in a disk which turns the respective regions in $\widetilde{f}$ and $\widetilde{g}$ where the annulus crosses under itself into the Φ-diagrams of Theorem 2.3.6. Moreover, there is a natural evaluation map (**eval** $\circ$ **rectify** of Section 1.4) on such diagrams taking values in ordinary morphisms $C(X \to Z)$ in the category C; the only problems can occur when there is a singular point in the sense of Theorem 2.3.6 (which is perfectly acceptable), or when a string intersects the distinguished line transversely. We handle the latter situation by (unambiguously) resolving transverse intersections using the braiding β with which $(X, \beta) \in \mathcal{Z}(C)$ is equipped.

It is sufficient to show that evaluation of these Φ-diagrams is an isotopy invariant of the underlying labelled string diagram, since the evaluation map itself is an invariant of isotopy and local replacement in disks (by Theorem 1.4.6). Hence by Theorem 2.3.6 it remains to show that evaluation is invariant under the Reidemeister 2 move and moving a vertex under the distinguished line. The former condition is exactly the content of Proposition 2.2.7, and the latter is verified by performing the diagrammatic replacement

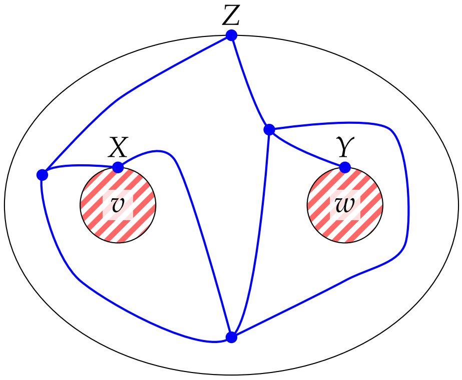


Here we have replaced an arbitrary vertex labelled with a morphism f with three vertices; a central vertex of valance 2 still labelled by f, flanked on either side by a vertex labelled with $\mathrm{id} : X_1 \otimes \cdots X_n \to X_1 \otimes \cdots X_n$—which just bundles the strings together. Proposition 2.2.5 asserts that valence 2 labelled vertices may be freely slid under the distinguished line, and Proposition 2.2.6 asserts (by induction) that the identities may pass under as well. This verifies the second required move, and so completes the proof. □

2.4 A braided monoidal structure on $\mathsf{Rep} \int_{S^1} C$

We saw in Chapter 1 that the Drinfeld center of any monoidal category is naturally a braided monoidal category. In fact, representations of the annular category of C also posses a natural braided monoidal structure. In this section we will see that when C is finitely semisimple the equivalence of the previous section extends to a braided monoidal equivalence.

Thus, let $V, W \in \mathsf{Rep} \int_{S^1} C$ be representations. For each $Z \in \int_{S_1} C$ we declare $(V \otimes W)(Z)$ to be a quotient of the free $\Bbbk$-vector space on all diagrams of the form



$$\text{with}\quad \begin{matrix} v \in V(X), \\ w \in W(Y) \end{matrix}\;. \tag{2.4.1}$$

That is, diagrams in a fixed disk with two interior disks removed (a *left* and *right* puncture), so that

- the outer boundary is labelled by Z,
- the boundary of each inner puncture is labelled with arbitrary element of $\int_{S^1} C$, and
- the left puncture with boundary X is labelled by a vector $v \in V(X)$, and the right puncture is labelled by a vector $w \in W(Y)$ similarly.

We already have a good theory of diagrams up to isotopy and local replacement drawn in manifolds such as a doubly-punctured disk, so we just impose one further relation to ensure compatibility with the vectors labelling each puncture. To describe the relation, note that given any annular neighbourhood $f : X \to Y$ of a puncture labelled with the vector v in $V(X)$ we obtain another vector $v' = V(f)(v)$ in $V(Y)$. We declare that we obtain an equal diagram by deleting the annular neighbourhood S and labelling the resulting puncture with the vector v' (here and henceforth we extend such identifications linearly in the free vector space on all diagrams).

It is easy to then see that $(V \otimes W)(Z)$ inherits the structure of a vector space; multiplying a puncture by a constant can be freely moved to morphism connected to that puncture by a string. We then define $(V \otimes W)(f : X \to Y)$ to just be the gluing of the outer X-boundary of each diagram $S \in (V \otimes W)(X)$ to the inner boundary of the annulus f, with the result interpreted in $(V \otimes W)(Y)$. This construction yields a linear map, and hence a linear bifunctor $\otimes$ on $\int_{S^1} C$ is defined on the morphisms.

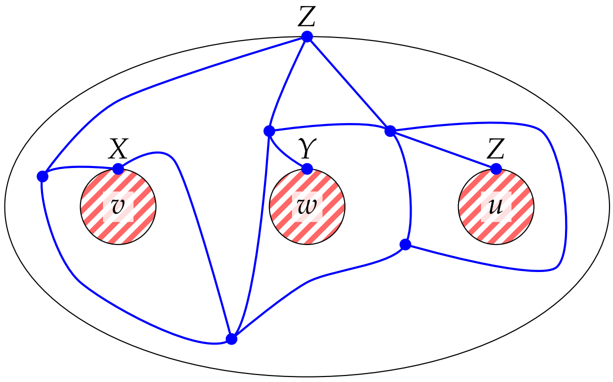
Now given two morphisms $\eta : V \to T$ and $\mu : W \to U$ their tensor product is for each $Z \in \int_{S^1} C$ a morphism $(\eta \otimes \mu)_Z : (V \otimes W)(X) \to (T \otimes U)(Z)$ which is natural in Z. We obtain such a map by sending diagrams of the form (2.4.1) to the diagram obtained by replacing the label $v \in V(X)$ with $\eta(v)$ and the label $w \in W(Y)$ with $\mu(w)$. Since $V \otimes W$ and $T \otimes U$ both send morphisms $f : Z \to Z'$ to the linear map which glues f around the outer boundaries of diagrams, we see directly that the family of morphisms $(\eta \otimes \mu)_Z$ is natural. This completes the definition of the product $\otimes$, since functoriality of the product is similarly clear.[6]

The monoidal unit for $\otimes$ is just the representation of $\int_{S^1} C$ which for each $Z \in \int_{S^1} C$ reports the vector space of diagrams in the disk (not annulus) with boundary Z—and with the action of morphisms defined in the usual way by gluing to the boundary of disk.

We also need an associator and unitors for the product $\otimes$, which we build by an appeal to some general machinery. Observe that the tensor product we have defined on representations

[6]Note that the monoidal product we have described is not the same as the product induced "pointwise" by the symmetric monoidal structure on Vec.

of the annular category naturally extends to a 3-fold product of three representations V, W, U just considering diagrams



$$\text{with } \begin{array}{r} v \in V(X), \\ w \in W(Y), \\ u \in U(Z) \end{array}$$

and again having morphisms in $\int_{S^1} C$ act by gluing around the outer boundary of any diagram. Moreover it is easy to see that there is an analogous product $\otimes_n : \left(\mathsf{Rep}\int_{S^1} C\right)^n \to \mathsf{Rep}\int_{S^1} C$ for all $n \geq 0$.[7] To reduce notational clutter we will write n-fold products with square brackets, so that $[] := \mathbb{1}$, $[V] := V$, $[V, W] := V \otimes W$, the 3-fold product is denoted $[V, W, U]$, and so on.

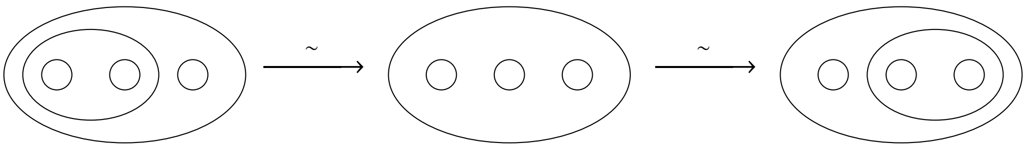
The key point is that each of these products is compatible, in that there is an isomorphism **flatten** from each product of products, e.g.

$$[[V_1], [V_2, V_3, V_4], [V_5, V_6]] \quad \text{to a single product} \quad [V_1, V_2, V_3, V_4, V_5, V_6]$$

which deletes brackets. In particular, **flatten** gives a composite of isomorphisms

$$\alpha_{V_1,V_2,V_3} : \ [[V_1, V_2], V_3] \xrightarrow[\sim]{\textbf{flatten}} [V_1, V_2, V_3] \xleftarrow[\sim]{\textbf{flatten}} [V_1, [V_2, V_3]] \tag{2.4.2}$$

which we declare to be a component of the associator of $\otimes$ (recall that a bracket of two representations is just the product $\otimes$). In our case these associators are schematically isomorphisms



between the vector spaces of diagrams with the indicated shape.

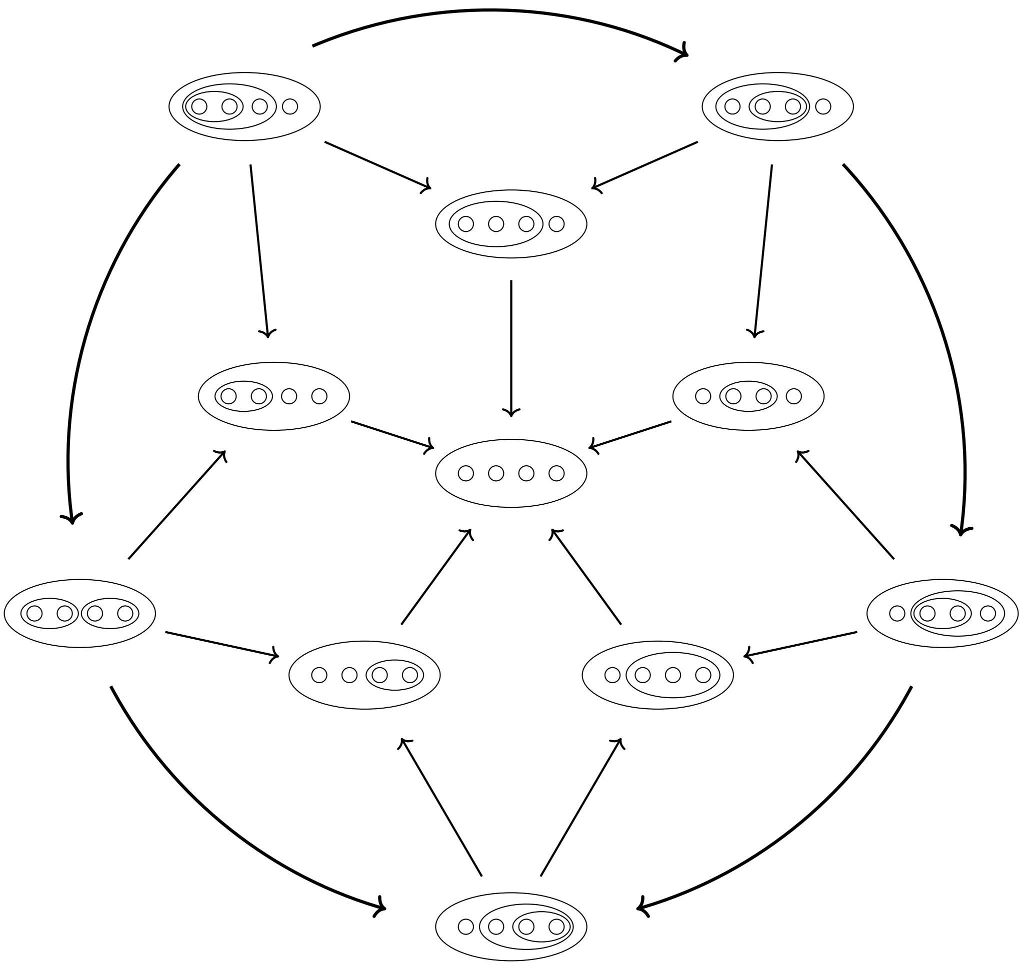
We claim that so long as **flatten** is natural in an appropriate generalised sense the isomorphisms (2.4.2) assemble into a natural transformation α which obeys the pentagon associativity constraint, and hence we obtain the desired associator. The notion of generalised naturality we mean is just that if $f_1, \ldots, f_n$ are morphisms of representations arranged in some product of products, then the resulting morphism commutes with **flatten** after deleting brackets. For example, we require

$$\textbf{flatten} \circ [[f_1], [f_2, f_3, f_4], [f_5], [f_6, f_7]] = [f_1, f_2, f_3, f_4, f_5, f_6, f_7] \circ \textbf{flatten}.$$

Now observe that to show α actually is an associator, schematically we must just verify that the

[7] The underlying structure here is essentially that we have an E_1-algebra in Cat (the 2-category of categories); [13] explains the monoidal $(\infty, 1)$-category situation this way. The case of symmetric monoidal categories is explicitly described in [6].

boundary of the following diagram[8] is commutative.



The outermost triangular faces commute by the definition of the associator to begin with. Each remaining (quadrilateral) face then commutes by the generalised naturality of **flatten** applied to **flatten** itself, so the boundary of the diagram is commutative as desired.

Suppose now that there is another map **strip** which for each n-fold product e.g. $[V_1, \mathbb{1}, \mathbb{1}, V_2, \mathbb{1}]$ gives an isomorphism to the same product with all of the copies of the tensor unit deleted, i.e. in this case $[V_1, V_2]$, and is again natural in our generalised sense in all of the entries which are not the tensor unit. Then we automatically have natural isomorphisms

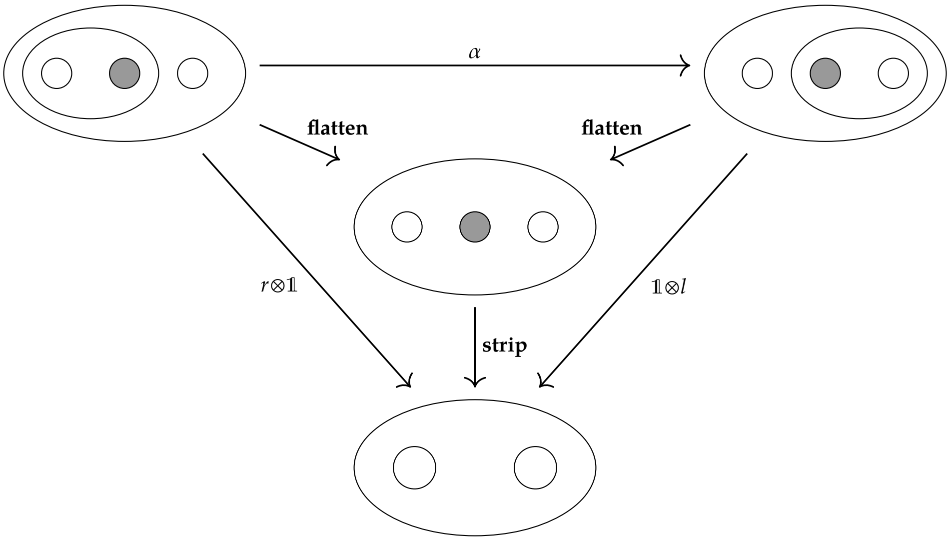
$$l_V : \ [\mathbb{1}, V] \xrightarrow[\sim]{\textbf{strip}} [V] = V$$

$$r_V : \ [V, \mathbb{1}] \xrightarrow[\sim]{\textbf{strip}} [V] = V.$$

We claim that if **strip** and **flatten** always commute[9], then these maps satisfy the triangular identity for unitors. This follows by commutativity of the boundary of the following schematic

[8] Note that this diagram makes sense for any bracket product by interpreting the punctured disks as brackets.
[9] It is easy to see that the source and target of these maps is such that this statement makes sense.

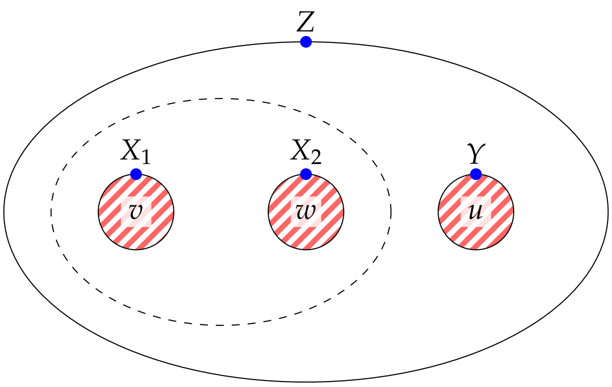
diagram



where punctures labelled by the tensor unit are shaded. The upper triangle commutes by the definition of the associator, and the right and left triangles are commutativity squares for **strip** and **flatten** which have been collapsed (one side is an equality in each case).

Proposition 2.4.1. *The product* $\otimes$ *on* $\int_{S^1} C$ *has natural* **flatten** *and* **strip** *maps which commute, and hence* $\int_{S^1} C$ *is a monoidal category.*

Proof. In order to avoid introducing new notation so that we can speak in full generality, we just illustrate the special case of a product $(V \otimes W) \otimes U = [[V, W], U]$ and assert that the general case is completely analogous. Now, a representative of any fixed $t \in ((V \otimes W) \otimes U)(Z)$ is a diagram in a doubly punctured disk with the boundaries of the inner punctures labelled by $X, Y \in \int_{S^1} C$, and respectively $s \in (V \otimes W)(X)$, and $u \in U(Y)$. In turn $s \in (V \otimes W)(X)$ is represented by another diagram in a doubly punctured disk with again $X_1, X_2 \in \int_{S^1} C$, and labels $v \in V(X_1)$ and $w \in W(X_2)$. By inserting the latter diagram into the former, we obtain a diagram t' of the form (with some string diagram in the interior)



which lies in $[V, W, U](Z)$. We claim that this construction gives a well-defined map sending any $t \in ((V \otimes W) \otimes U)(Z)$ to the corresponding in $t' \in [V, W, U](Z)$. This follows since in $((V \otimes W) \otimes U)(Z)$ we imposed the relation that we can "collapse" annular neighbourhoods of any puncture into the puncture. Consequently any isotopy or local replacement move of a representative of a diagram in $((V \otimes W) \otimes U)(Z)$ gives rise to a corresponding move of the corresponding diagram in $[V, W, U](Z)$, and vice versa. We see directly that the morphisms

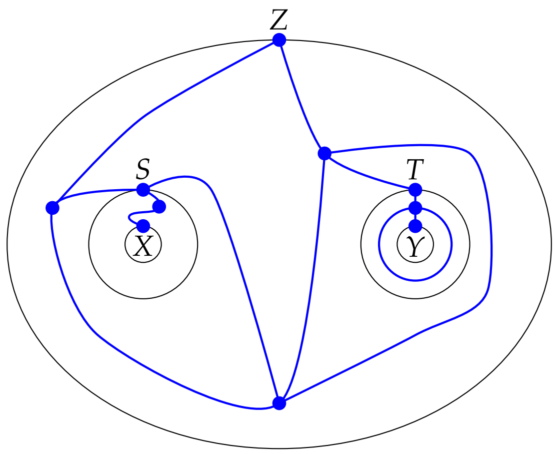
flatten which we obtain in full generality from this construction are natural in morphisms of representations, just by unravelling our definition of the tensor product of morphisms.

The definition of **strip** is similarly straightforward; suppose that some number of copies of $\mathbb{1}$ occurs in an n-fold product $T = [V_1, \ldots, V_n]$ and T' is the result of deleting every copy of $\mathbb{1}$ in the product. In addition let v be a diagram representing a vector in the value of T on some $Z \in \int_{S^1} C$. Then again the fact that we can collapse disk neighbourhoods of punctures of v means that we can replace v with a representative where each puncture corresponding to $\mathbb{1}$ is labelled by the empty diagram in the disk! Consequently we obtain a way to construct a diagram representing a vector in $T'(Z)$ just by "filling in" each of the punctures labelled with $\mathbb{1}$ with an empty disk. Naturality is evident for the same reason as for **flatten** (morally, morphisms of non-unit objects act "far away" from where **strip** modifies the diagrams).

Finally, commutativity of **strip** and **flatten** is just the statement that we can fill in punctures associated to the unit $\mathbb{1}$ before and after **flatten**ing and obtain the same diagram; and this is manifest. □

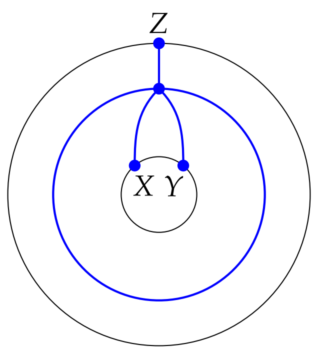
Proposition 2.4.2. *The functor G is monoidal.*

Proof. First, in order to equip G with a monoidal structure we must provide a morphism $K_{(X,\beta),(Y,\gamma)} : G(X, \beta) \otimes G(Y, \gamma) \to G((X, \beta) \otimes (Y, \gamma))$. Recall that for any $Z \in C$ vectors in $(G(X, \beta) \otimes G(Y, \gamma))(Z)$ are represented by diagrams of the form



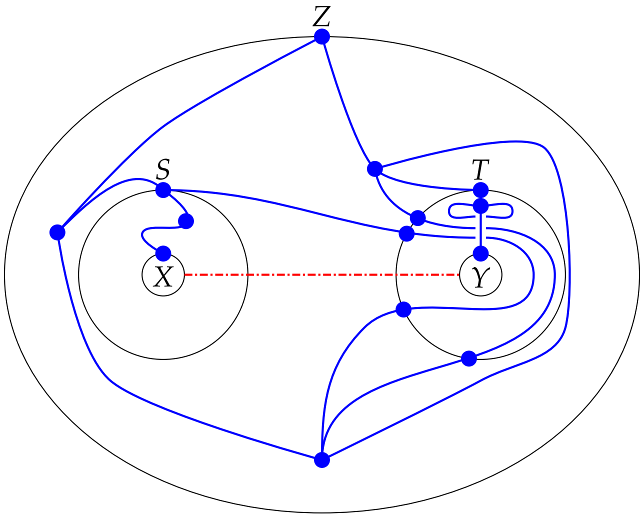
(2.4.3)

with $v \in G(X, \beta)(S)$ and $w \in G(Y, \gamma)(T)$ for some $S, T \in C$. In particular recall that by the definition of G, in both of the inner annuli it is permissible to pull a string over the puncture via the respective half-braiding β or γ. On the other hand, vectors in $G((X, \beta) \otimes (Y, \gamma))(Z)$ are represented by diagrams of the form

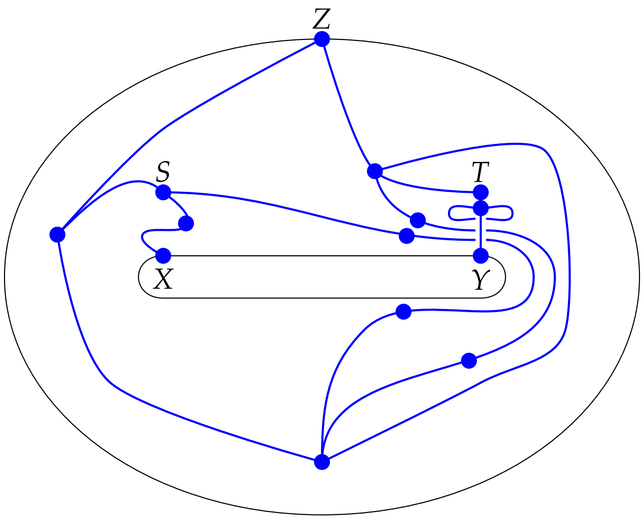


. (2.4.4)

We build a linear map $L : (G(X, \beta) \otimes G(Y, \gamma))(Z) \to G((X, \beta) \otimes (Y, \gamma))(Z)$ in two steps. First, we begin with any diagram of the form (2.4.3) and inscribe a line connecting the two punctures (depicted below in dotted red). We then pull any strings which intersect this line over the right-hand puncture. For example beginning with the diagram (2.4.3) we would obtain



As a second step we can then remove the (dotted red) line connecting the punctures in order to obtain a single annular diagram representing a morphism $XY \to Z$ in $\int_{S^1} C$, exactly of the form (2.4.4):



This construction gives a well defined linear map, because the braiding of $(X, \beta) \otimes (Y, \gamma)$ exactly corresponds to using β, then γ. Hence pulling a string over the X-puncture, then the Y-puncture, then coalescing the punctures gives the same result as coalescing the punctures and then pulling a string over the single remaining puncture. Moreover L is obviously surjective; any morphism $f : X \otimes Y \to Z$ in $\int_{S^1} C$ can have its inner puncture pulled apart and separated into two punctures (exactly undoing the "second step" used to construct L) in order to find an element of $(G(X, \beta) \otimes G(Y, \gamma))(Z)$ mapping to f.

To show that L is injective we appeal to the natural generalisation of Theorem 2.3.6 which holds for 2 distinguished lines. The only required modification to the proof amounts to the observation that isotopies of diagrams with two distinguished lines can be arranged so that there is only ever a singular point at one of the distinguished lines at any point in time. Doubly-punctured disks may be similarly "mushroom-ified" (self-embedded) in the obvious way, so as a result injectivity of L is again implied by Propositions 2.2.5, 2.2.6, and 2.2.7 together. Naturality of the resulting components $K_{(X,\beta),(Y,\gamma)}$ of the tensorator is just the statement that the same diagram is obtained if; any pair of morphisms $f : X \to Z$ and $g : Y \to W$ are glued along the punctures of a doubly punctured disk and then are coalesced together, or instead the punctures are coalesced and the single annulus $J(f \otimes g)$ is glued inside the single remaining puncture.

We now translate the associativity constraint for the tensorator $K_{(X,\beta),(Y,\gamma)}$ into the language of diagrams. After applying **flatten** and traversing each arm of the associativity constraint axiom, we just have to verify equality of the pair of diagrams on the last row of Figure 2.2.

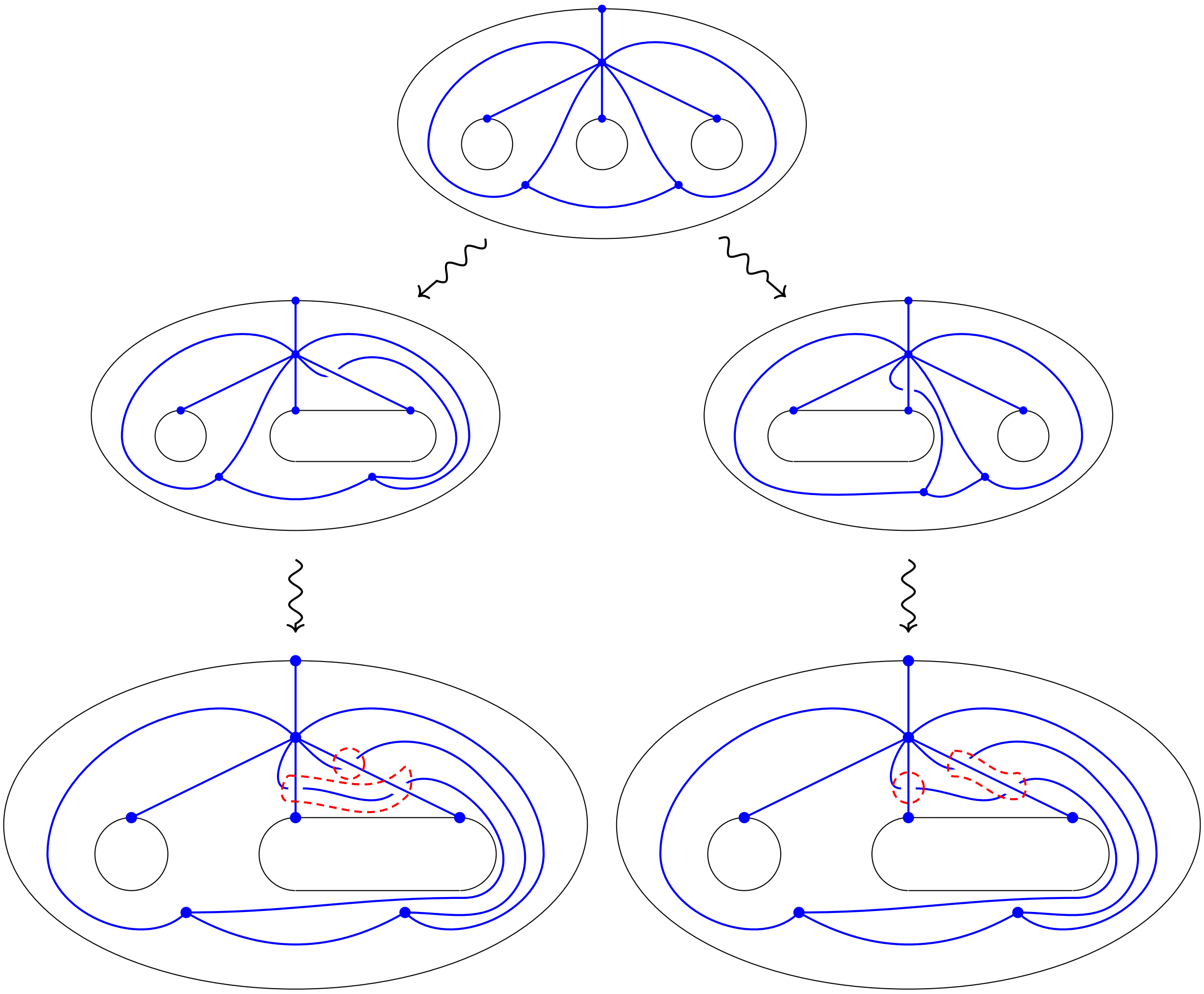
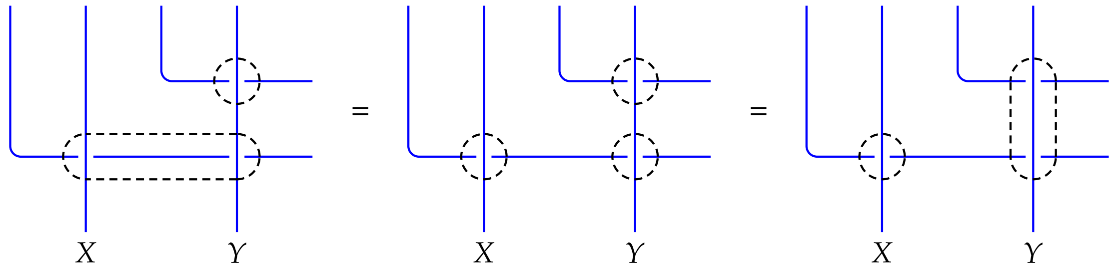


Figure 2.2: A diagrammatic representation of each side of the associativity constraint hexagon for K.

Superficially they are equal, but they differ by the encircled dashed regions which represent places where multiple string(s) have been braided over other(s) all in one go. Their equality is geometrically evident if we can show that both of the identities



hold in our diagrammatic calculus given objects (X, β) and (Y, γ) of $\mathcal{Z}(\mathcal{C})$. Thus we are done, since the first equality follows from just the definition of the half-braiding on a tensor product in $\mathcal{Z}(\mathcal{C})$, and the second is an application of Proposition 2.2.6.

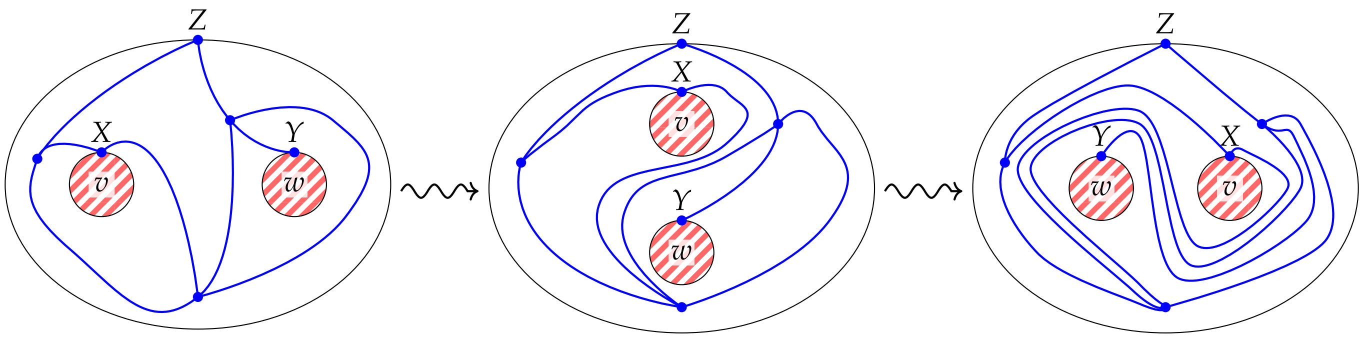
We have already seen above that the monoidal unit of $\mathsf{Rep} \int_{S^1} \mathcal{C}$ assigns to each $Z \in \int_{S^1} \mathcal{C}$ the vector space of diagrams drawn in the disk with boundary Z. In turn, the identitor $\kappa : \mathbb{1} \to G(\mathbb{1})$ is just the map which given any disk diagram makes a puncture somewhere away from a string, subject to the relation that strings may freely pass over the puncture!

Finally, we verify the identity constraints. Consider a doubly punctured disk of the form (2.4.1) representing a vector in $(\mathbb{1} \otimes V)(Z)$ for $Z \in \int_{S^1} \mathcal{C}$. Then in order to obtain a representative of a vector in $V(Z)$ we could choose a representing vector for the left puncture which is an empty disk, and then fill the left puncture in with this disk (this is the left unitor). Alternatively, we could coalesce the punctures as usual (the tensorator). The left identity constraint says that these two results must be equal, and it holds by unfolding the definition of the braiding on the tensor product $\mathbb{1} \otimes (X, \beta)$ in $\mathcal{Z}(\mathcal{C})$. The right identity constraint is even simpler (the asymmetry arises because in the definition of $\otimes$ when coalescing we only pull strings over the right puncture and never the left), so this completes the proof. □

Corollary 2.4.2.1. *When $\mathcal{C}$ is finitely semisimple and the functor F exists, it is monoidal.*

Proof. Recall that F is only defined up to the structure of an equivalence on the inclusion $E : \mathcal{D} \hookrightarrow \mathsf{Rep} \int_{S^1} \mathcal{C}$. Let us arrange that E is part of an adjoint equivalence, so that F and G are part of an adjoint equivalence. Then F acquires a monoidal structure from G as in Proposition 1.2.16. □

A braiding on $\mathsf{Rep} \int_{S^1} \mathcal{C}$ may be similarly defined diagrammatically. We define an isomorphism $(b_{V,W})_Z : (V \otimes W)(Z) \to (W \otimes V)(Z)$ for each $Z \in \int_{S^1} \mathcal{C}$ and diagram $v \in (V \otimes W)(Z)$ by "sticking our fingers in both punctures and rotating clockwise one-half revolution". For example, through two quarter revolutions the diagram (2.4.1) would become

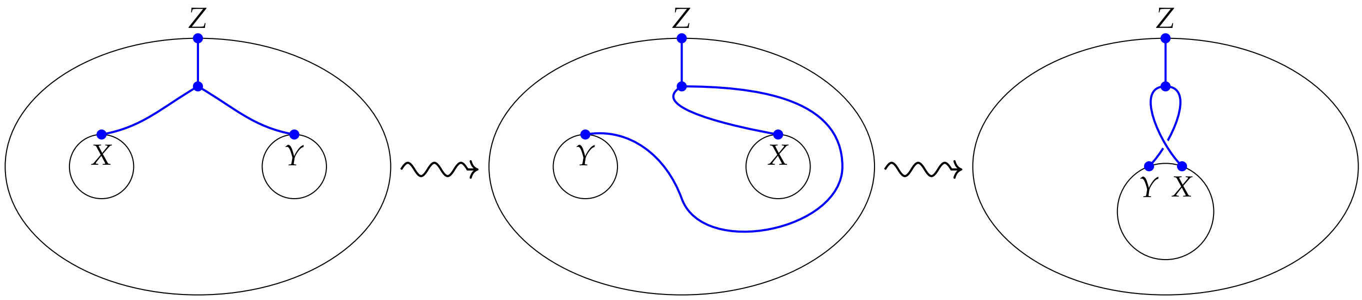


This construction defines a linear isomorphism for each Z, with inverse "inserting your fingers back into the holes and rotating counterclockwise" (i.e. playing the isotopy in reverse). Moreover naturality of $b_{V,W}$ follows by inspection, just because isotoping punctures around and then replacing their vector labels commutes with replacing vector labels and then isotoping the punctures.

The easiest way to verify the hexagon axioms for b is to apply the **flatten** map to each vertex of each axiom, so we can view each as a statement about isotopies in triply punctured disks (with the punctures respectively labelled by a vector from a fixed representation). Viewed this way, the two arms of each hexagon give different isotopies of a triply punctured disk which each have the same starting and ending locations for the labelled punctures. However, string diagrams inscribed on each punctured disk might have been distorted differently—though there is clearly an isotopy which rearranges the punctures and takes one such distortion to the other distortion of the same diagram (we just compose the isotopy for one of the arms of the hexagon with reverse of the other.). The question is just whether there is always an isotopy between these two distorted diagrams which fixes the punctures in place (i.e. rel boundary), and this is clear by inspection.

Proposition 2.4.3. *The functor G is braided.*

Proof. Let us first compute the result of braiding $G(X, \beta)$ past $G(Y, \gamma)$, and the coalescing the two together using the tensorator of G. By the relation permitting pulling-across the annulus, every diagram in $(G(X, \beta) \otimes G(Y, \gamma))(Z)$ is represented by a diagram of the leftmost form below, whereupon braiding then coalescing we obtain a diagram of the second form. In particular, the sole interior vertex retains the same label.



The claim that G is braided is the assertion that this last diagram is equal to coalescing without braiding and then gluing $J(\beta_Y)$ along the inside of the resulting diagram. This holds by inspection. □

Corollary 2.4.3.1. *When C is finitely semisimple and the functor F exists, it is braided.*

Proof. Again use Proposition 1.2.16. □

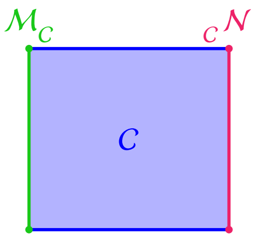
Assembling all of the results of this chapter together, we have established the following theorem.

Theorem 2.4.4. *Let C be finitely semisimple with $\mathrm{End}(\mathbb{1}) \cong \Bbbk$. Then $\mathcal{Z}(C)$ and $\mathsf{Rep}^{\mathrm{op}} \int_{S^1} C$ are equivalent as braided monoidal categories.*

Proof. Combine Theorem 2.3.5, Proposition 2.4.3, and Corollary 2.4.3.1; the functors F and G witness a braided monoidal equivalence. □

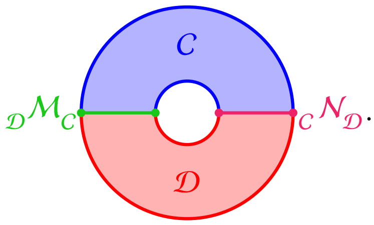
Balanced tensor products of (bi)module categories

Given a pair of module categories $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$, under suitable hypotheses we can take their Deligne product $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ (introduced in Definition 1.3.11). On the other hand, we can form a category of diagrams drawn in the 2-manifold



(3.0.1)

and consider representations of this category. Similarly, when ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ are bimodule categories we have a Deligne product $\mathcal{M} \underset{\mathcal{C}\otimes\mathcal{D}^{\mathrm{mop}}}{\boxtimes} \mathcal{N}$ (after taking flips). When $\mathcal{C}$ and $\mathcal{D}$ are pivotal there is now an associated category of diagrams of the form



(3.0.2)

In this chapter we prove equivalence theorems between representations of the diagram categories just described and the corresponding purely algebraic constructions. We do this by passing via our *balanced tensor product* $\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}$ of $\mathcal{C}$-module categories, and its sister construction the category $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ of *bibalanced objects* of $\mathcal{M} \otimes \mathcal{N}$.

We define functors from $\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ to representations of their associated diagram categories (respectively (3.0.1) and (3.0.2)), and also to categories of module and bimodule functors. When all of our categories are finitely semisimple all of these functors are equivalences. In turn the Deligne product of module categories is known to be modelled by a functor category in the finitely semisimple case [11], so this will connect all of our constructions. When $\mathcal{C} = \mathcal{D}$ we also obtain an equivalence between the Drinfeld center $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ and representations of a special case of the annular category with only one equatorial boundary line.

3.1 Properties of the category $\mathcal{M} \otimes \mathcal{N}$

This section contains basic results on the naïve tensor product $\mathcal{M} \otimes \mathcal{N}$ when $\mathcal{M}$ and $\mathcal{N}$ are each module categories. We collect together some important definitions and technical facts which we will reference in the sequel.

Thus in this section let $\mathcal{C}$ be a $\Bbbk$-linear rigid monoidal category, and let $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ be a pair of right and left module categories for $\mathcal{C}$.

Definition 3.1.1. For every object $X \in \mathcal{C}$ there are natural functors $L_{\mathcal{C}}$ and ${}_{\mathcal{C}}R : \mathcal{C} \times (\mathcal{M} \otimes \mathcal{N}) \to \mathcal{M} \otimes \mathcal{N}$ defined for $X \in \mathcal{C}$ by

$$L_{\mathcal{C}}(X)(M \otimes N) = (M \triangleleft X) \otimes N \quad \text{and} \quad {}_{\mathcal{C}}R(X)(M \otimes N) = M \otimes (X \triangleright N)$$

and extending to direct sums.

Note that if $\mathcal{M}$ and $\mathcal{N}$ were instead respective left and right $\mathcal{D}$-module categories then we have analogous functors ${}_{\mathcal{D}}L$ and $R_{\mathcal{D}} : \mathcal{D} \times (\mathcal{M} \otimes \mathcal{N}) \to \mathcal{M} \otimes \mathcal{N}$.

Remark 3.1.2. By unravelling definitions for all $X, Y \in \mathcal{C}$ and $S \in \mathcal{M} \otimes \mathcal{N}$ we have a literal equality

$$(L_{\mathcal{C}}(X) \circ {}_{\mathcal{C}}R(Y))(S) = ({}_{\mathcal{C}}R(Y) \circ L_{\mathcal{C}}(X))(S).$$

The module associators $n_{X,Y,M} : M \triangleleft (X \otimes Y) \to (M \triangleleft X) \triangleleft Y$ and $m_{X,Y,N} : (X \otimes Y) \triangleright N \to X \triangleright (Y \triangleright N)$ further give rise to natural isomorphisms

$$\begin{aligned} n^L_{X,Y,S} &: L_{\mathcal{C}}(X \otimes Y)(S) \to (L_{\mathcal{C}}(Y) \circ L_{\mathcal{C}}(X))(S), \quad \text{and} \\ m^R_{X,Y,S} &: {}_{\mathcal{C}}R(X \otimes Y)(S) \to ({}_{\mathcal{C}}R(X) \circ {}_{\mathcal{C}}R(Y))(S) \end{aligned} \tag{3.1.1}$$

in the obvious way (note the interchange of X and Y in $n^L_{X,Y,S}$ compared to $m^R_{X,Y,S}$). Namely $n^L_{X,Y,S}$ is constructed just by tensoring $n_{X,Y,M}$ on the left with the identity, and $m^R_{X,Y,S}$ is similarly obtained by tensoring $m_{X,Y,N}$ on the right instead.

The unitors $r_M : M \triangleleft \mathbb{1} \to M$ and $l_N : \mathbb{1} \triangleright N \to N$ similarly give rise to natural isomorphisms

$$u^L_S : L_{\mathcal{C}}(\mathbb{1})(S) \to S \quad \text{and} \quad u^R_S : {}_{\mathcal{C}}R(\mathbb{1})(S) \to S.$$

Definition 3.1.3. The *hom-pairing* on $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ is the functor $\mathrm{Hom}_{\mathcal{M}} : \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \to \mathsf{Vec}$ defined on objects by

$$\mathrm{Hom}_{\mathcal{M}}(M_1 \otimes M_2) = \mathcal{M}(M_1 \to M_2)$$

and extending to direct sums. We define $\mathrm{Hom}_{\mathcal{M}}$ on morphisms in $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ by pre- and post-composition in the obvious way.

The fact that $\mathcal{M}$ is tensored over Vec means that the hom-pairing can be used to build a functor (we suppress an association of the tensor product $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \otimes \mathcal{N} \to \mathcal{N}$, there being an obvious equivalence between the two possibilities)

$$\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N} : \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \otimes \mathcal{N} \to \mathcal{N},$$

which for example sends $M_1 \otimes M_2 \otimes N$ to $\mathcal{M}(M_1 \to M_2)N$. The functors of Definition 3.1.1 naturally interact with $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$ via several structure maps, which we now construct. Of course there is also a dual functor $\mathcal{M}\,\mathrm{Hom}_{\mathcal{N}} : \mathcal{M} \otimes \mathcal{N}^{\mathrm{op}} \otimes \mathcal{N} \to \mathcal{M}$ with its own analogous structure maps obtained formally from those which we give below.

The following proposition will permit us at first to simplify the statements of the structure maps, and will play a much more significant role in Chapter 4.

Proposition 3.1.4 (Dual bimodule category)**.** *If* $\mathcal{M}$ *is a* $(\mathcal{C}, \mathcal{D})$*-bimodule category, then* $\mathcal{M}^{\mathrm{op}}$ *is naturally a* $(\mathcal{D}, \mathcal{C})$*-bimodule category in two different ways, which we call* $\mathcal{M}^*$ *and* ${}^*\mathcal{M}$*. Moreover, the iterated duals* ${}^*(\mathcal{M}^*)$ *and* $({}^*\mathcal{M})^*$ *recover the original bimodule category up to equivalence.*

Proof. Each bimodule category structure ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ is precisely the data of a bimodule category structure ${}_{\mathcal{D}^{\mathrm{mop}}}\mathcal{M}_{\mathcal{C}^{\mathrm{mop}}}$. Also, the opposite of the underlying ordinary category of $\mathcal{M}$ gives a bimodule structure ${}_{\mathcal{C}^{\mathrm{op}}}\mathcal{M}^{\mathrm{op}}{}_{\mathcal{D}^{\mathrm{op}}}$. Combining these two facts together yields a $(\mathcal{D}^{\mathrm{op,mop}}, \mathcal{C}^{\mathrm{op,mop}})$-bimodule structure on $\mathcal{M}^{\mathrm{op}}$. Since rigid categories are monoidally equivalent to their dual opposite (Corollary 1.2.9.1) we obtain the desired $(\mathcal{D}, \mathcal{C})$-bimodule category structures $\mathcal{M}^*$ and ${}^*\mathcal{M}$ by respectively using the right dual functors in $\mathcal{C}$ and $\mathcal{D}$, and the left dual functors in $\mathcal{C}$ and $\mathcal{D}$.

Note that in a precise 3-categorical sense these are the correct notions of left and right duals of bimodule categories (see Definition 2.4.4 of [8] and the remarks subsequent). □

It is easy to see that every left or right $\mathcal{C}$-module category is a bimodule category with Vec acting on the other side, essentially just because all linear functors preserve direct sums. Thus we obtain that the analogous fact holds for module categories as well. For the remainder of this section, as in the next proposition, we take the (bi)module category structure induced by Proposition 3.1.4 for granted. To avoid a mess of nested parentheses below we write $X \triangleright M \otimes S$ for $(X \triangleright M) \otimes S$ (with $M \in \mathcal{M}^*$, $X \in \mathcal{C}$, and $S \in \mathcal{M} \otimes \mathcal{N}$).

Proposition 3.1.5 (Swap isomorphism)**.** *There is a canonical natural isomorphism (with* $M \in \mathcal{M}^*$*,* $X \in \mathcal{C}$*, and* $S \in \mathcal{M} \otimes \mathcal{N}$*)*

$$\phi_{M,X,S} : (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{\mathcal{C}}(X)(S)) \xrightarrow{\sim} (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright M \otimes S).$$

Proof. Recall that for $X \in \mathcal{C}$ and $M \in \mathcal{M}^*$ the object $X \triangleright M$ is by definition $M \triangleleft {}^*X$, and hence it is the content of Proposition 1.3.10 that there is a natural isomorphism δ_{X,M_1,M_2} : $\mathrm{Hom}_{\mathcal{M}}(M_1 \triangleleft {}^*X \otimes M_2) \to \mathrm{Hom}_{\mathcal{M}}(M_1 \otimes M_2 \triangleleft X)$. The functor $\mathrm{Hom}_{\mathcal{M}}(X \triangleright M_1 \otimes M_2)$ factors as $(\mathrm{Hom}_{\mathcal{M}} \circ (X \triangleright - \otimes \mathcal{M}))(M_1 \otimes M_2)$ and $\mathrm{Hom}_{\mathcal{M}}(M_1 \otimes M_2 \triangleleft X)$ factors similarly, so whiskering δ_{X,M_1,M_2} with the identity functor on $\mathcal{N}$ on the right we obtain the desired natural transformation $\phi_{M,X,S}$. □

If $S \in \mathcal{M} \otimes \mathcal{N}$ is a direct sum

$$S = \bigoplus_i M_i \otimes N_i$$

with $M_i \in \mathcal{M}$, $N_i \in \mathcal{N}$, then in explicit terms we have just constructed the natural map (for $M \in \mathcal{M}^*$)

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}\left(\bigoplus_i M \otimes (M_i \triangleleft X) \otimes N_i\right) &= \bigoplus_i \mathcal{M}(M \to M_i \triangleleft X)N_i \\ &\xrightarrow{\sim} \bigoplus_i \mathcal{M}(X \triangleright M \to M_i)N_i = (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})\left(\bigoplus_i (X \triangleright M) \otimes M_i \otimes N_i\right)\end{aligned}$$

using Proposition 1.3.10.

Proposition 3.1.6. *There is a canonical natural isomorphism (with* $M \in \mathcal{M}^*$*,* $X \in \mathcal{C}$*, and* $S \in \mathcal{M} \otimes \mathcal{N}$*)*

$$t_{M,X,S} : ((\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}))(M \otimes {}_{\mathcal{C}}R(X)(S)) \xrightarrow{\sim} X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S).$$

Proof. This follows directly from the fact that the module action preserves direct sums (and thus respects the tensoring of $\mathcal{N}$ over Vec). □

We now establish that the natural isomorphisms ϕ and t just introduced are each compatible with one another, and are compatible with the module structure in a certain special sense. Each claim follows from elementary properties of duals or direct sums in module categories, and so the proof of each is omitted.

Proposition 3.1.7 (Compatibility of ϕ and t)**.** *For all $M \in \mathcal{M}^*$, $X, Y \in \mathcal{C}$, and $S \in \mathcal{M} \otimes \mathcal{N}$, the diagram*

$$\begin{tikzcd}
 & X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y \triangleright M \otimes S) & \\
X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(Y)(S)) \arrow[ur, "X \triangleright \phi_{M,Y,S}"] & & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y \triangleright M \otimes L_{\mathcal{C}}(X)(S)) \arrow[ul, "t_{Y \triangleright M,X,S}"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes (L_{\mathcal{C}}(X) \circ {}_{\mathcal{C}}R(Y))(S)) \arrow[u, "t_{M,X,{}_{\mathcal{C}}R(Y)(S)}"] \arrow[rr, equal] & & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes ({}_{\mathcal{C}}R(Y) \circ L_{\mathcal{C}}(X))(S)) \arrow[u, "\phi_{M,Y,L_{\mathcal{C}}(X)(S)}"']
\end{tikzcd}$$

is commutative.

Proposition 3.1.8 (Module associativity compatibility of ϕ)**.** *For all $M \in \mathcal{M}^*$, $X, Y \in \mathcal{C}$, and $S \in \mathcal{M} \otimes \mathcal{N}$, the diagram*

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(XY \triangleright M \otimes S) \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(n^{*-1}_{X,Y,M} \otimes S)"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright (Y \triangleright M) \otimes S) \\
 & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y \triangleright M \otimes L_{\mathcal{C}}(X)(S)) \arrow[u, "\phi_{Y \triangleright M,X,S}"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{\mathcal{C}}(XY)(S)) \arrow[uu, "\phi_{M,XY,S}"] \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes n^{L}_{X,Y,S})"'] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes (L_{\mathcal{C}}(Y) \circ L_{\mathcal{C}}(X))(S)) \arrow[u, "\phi_{M,Y,L_{\mathcal{C}}(X)(S)}"']
\end{tikzcd}$$

is commutative.

Proposition 3.1.9 (Module identity compatibility of ϕ)**.** *For all $M \in \mathcal{M}^*$ and $S \in \mathcal{M} \otimes \mathcal{N}$ the diagram*

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(\mathbb{1} \triangleright M \otimes S) \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(r^{*-1}_{M} \otimes S)"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(\mathbb{1})(S)) \arrow[u, "\phi_{M,\mathbb{1},S}"] \arrow[ur, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes u^{L}_{S})"'] &
\end{tikzcd}$$

is commutative.

Proposition 3.1.10 (Module associativity compatibility of t)**.** *For all $M \in \mathcal{M}^*$, $X, Y \in \mathcal{C}$, and*

$S \in \mathcal{M} \otimes \mathcal{N}$, the diagram

$$\begin{tikzcd}
XY \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \arrow[r, "m_{X,Y,(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes S)}"] & X \triangleright (Y \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S)) \\
& X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(Y)(S)) \arrow[u, "X \triangleright t_{M,Y,S}"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(XY)(S)) \arrow[uu, "t_{M,XY,S}"] \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes m^R_{X,Y,S})"'] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes ({}_{\mathcal{C}}R(X) \circ {}_{\mathcal{C}}R(Y))(S)) \arrow[u, "t_{M,X,{}_{\mathcal{C}}R(Y)(S)}"']
\end{tikzcd}$$

is commutative.

Proposition 3.1.11 (Module identity compatibility of t)**.** *For all $M \in \mathcal{M}^*$ and $S \in \mathcal{M} \otimes \mathcal{N}$ the diagram*

$$\begin{tikzcd}
\mathbb{1} \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \arrow[r, "l_{(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes S)}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(\mathbb{1})(S)) \arrow[u, "t_{M,\mathbb{1},S}"] \arrow[ur, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes u^R_S)"']
\end{tikzcd}$$

is commutative.

In the following proposition we define the key functor which we will use to relate the balanced tensor product (once defined) to the other notions of tensor products of module categories.

Proposition 3.1.12. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\Bbbk$-linear categories. Then there is a functor*

$$P : \mathcal{M} \otimes \mathcal{N} \to [\mathcal{M}^{\mathrm{op}} \to \mathcal{N}].$$

Proof. Let $S \in \mathcal{M} \otimes \mathcal{N}$ be any object. Then we can construct a functor $P(S) : \mathcal{M}^{\mathrm{op}} \to \mathcal{N}$ by forming the composite

$$\mathcal{M}^{\mathrm{op}} \xrightarrow{\mathcal{M}^{\mathrm{op}}\otimes S} \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \otimes \mathcal{N} \xrightarrow{\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}} \mathcal{N}.$$

Concretely, if

$$S = \bigoplus_i M_i \otimes N_i.$$

then the functor $P(S)$ sends some $M \in \mathcal{M}^{\mathrm{op}}$ to

$$M \longmapsto M \otimes S = \bigoplus_i M \otimes M_i \otimes N_i \longmapsto \bigoplus_i \mathcal{M}(M \to M_i)N_i.$$

Now given a morphism $f : S \to T$ in $\mathcal{M} \otimes \mathcal{N}$, we need to produce a natural transformation $P(f) : P(S) \to P(T)$. This is the data for each $M \in \mathcal{M}^{\mathrm{op}}$ of a morphism

$$P(S)(M) = (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \to (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes T) = P(T)(M),$$

which we produce in the obvious way; namely, we define $P(f)$ componentwise by setting $P(f)_M = (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes f)$. Of course, we must verify that for every morphism $g : M_1 \to M_2$

in $\mathcal{M}^{\mathrm{op}}$ the naturality square

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M_1 \otimes T) \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(g\otimes T)"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M_2 \otimes T) \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M_1 \otimes S) \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(g\otimes S)"] \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M_1\otimes f)"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M_2 \otimes S) \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M_2\otimes f)"']
\end{tikzcd}$$

commutes, but this is immediate from the fact that $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$ is a functor. □

Remark 3.1.13. Dual to the functor P there is a functor

$$P' : \mathcal{M} \otimes \mathcal{N}^{\mathrm{op}} \to [\mathcal{N} \to \mathcal{M}],$$

given for each $T \in \mathcal{M} \otimes \mathcal{N}^{\mathrm{op}}$ by the composite

$$\mathcal{N} \xrightarrow{T\otimes\mathcal{N}} \mathcal{M} \otimes \mathcal{N}^{\mathrm{op}} \otimes \mathcal{N} \xrightarrow{\mathcal{M}\,\mathrm{Hom}_{\mathcal{N}}} \mathcal{M}.$$

Of course P' can be interpreted a functor $\mathcal{M} \otimes \mathcal{N} \to [\mathcal{N}^{\mathrm{op}} \to \mathcal{M}]$ instead since $(\mathcal{N}^{\mathrm{op}})^{\mathrm{op}}$ is just $\mathcal{N}$, and the results of this section are all naturally translated into corresponding results for P'.

As a consequence, we have a pair of maps P and P' from $\mathcal{M} \otimes \mathcal{N}$ into the functor categories $[\mathcal{M}^{\mathrm{op}} \to \mathcal{N}]$ and $[\mathcal{N}^{\mathrm{op}} \to \mathcal{M}]$. These two functor categories are in general not equivalent, nor even comparable. Nonetheless, it will follow from the results of Section 3.3 (see Remark 3.3.4) that when both $\mathcal{M}$ and $\mathcal{N}$ are finitely semisimple the categories $[\mathcal{M}^{\mathrm{op}} \to \mathcal{N}]$ and $[\mathcal{N}^{\mathrm{op}} \to \mathcal{M}]$ are equivalent.

3.2 Balancings and bibalancings

Let $\mathcal{C}$ and $\mathcal{D}$ be $\Bbbk$-linear rigid monoidal categories.

Definition 3.2.1. Let $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ be module categories, and let $S \in \mathcal{M} \otimes \mathcal{N}$. A *balancing of S* (with respect to the $\mathcal{C}$-module structures on $\mathcal{M}$ and $\mathcal{N}$) is a natural isomorphism

$$\tau_X : L_{\mathcal{C}}(X)(S) \xrightarrow{\sim} {}_{\mathcal{C}}R(X)(S)$$

such that the *associativity constraint diagram*

$$\begin{tikzcd}
{}_{\mathcal{C}}R(XY)(S) \arrow[r, "m^R_{X,Y,S}"] & ({}_{\mathcal{C}}R(X) \circ {}_{\mathcal{C}}R(Y))(S) \\
 & ({}_{\mathcal{C}}R(X) \circ L_{\mathcal{C}}(Y))(S) \arrow[u, "{}_{\mathcal{C}}R(X)(\tau_Y)"'] \\
 & (L_{\mathcal{C}}(Y) \circ {}_{\mathcal{C}}R(X))(S) \arrow[u, equal] \\
L_{\mathcal{C}}(XY)(S) \arrow[uuu, "\tau_{XY}"] \arrow[r, "n^L_{X,Y,S}"] & (L_{\mathcal{C}}(Y) \circ L_{\mathcal{C}}(X))(S) \arrow[u, "L_{\mathcal{C}}(Y)(\tau_X)"']
\end{tikzcd} \tag{3.2.1}$$

commutes for all $X, Y \in \mathcal{C}$.

Suppose for a moment that we have bimodule categories ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$, with middle associators $r_{Y,M,X}$ and $s_{X,N,Y}$ respectively. Then just as we had the natural isomorphisms n^L and m^R, there are natural isomorphisms

$$\begin{aligned} s^L_{X,Y} &: ({}_{\mathcal{D}}L(Y) \circ L_{\mathcal{C}}(X))(S) \to (L_{\mathcal{C}}(X) \circ {}_{\mathcal{D}}L(Y))(S), \quad \text{and} \\ s^R_{X,Y} &: (R_{\mathcal{D}}(Y) \circ {}_{\mathcal{C}}R(X))(S) \to ({}_{\mathcal{C}}R(X) \circ R_{\mathcal{D}}(Y))(S). \end{aligned}$$

Definition 3.2.2. Suppose ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ are bimodule categories. Then a *bibalancing of* $S \in \mathcal{M} \otimes \mathcal{N}$ (with respect to this bimodule structure) is a pair

$$(\tau_X : L_{\mathcal{C}}(X)(S) \xrightarrow{\sim} {}_{\mathcal{C}}R(X)(S),\ \sigma_Y : L_{\mathcal{D}}(Y)(S) \xrightarrow{\sim} {}_{\mathcal{D}}R(D)(S))$$

of balancings of S with respect to the right $\mathcal{C}$- and $\mathcal{D}^{\mathrm{mop}}$-module category structures on $\mathcal{M}$ and $\mathcal{N}$ so that for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ the *bibalancing compatibility constraint diagram*

$$\begin{tikzcd}
({}_{\mathcal{C}}R(X) \circ R_{\mathcal{D}}(Y))(S) \arrow[r, "s^R_{X,Y,S}"] & (R_{\mathcal{D}}(Y) \circ {}_{\mathcal{C}}R(X))(S) \\
({}_{\mathcal{C}}R(X) \circ {}_{\mathcal{D}}L(Y))(S) \arrow[u, "{}_{\mathcal{C}}R(X)(\sigma_Y)"] \arrow[d, equal] & (R_{\mathcal{D}}(Y) \circ L_{\mathcal{C}}(X))(S) \arrow[u, "R_{\mathcal{D}}(Y)(\tau_X)"'] \arrow[d, equal] \\
({}_{\mathcal{D}}L(Y) \circ {}_{\mathcal{C}}R(X))(S) & (L_{\mathcal{C}}(X) \circ R_{\mathcal{D}}(Y))(S) \\
({}_{\mathcal{D}}L(Y) \circ L_{\mathcal{C}}(X))(S) \arrow[u, "{}_{\mathcal{D}}L(Y)(\tau_X)"] \arrow[r, "s^L_{X,Y,S}"'] & (L_{\mathcal{C}}(X) \circ {}_{\mathcal{D}}L(Y))(S) \arrow[u, "L_{\mathcal{C}}(X)(\sigma_Y)"']
\end{tikzcd} \tag{3.2.2}$$

commutes.

Proposition 3.2.3. *Let* (S, τ) *be an arbitrary balanced object of* $\mathcal{M} \otimes \mathcal{N}$*. Then the diagram*

$$\begin{tikzcd}
{}_{\mathcal{C}}R(\mathbb{1})(S) \arrow[r, "u^R_S"] & S \\
L_{\mathcal{C}}(\mathbb{1})(S) \arrow[u, "\tau_{\mathbb{1}}"] \arrow[ur, "u^L_S"'] &
\end{tikzcd} \tag{3.2.3}$$

is commutative. Thus $\tau_{\mathbb{1}}$ *is fixed by* S*.*

Proof. Setting $X = Y = \mathbb{1}$ in (3.2.1) and drawing three naturality squares yields a diagram

$$\begin{tikzcd}
{}_C R(\mathbb{1})(S) \arrow[rrr, bend left, equal] & {}_C R(\mathbb{1}\mathbb{1})(S) \arrow[l, "{}_C R(l_{\mathbb{1}})(S)"'] \arrow[r, "m^R_{\mathbb{1},\mathbb{1},S}"'] & ({}_C R(\mathbb{1}) \circ {}_C R(\mathbb{1}))(S) \arrow[r, "u^R_{{}_C R(\mathbb{1})(S)}"'] & {}_C R(\mathbb{1})(S) \\
 & & ({}_C R(\mathbb{1}) \circ L_C(\mathbb{1}))(S) \arrow[u, "{}_C R(\mathbb{1})(\tau_{\mathbb{1}})"] \arrow[r, "u^R_{L_C(\mathbb{1})(S)}"] \arrow[d, equal] & L_C(\mathbb{1})(S) \arrow[u, "\tau_{\mathbb{1}}"'] \\
 & & (L_C(\mathbb{1}) \circ {}_C R(\mathbb{1}))(S) \arrow[r, "u^L_{{}_C R(\mathbb{1})(S)}"] & {}_C R(\mathbb{1})(S) \\
L_C(\mathbb{1})(S) \arrow[uuu, "\tau_{\mathbb{1}}"] \arrow[rrr, bend right, equal] & L_C(\mathbb{1}\mathbb{1})(S) \arrow[l, "L_C(l_{\mathbb{1}})(S)"'] \arrow[uuu, "\tau_{\mathbb{1}\mathbb{1}}"] \arrow[r, "n^L_{\mathbb{1},\mathbb{1},S}"] & (L_C(\mathbb{1}) \circ L_C(\mathbb{1}))(S) \arrow[u, "L_C(\mathbb{1})(\tau_{\mathbb{1}})"] \arrow[r, "u^L_{L_C(\mathbb{1})(S)}"] & L_C(\mathbb{1})(S). \arrow[u, "\tau_{\mathbb{1}}"']
\end{tikzcd}$$

The topmost and bottommost faces commute by the coherence theorem for module categories, and hence the entire diagram is commutative. Since $\tau_{\mathbb{1}}$ is an isomorphism this implies that the upper square in the diagram below commutes.

$$\begin{tikzcd}
({}_C R(\mathbb{1}) \circ L_C(\mathbb{1}))(S) \arrow[rr, equal] \arrow[d, dotted, "u^R_{L_C(\mathbb{1})(S)}"'] \arrow[drr, dotted, "{}_C R(\mathbb{1})(u^L_S)"] & & (L_C(\mathbb{1}) \circ {}_C R(\mathbb{1}))(S) \arrow[d, "u^L_{{}_C R(\mathbb{1})(S)}"] \\
L_C(\mathbb{1})(S) \arrow[rr, "\tau_{\mathbb{1}}"] \arrow[dr, dotted, "u^L_S"'] & & {}_C R(\mathbb{1})(S) \arrow[dl, dotted, "u^R_S"] \\
 & S &
\end{tikzcd}$$

The four dotted morphisms form a quadrilateral which commutes by naturality of u^R, and the upper right triangular face also commutes by definition. Thus it follows that the bottom triangular face commutes as well, as desired. $\square$

Definition 3.2.4. Given any pair of module categories $\mathcal{M}_C$ and ${}_C\mathcal{N}$ there is a category $\mathrm{Bal}_C(\mathcal{M}, \mathcal{N})$ of *C-balanced objects of $\mathcal{M} \otimes \mathcal{N}$*. That is, the objects of $\mathrm{Bal}_C(\mathcal{M}, \mathcal{N})$ are pairs (S, τ) with S an object of $\mathcal{M} \otimes \mathcal{N}$ equipped with a balancing $\tau_X : L_C(X)(S) \to {}_C R(X)(S)$ of S. A morphism of balanced objects $(S, \tau) \to (T, \kappa)$ is just morphism $f : S \to T$ in $\mathcal{M} \otimes \mathcal{N}$ such that the diagram

$$\begin{tikzcd}
L_C(X)(T) \arrow[r, "\kappa_X"] & {}_C R(X)(T) \\
L_C(X)(S) \arrow[u, "L_C(X)(f)"] \arrow[r, "\tau_X"] & {}_C R(X)(S) \arrow[u, "{}_C R(X)(f)"']
\end{tikzcd} \tag{3.2.4}$$

is commutative for all $X \in \mathcal{C}$. It is clear that morphisms of $\mathcal{C}$-balanced objects are closed under composition, and that the identity is always a morphism of $\mathcal{C}$-balanced objects.

Similarly when ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ are bimodule categories there is a category $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ of $(\mathcal{D},\mathcal{C})$-*bibalanced objects of* $\mathcal{M} \otimes \mathcal{N}$ with objects triplets (S,τ,σ) with (τ,σ) a bibalancing of S in $\mathcal{M}$. A morphism $(S,\tau,\sigma) \to (T,\kappa,\chi)$ in $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ is just a morphism $f : S \to T$ which is simultaneously a morphism of the respective pairs of $\mathcal{C}$- and $\mathcal{D}^{\mathrm{mop}}$-balanced objects (S,τ) and (T,κ), and (S,σ) and (T,χ).

The nature of the balancing and bibalancing constraints (3.2.1) and (3.2.2), along with the morphism constraint (3.2.4), imply that the categories $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ each have direct sums. We simply take direct sums of underlying objects and morphisms, and direct sums of the components of the balancings of objects. It is easy to then see that $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ are both $\Bbbk$-linear categories.

Proposition 3.2.5. *Let $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ be $\mathcal{C}$-module categories. Then the functor P of Proposition 3.1.12 extends to a functor $P : \mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N}) \to [{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$.*

Proof. Let $(S,\tau) \in \mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ be any balanced object; then Proposition 3.1.12 already generates a functor $P(S) : \mathcal{M}^{\mathrm{op}} \to \mathcal{N}$.

It remains to construct a modulator in order to turn $P(S)$ into a left $\mathcal{C}$-module functor. For this purpose, observe that we have an isomorphism

$$M \otimes L_{\mathcal{C}}(X)(S) \xrightarrow{M\otimes\tau_X} M \otimes {}_{\mathcal{C}}R(X)(S) \tag{3.2.5}$$

in $\mathcal{M}^* \otimes \mathcal{M} \otimes \mathcal{N}$ coming from the $\mathcal{C}$-balancing τ_X of S. Applying the functor $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$ to (3.2.5) we can then form the composite of isomorphisms

$$\begin{aligned}(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright M \otimes S) &\xrightarrow{\phi^{-1}_{M,X,S}} (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{\mathcal{C}}(X)(S)) \\ &\xrightarrow{(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes\tau_X)} (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(X)(S)) \\ &\xrightarrow{t_{M,X,S}} X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S).\end{aligned} \tag{3.2.6}$$

Since by definition

$$P(S)(X \triangleright M) = (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright M \otimes S) \quad \text{and} \quad X \triangleright P(S)(M) = X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S),$$

we obtain the left modulator $c_{X,M} : P(S)(X \triangleright M) \to X \triangleright P(S)(M)$ by taking (3.2.6) verbatim.

The modulator identity constraint is encoded in the boundary of Figure 3.1. The triangle $\mathbf{J_1}$ commutes by Proposition 3.1.9, and the triangle $\mathbf{J_3}$ commutes by Proposition 3.1.11 similarly. The square $\mathbf{J_2}$ is obtained by tensoring with M and applying the functor $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$ to the identity triangle for the balancing τ of Proposition 3.2.3. This establishes commutativity of the outer boundary, and thus the modulator identity constraint for $P(S)$.

The associativity constraint for the modulator $c_{X,M}$ then amounts to verification that external boundary of the diagram of Figure 3.2 commutes. It is the content of Proposition 3.1.8 that the lower pentagon $\mathbf{H_1}$ commutes, and the upper pentagon $\mathbf{H_6}$ commutes by Proposition 3.1.10 similarly. Squares $\mathbf{H_2}$ and $\mathbf{H_4}$ commute by the respective naturality of ϕ and t. The pentagon $\mathbf{H_3}$ is the compatibility between ϕ and t asserted by Proposition 3.1.7. Finally, the hexagon $\mathbf{H_5}$ is obtained by tensoring with M and applying the functor $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$ to the associativity constraint (3.2.1) for the balancing τ. Therefore the outer polygon commutes, and thus $P(S)$ really becomes a left $\mathcal{C}$-module functor when equipped with the modulator (3.2.6).

$$
\begin{tikzcd}[column sep=large]
 & \mathbb{1} \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \arrow[dl, "l^{\mathcal{N}}_{(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes S)}"'] \arrow[d, phantom, "\mathbf{J_3}" near end, shift right=8ex] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \arrow[d, equal] \arrow[dr, phantom, "\mathbf{J_2}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{C}R(\mathbb{1})(S)) \arrow[l, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes u^{L}_{S})"] \arrow[u, "t_{M,\mathbb{1},S}"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{C}(\mathbb{1})(S)) \arrow[l, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes u^{R}_{S})"'] \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(\tau_{\mathbb{1}}\otimes M)"'] \arrow[d, phantom, "\mathbf{J_1}", shift right=8ex] \\
 & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(\mathbb{1} \triangleright M \otimes S) \arrow[ul, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(l^{\mathcal{M}^*}_{M}\otimes S)"] \arrow[u, "\phi^{-1}_{M,\mathbb{1},S}"']
\end{tikzcd}
$$

Figure 3.1: The modulator identity constraint for $P(S)$.

$$
\begin{tikzcd}
X \triangleright (Y \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S)) \arrow[rr, equal] & & X \triangleright (Y \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S)) \\
& \mathbf{H_6} & XY \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \arrow[u, "m_{X,Y,(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes S)}"'] \\
X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{C}R(Y)(S)) \arrow[uu, "X\triangleright t_{M,Y,S}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{C}R(X)({}_{C}R(Y)(S))) \arrow[l, "t_{M,X,{}_{C}R(Y)(S)}"'] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{C}R(XY)(S)) \arrow[l, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes n^{L}_{X,Y,S})"'] \arrow[u, "t_{M,XY,S}"'] \\
X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{C}(Y)(S)) \arrow[u, "X\triangleright(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes\tau_Y)"] \arrow[ur, phantom, "\mathbf{H_4}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{C}R(X)(L_{C}(Y)(S))) \arrow[l, "t_{M,X,L_{C}(Y)(S)}"] \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes {}_{C}R(X)(\tau_Y))"'] & \\
X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y \triangleright M \otimes S) \arrow[u, "X\triangleright\phi^{-1}_{M,Y,S}"] \arrow[r, phantom, "\mathbf{H_3}"] & {} \arrow[r, phantom, "\mathbf{H_5}"] & {} \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y \triangleright M \otimes {}_{C}R(X)(S)) \arrow[u, "t_{Y\triangleright M,X,S}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{C}(Y)({}_{C}R(X)(S))) \arrow[l, "\phi_{M,Y,{}_{C}R(X)(S)}"'] \arrow[uu, equal] & \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y \triangleright M \otimes L_{C}(X)(S)) \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(Y\triangleright M\otimes\tau_X)"] \arrow[ur, phantom, "\mathbf{H_2}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{C}(Y)(L_{C}(X)(S))) \arrow[l, "\phi_{M,Y,L_{C}(S)}"] \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes L_{C}(Y)(\tau_X))"'] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{C}(XY)(S)) \arrow[l, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes m^{R}_{X,Y,S})"] \arrow[uuuu, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes\tau_{XY})"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright (Y \triangleright M) \otimes S) \arrow[u, "\phi^{-1}_{Y\triangleright M,X,S}"] & \mathbf{H_1} & \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(XY \triangleright M \otimes S) \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(m_{X,Y,M}\otimes S)"] \arrow[rr, equal] & & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(XY \triangleright M \otimes S) \arrow[uu, "\phi^{-1}_{M,XY,S}"']
\end{tikzcd}
$$

Figure 3.2: The modulator associativity constraint for $P(S)$.

Given a morphism $f : (S, \tau) \to (T, \kappa)$ of balanced objects, we need to check that the natural transformation $P(f) : P(S) \to P(T)$ is a morphism of left $\mathcal{C}$-module functors. This amounts to verifying commutativity of the boundary of Figure 3.3.

$$\begin{tikzcd}
X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes S) \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(f\otimes M)"] & X \triangleright (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes T) \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(X)(S)) \arrow[u, "t_{M,X,S}"] \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes_{\mathcal{C}}R(X)(f))"] \arrow[ur, phantom, "\mathbf{K_3}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes {}_{\mathcal{C}}R(X)(T)) \arrow[u, "t_{M,X,T}"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{\mathcal{C}}(X)(S)) \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes\tau_X)"] \arrow[r, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes L_{\mathcal{C}}(X)(f))"] \arrow[ur, phantom, "\mathbf{K_2}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M \otimes L_{\mathcal{C}}(X)(T)) \arrow[u, "(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(M\otimes\kappa_X)"'] \\
(\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright M \otimes S) \arrow[u, "\phi^{-1}_{M,X,S}"] \arrow[r, "\mathrm{Hom}_{\mathcal{M}}(f\otimes X\triangleright M)"] \arrow[ur, phantom, "\mathbf{K_1}"] & (\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N})(X \triangleright M \otimes T) \arrow[u, "\phi^{-1}_{M,X,T}"']
\end{tikzcd}$$

Figure 3.3: The module functor morphism constraint for $P(f : (S, \sigma) \to (T, \kappa))$.

The square $\mathbf{K_1}$ commutes by naturality of ϕ, and the square $\mathbf{K_3}$ commutes by naturality of t. The final square $\mathbf{K_2}$ is obtained by tensoring the balancing morphism constraint (3.2.4) with M and then applying the functor $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$. □

Proposition 3.2.6. *Let ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ be bimodule categories. Then the functor P of Proposition 3.1.12 extends to a functor $P : \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N}) \to [{}_{\mathcal{C}}\mathcal{M}^*_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}]$.*

Proof. There is an evident forgetful functor $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N}) \to \mathrm{Bal}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ obtained from forgetting the respective side of the bimodule category structure of $\mathcal{M}$ and $\mathcal{N}$ on which $\mathcal{D}$ acts. Thus (together with the forgetful functor corresponding to forgetting a $\mathcal{C}$-module structure) Proposition 3.2.5 implies that for each $(S, \sigma, \tau) \in \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$ the functor $P(S) : \mathcal{M}^* \to \mathcal{N}$ canonically becomes a left $\mathcal{D}$- and right $\mathcal{C}$-module functor.

Hence it remains to just verify that the left $\mathcal{C}$- and right $\mathcal{D}$-module functor structures are compatible and extend to the data of a bimodule functor. That is, we must verify commutativity of the boundary of Figure 3.4 (in which we write $\mathrm{H}\,\mathcal{N}$ for $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$ for brevity).

$$
\begin{tikzcd}
(X \triangleright \mathrm{H}\mathcal{N}(M \otimes S)) \triangleleft Y \arrow[rrr, "s_{X,\mathrm{H}\mathcal{N}(M\otimes S),Y}"'] & & & X \triangleright (\mathrm{H}\mathcal{N}(M \otimes S) \triangleleft Y) \\
\mathrm{H}\mathcal{N}(M \otimes {}_{\mathcal{C}}R(X)(S)) \triangleleft Y \arrow[u, "t^{\mathcal{C}}_{M,X,S}\triangleleft Y"] & \mathrm{H}\mathcal{N}(M \otimes (R_{\mathcal{D}}(Y) \circ {}_{\mathcal{C}}R(X))(S)) \arrow[l, "t^{\mathcal{D}}_{M,Y,{}_{\mathcal{C}}R(X)(S)}"'] \arrow[r, "\mathrm{H}\mathcal{N}(M\otimes s^{L}_{X,Y,S})"] \arrow[ur, phantom, "\mathbf{L_9}"] & \mathrm{H}\mathcal{N}(M \otimes ({}_{\mathcal{C}}R(X) \circ R_{\mathcal{D}}(Y))(S)) \arrow[r, "t^{\mathcal{C}}_{M,X,R_{\mathcal{D}}(Y)(S)}"] & X \triangleright \mathrm{H}\mathcal{N}(M \otimes R_{\mathcal{D}}(Y)(S)) \arrow[u, "X\triangleright t^{\mathcal{D}}_{M,Y,S}"'] \\
\mathrm{H}\mathcal{N}(M \otimes L_{\mathcal{C}}(X)(S)) \triangleleft Y \arrow[u, "\mathrm{H}\mathcal{N}(M\otimes\tau_X)\triangleleft Y"] \arrow[ur, phantom, "\mathbf{L_4}"] & \mathrm{H}\mathcal{N}(M \otimes (R_{\mathcal{D}}(Y) \circ L_{\mathcal{C}}(X))(S)) \arrow[l, "t^{\mathcal{D}}_{M,Y,L_{\mathcal{C}}(X)(S)}"] \arrow[u, "\mathrm{H}\mathcal{N}(M\otimes R_{\mathcal{D}}(Y)(\tau_X))"'] & \mathrm{H}\mathcal{N}(M \otimes ({}_{\mathcal{C}}R(X) \circ {}_{\mathcal{D}}L(Y))(S)) \arrow[u, "\mathrm{H}\mathcal{N}(M\otimes {}_{\mathcal{C}}R(X)(\kappa_Y))"] \arrow[r, "t^{\mathcal{C}}_{M,X,{}_{\mathcal{D}}L(Y)(S)}"'] \arrow[ur, phantom, "\mathbf{L_8}"] & X \triangleright \mathrm{H}\mathcal{N}(M \otimes {}_{\mathcal{D}}L(Y)(S)) \arrow[u, "X\triangleright \mathrm{H}\mathcal{N}(M\otimes\kappa_Y)"'] \\
\mathrm{H}\mathcal{N}(X \triangleright M \otimes S) \triangleleft Y \arrow[u, "\phi^{\mathcal{C}-1}_{M,X,S}\triangleleft Y"] \arrow[r, phantom, "\mathbf{L_3}"] & \arrow[r, phantom, "\mathbf{L_5}"] & \arrow[r, phantom, "\mathbf{L_7}"] & X \triangleright \mathrm{H}\mathcal{N}(M \triangleleft Y \otimes S) \arrow[u, "X\triangleright\phi^{\mathcal{D}-1}_{M,S,Y}"'] \\
\mathrm{H}\mathcal{N}(X \triangleright M \otimes R_{\mathcal{D}}(Y)(S)) \arrow[u, "t^{\mathcal{D}}_{X\triangleright M,Y,S}"] & \mathrm{H}\mathcal{N}(M \otimes (L_{\mathcal{C}}(X) \circ R_{\mathcal{D}}(Y))(S)) \arrow[l, "\phi^{\mathcal{C}}_{M,X,R_{\mathcal{D}}(Y)(S)}"'] \arrow[uu, equal] & \mathrm{H}\mathcal{N}(M \otimes ({}_{\mathcal{D}}L(Y) \circ {}_{\mathcal{C}}R(X))(S)) \arrow[uu, equal] \arrow[r, "\phi^{\mathcal{D}}_{M,Y,{}_{\mathcal{C}}R(X)(S)}"] & \mathrm{H}\mathcal{N}(M \triangleleft Y \otimes {}_{\mathcal{C}}R(X)(S)) \arrow[u, "t^{\mathcal{C}}_{M\triangleleft Y,X,S}"'] \\
\mathrm{H}\mathcal{N}(X \triangleright M \otimes {}_{\mathcal{D}}L(Y)(S)) \arrow[u, "\mathrm{H}\mathcal{N}(X\triangleright M\otimes\kappa_Y)"] \arrow[ur, phantom, "\mathbf{L_2}"] & \mathrm{H}\mathcal{N}(M \otimes (L_{\mathcal{C}}(X) \circ {}_{\mathcal{D}}L(Y))(S)) \arrow[l, "\phi^{\mathcal{C}}_{M,X,{}_{\mathcal{D}}L(Y)(S)}"] \arrow[r, "\mathrm{H}\mathcal{N}(M\otimes s^{R}_{X,Y,S})"'] \arrow[u, "\mathrm{H}\mathcal{N}(M\otimes L_{\mathcal{C}}(X)(\kappa_Y))"'] \arrow[dr, phantom, "\mathbf{L_1}"] & \mathrm{H}\mathcal{N}(M \otimes ({}_{\mathcal{D}}L(Y) \circ L_{\mathcal{C}}(X))(S)) \arrow[u, "\mathrm{H}\mathcal{N}(M\otimes {}_{\mathcal{D}}L(Y)(\tau_X))"] \arrow[r, "\phi^{\mathcal{D}}_{M,Y,{}_{\mathcal{C}}R(X)(S)}"'] \arrow[ur, phantom, "\mathbf{L_6}"] & \mathrm{H}\mathcal{N}(M \triangleleft Y \otimes L_{\mathcal{C}}(X)(S)) \arrow[u, "\mathrm{H}\mathcal{N}(M\triangleleft Y\otimes\tau_X)"'] \\
\mathrm{H}\mathcal{N}((X \triangleright M) \triangleleft Y \otimes S) \arrow[u, "\phi^{\mathcal{D}-1}_{X\triangleright M,Y,S}"] \arrow[rrr, "\mathrm{H}\mathcal{N}(s_{X,M,Y\otimes S})"] & & & \mathrm{H}\mathcal{N}(X \triangleright (M \triangleleft Y) \otimes S) \arrow[u, "\phi^{\mathcal{C}-1}_{M\triangleleft Y,X,S}"']
\end{tikzcd}
$$

Figure 3.4: The bimodule functor compatibility constraint (1.3.1) for $P(S)$.

The squares $\mathbf{L_2}$, $\mathbf{L_4}$, $\mathbf{L_6}$, and $\mathbf{L_8}$ are all consequences of the naturality of either ϕ or t. Squares $\mathbf{L_3}$ and $\mathbf{L_7}$ commute by the variants of Proposition 3.1.7 for when ϕ and t come from different module structures. Commutativity of the squares $\mathbf{L_1}$ and $\mathbf{L_9}$ constitute the compatibility results for ϕ and t for the bimodule associator which are analogous to Propositions 3.1.8 and 3.1.10. Finally, the hexagon $\mathbf{L_5}$ is obtained from tensoring the bibalancing compatibility constraint (3.2.2) with M and applying the functor $\mathrm{Hom}_{\mathcal{M}}\,\mathcal{N}$.

Since a morphism of bimodule functors is just a morphism of each left and right module functor structure separately, this completes the proof. □

3.3 Algebraic consequences of finite semisimplicity

Henceforth fix a distinguished collection $\{X_i\}$ of representatives of the isomorphism classes of simple objects of $\mathcal{M}$, and another such collection $\{Y_j\}$ for $\mathcal{N}$. Suppose also that $\mathcal{M}$ (but not $\mathcal{N}$) is finitely semisimple.

Proposition 3.3.1. *There is a functor $Q : [\mathcal{M}^{\mathrm{op}} \to \mathcal{N}] \to \mathcal{M} \otimes \mathcal{N}$ which is pre-inverse to $P : \mathcal{M} \otimes \mathcal{N} \to [\mathcal{M}^{\mathrm{op}} \to \mathcal{N}]$ of Proposition 3.1.12 (up to natural isomorphism). If $\mathcal{N}$ is semisimple then P and Q together witness an equivalence of categories.*

Proof. Given a functor $F : \mathcal{M}^{\mathrm{op}} \to \mathcal{N}$, we get an object of $\mathcal{M} \otimes \mathcal{N}$ by the formula

$$Q(F) = \bigoplus_i X_i \otimes F(X_i).$$

Then given a natural transformation $\eta : F \to G$ of functors $\mathcal{M}^{\mathrm{op}} \to \mathcal{N}$ we can similarly form the direct sum

$$Q(\eta) = \bigoplus_i X_i \otimes \eta_{X_i} : Q(F) \to Q(G),$$

obtaining a morphism in $\mathcal{M} \otimes \mathcal{N}$; this assignment is obviously functorial, so we obtain a functor $Q : [\mathcal{M}^{\mathrm{op}} \to \mathcal{N}] \to \mathcal{M} \otimes \mathcal{N}$.

Now, in order to provide a natural isomorphism $\alpha_F : F \to PQ(F)$, we must determine a natural isomorphism $(\alpha_F)_{X_i}$ for each simple object X_i. This is just the data of a map

$$(\alpha_F)_{X_i} : F(X_i) \xrightarrow{\sim} \bigoplus_j \mathcal{M}(X_i \to X_j)F(X_j),$$

which we build by taking an isomorphism $F(X_i) \to \mathcal{M}(X_i \to X_i)F(X_i)$ and ignoring all other summands. Since $\mathcal{M}(X_i \to X_j) \cong 0$ for all $i \neq j$, the complete map $(\alpha_F)_{X_i}$ so assembled is an isomorphism, too. These components yield a natural isomorphism α_F just because of the fact that the tensored action of Vec on $\mathcal{M}$ is functorial. Naturality of α itself asserts in components that for any $f : F \to G$ in $\mathrm{End}(\mathcal{M})$ the outer boundary of the rectangle

$$\begin{tikzcd}
\bigoplus_i \mathcal{M}(X_i \to X_j)G(X_i) \arrow[r, equal] & \bigoplus_i \mathcal{M}(X_i \to X_j)G(X_i) \\
\bigoplus_i \mathcal{M}(X_i \to X_j)F(X_i) \arrow[u, "PQ(f)_{X_j}"] \arrow[r, "\widetilde{f_{X_j}}"] & G(X_j) \arrow[u, "(\alpha_G)_{X_j}"'] \\
F(X_j) \arrow[u, "(\alpha_F)_{X_j}"] \arrow[r, equal] & F(X_j) \arrow[u, "f_{X_j}"']
\end{tikzcd}$$

is commutative, with the middle horizontal morphism being assembled from the zero map for every summand of $\bigoplus_i \mathcal{M}(X_i \to X_j)F(X_i)$ with $i \neq j$, and from f_{X_j} for $i = j$. Both squares commute by applying elementary properties of direct sums to the definition of α.

Finally, assume that $\mathcal{N}$ is semisimple. We must construct a second natural isomorphism $\beta_S : S \to QP(S)$. Now, it is clear that every such $S \in \mathcal{M} \otimes \mathcal{N}$ is isomorphic to a direct sum of the form (for some sequences of indices a_i and b_i)

$$\bigoplus_i X_{a_i} \otimes Y_{b_i}, \tag{3.3.1}$$

i.e. that $\mathcal{M} \otimes \mathcal{N}$ is then semisimple as well. Thus we will just define the components of β assuming that $S = X_i \otimes Y_j$ (a summand of (3.3.1)), and will then extend using semisimplicity and the axiom of choice. In this situation, the component β_S is the data of a morphism

$$\begin{aligned}\beta_S : X_i \otimes Y_j \xrightarrow{\sim} & \bigoplus_k X_k \otimes (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(X_k \otimes X_i \otimes Y_j) \\ \cong & \bigoplus_k X_k \otimes \mathcal{M}(X_k \to X_i) Y_j.\end{aligned}$$

Thus as in the previous case it is sufficient to now choose an isomorphism $Y_j \to \mathcal{M}(X_i \to X_i)Y_j$ for each X_i and Y_j, and elementary properties of direct sums then again yield that a natural transformation may be built by extending the definition of β_S to all direct sums and isomorphic objects. □

Proposition 3.3.2. *The functor Q of Proposition 3.3.1 extends to a functor $Q : [{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}] \to \mathrm{Bal}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, again part of an equivalence when $\mathcal{N}$ is semisimple.*

Proof. In order to extend Q to a functor into $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, let F be any left $\mathcal{C}$-module functor $\mathcal{M}^* \to \mathcal{N}$ with modulator isomorphism $c_{X,M} : F(X \triangleright M) \to X \triangleright F(M)$. Then together with the isomorphism $\alpha_F : F \to PQ(F)$ of Proposition 3.3.1, the modulator $c_{X,M}$ for F may be transported in the obvious way (filling in the missing arrow in the commutative diagram which asserts that α_F is a morphism of module functors) to obtain a modulator $d_{X,M} : PQ(F)(X \triangleright M) \to X \triangleright PQ(F)(M)$ for $PQ(F)$.

Recalling the definition of P, the modulator d consists in components of morphisms

$$d_{X,M} : (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(X \triangleright M \otimes Q(F)) \to X \triangleright (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(M \otimes Q(F)).$$

Thus by using the natural isomorphisms ϕ and t^{-1} we may build the composite (again, a natural isomorphism)

$$\begin{aligned}\xi_{M,X} : (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(M \otimes L_{\mathcal{C}}(X)(Q(F))) &\xrightarrow{\phi_{M,X,Q(F)}} (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(X \triangleright M \otimes Q(F)) \\ &\xrightarrow{d_{X,M}} X \triangleright (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(M \otimes Q(F)) \\ &\xrightarrow{t^{-1}_{M,X,Q(F)}} (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(M \otimes {}_{\mathcal{C}}R(X)(Q(F)))\end{aligned}$$

from which we will imminently extract a balancing of $Q(F)$. Fixing $X \in \mathcal{C}$ and taking $S = L_{\mathcal{C}}(X)(Q(F))$ and $T = {}_{\mathcal{C}}R(X)(Q(F))$, we recognise $\xi_{M,X}$ as a natural transformation $P(S) = (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(M \otimes S) \to (\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(T \otimes M) = P(T)$. The fact that P is fully faithful as a functor to $\mathcal{M} \otimes \mathcal{N}$ (by Proposition 3.3.1) yields a unique lift of $\xi_{-,X}$ to an ordinary isomorphism $\tau_X : S = L_{\mathcal{C}}(X)(Q(F)) \to {}_{\mathcal{C}}R(X)(Q(F)) = T$ in $\mathcal{M} \otimes \mathcal{N}$. Allowing $X \in \mathcal{C}$ to vary then yields morphisms τ_X which assemble into a natural transformation.

We now claim that the natural transformation τ_X is balancing of $Q(F)$, and because by construction $(\mathrm{Hom}_{\mathcal{M}}\, \mathcal{N})(M \otimes \tau_X) = \xi_{M,X}$, we have essentially given the argument already. The associativity condition to be verified is precisely the interior region $\mathbf{H_5}$ of Figure 3.2, and since the exterior boundary commutes by the associativity condition for the modulator isomorphism $d_{X,M}$, the associativity condition for τ_X is verified (all other internal faces commute unconditionally). Similarly the interior region $\mathbf{J_2}$ of Figure 3.1 is the identity constraint to be verified for τ_X, and commutativity follows again by the identity axiom for the modulator $d_{X,M}$. Hence the natural isomorphism τ_X is a balancing of $Q(F)$, and thus extends the definition of Q on the objects of $[{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$.

It remains to check that morphisms $\eta : F \to G$ in $[{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$ are sent by Q to morphisms of balanced objects. But since the module functor morphism condition which any such η obeys asserts that the exterior boundary of Figure 3.3 commutes, the face $\mathbf{K}_2$ commutes as well because all others commute unconditionally. This completes the proof. □

Proposition 3.3.3. *The functor Q of Proposition 3.3.2 further extends to a functor $Q : [{}_{\mathcal{C}}\mathcal{M}^*_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}] \to \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$, again part of an equivalence when $\mathcal{N}$ is semisimple.*

Proof. By two applications of the previous proposition a bimodule functor $F : {}_{\mathcal{C}}\mathcal{M}^*_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ is canonically assigned an object $S \in \mathcal{M} \otimes \mathcal{N}$ and a pair of balancings

$$\tau_X : L_{\mathcal{C}}(X)(S) \xrightarrow{\sim} {}_{\mathcal{C}}R(X)(S) \quad \text{and} \quad \sigma_Y : L_{\mathcal{D}}(Y)(S) \xrightarrow{\sim} {}_{\mathcal{D}}R(D)(S)$$

with respect to the left $\mathcal{C}$- and right $\mathcal{D}$-module structure. We are just left to verify that these two balancings are compatible in the sense of (3.2.2); observe that unravelling the construction of the two balancings τ and σ, the bimodule functor condition on F asserts that the exterior boundary of Figure 3.4 commutes. Thus since all of the internal faces with the exception of $\mathbf{L}_5$ commute unconditionally, we obtain commutativity of the face $\mathbf{L}_5$ as well, and the proof is complete. □

Remark 3.3.4. In Remark 3.1.13 we saw that there was a functor P' dual to P and mapping $\mathcal{M} \otimes \mathcal{N}$ into $[\mathcal{N}^{\mathrm{op}} \to \mathcal{M}]$, and it is not difficult to see that the results of the last two sections translate into analogous statements for P'. In particular, when $\mathcal{M}$ is finitely semisimple and $\mathcal{N}$ is semisimple Proposition 3.3.1 and its counterpart for P' yield a composite of equivalences

$$[\mathcal{M}^* \to \mathcal{N}] \xrightarrow{\quad Q \quad} \mathcal{M} \otimes \mathcal{N} \xrightarrow{\quad P \quad} [{}^*\mathcal{N} \to \mathcal{M}].$$

We also find that the module functor categories $[{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$ and $[{}^*\mathcal{N}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}]$ are equivalent, and similarly when $\mathcal{M}$ and $\mathcal{N}$ are bimodule categories.

This is a manifestation of the fact that functors between finitely semisimple categories have all left and right adjoints, and that this does not hold in general.

3.4 The definition of the balanced tensor product

Definition 3.4.1. The *balanced tensor product* $\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}$ is just the category $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$.

Thinking of the category of balanced objects of a pair of $\mathcal{C}$-module categories as a tensor product, we expect the following results of standard type.

Proposition 3.4.2. *The balanced tensor product ${}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}_{\mathcal{E}}$ of bimodule categories $\mathcal{M}$ and $\mathcal{N}$ is a $(\mathcal{D},\mathcal{E})$-bimodule category. (This justifies our notation.)*

Proof. Observe that ${}_{\mathcal{D}}L$ is a functor $\mathcal{D} \to \mathrm{End}(\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N})$, and it essentially gives the left $\mathcal{D}$-action. Given an object (S,τ) of ${}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}_{\mathcal{E}}$ we construct a balancing isomorphism for ${}_{\mathcal{D}}L(Y)(S)$ by the composite

$$\begin{aligned} \kappa_X : L_{\mathcal{C}}(X)({}_{\mathcal{D}}L(Y)(S)) &\xrightarrow{\quad s^{L-1}_{Y,X,S} \quad} {}_{\mathcal{D}}L(Y)(L_{\mathcal{C}}(X)(S)) \\ &\xrightarrow{\quad {}_{\mathcal{D}}L(Y)(\tau_Y) \quad} {}_{\mathcal{D}}L(Y)({}_{\mathcal{C}}R(X)(S)) = {}_{\mathcal{C}}R(X)({}_{\mathcal{D}}L(Y)(S)). \end{aligned}$$

On objects we set $Y \triangleright (S,\tau) := ({}_{\mathcal{D}}L(Y)(S), \kappa_X)$, and on morphisms $f \triangleright g := {}_{\mathcal{D}}L(f)(g)$; the balancing associativity condition for (S,τ) and the condition on morphisms of balancings g are verified by applying the functor ${}_{\mathcal{D}}L(Y)$ to the corresponding diagrams for (S,τ) and g, then juxtaposing some naturality squares and the middle associativity pentagon for s^L. The modulator for the $\mathcal{D}$-action is just $m^L_{X,Y,M} : {}_{\mathcal{D}}L(XY)(M) \to ({}_{\mathcal{D}}L(X) \circ {}_{\mathcal{D}}L(Y))(M)$, which obeys the required identities just because of the same identities for the ordinary left $\mathcal{D}$-module structure on $\mathcal{M}$.

Similarly we obtain a formally dual right $\mathcal{E}$-module structure from $R_{\mathcal{E}}$. The $\mathcal{D}$- and $\mathcal{E}$-module structures strictly commute, so in particular we obtain a $(\mathcal{D},\mathcal{E})$-bimodule category. □

Proposition 3.4.3. *Let ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ be bimodule categories. Then upon equipping each with its respective canonically associated right or left module category structure (i.e. flip), there is an equivalence of categories $\mathcal{M} \underset{\mathcal{D}^{\mathrm{mop}}\otimes\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N} \simeq \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$.*

Proof. We just translate the correspondence between bimodule categories ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and right (or left) module categories $\mathcal{M}_{\mathcal{D}^{\mathrm{mop}}\otimes\mathcal{C}}$ into the language of (bi)balancings. The bibalancing compatibility condition (3.2.2) exactly says that the balancings τ and σ of any $(S,\tau,\sigma) \in \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ assemble into a single right $(\mathcal{C}\otimes\mathcal{D}^{\mathrm{mop}})$-balancing. Simultaneous morphisms of τ and σ are consequently morphisms of the resulting $(\mathcal{C}\otimes\mathcal{D}^{\mathrm{mop}})$-balancing by superimposing the two compatibility squares and tensoring. □

There is also the following direct connection between arbitrary categories of bibalanced objects and the Drinfeld center of their balanced tensor product.

Proposition 3.4.4. *There is an equivalence of categories $\mathcal{Z}({}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}_{\mathcal{D}}) \simeq \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$.*

Proof. Objects of $\mathcal{Z}({}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}_{\mathcal{D}})$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ can both be interpreted as objects of $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ equipped with various extra data, and likewise morphisms in each category can be interpreted as morphisms in $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ each satisfying some condition(s). We will just show that such additional data and the conditions on morphisms are in functorial bijection.

Thus let $((S,\tau_Y),\beta_X : X \triangleright (S,\tau) \to (S,\tau) \triangleleft X)$ be an object of $\mathcal{Z}({}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}_{\mathcal{D}})$. Since $X \triangleright S = {}_{\mathcal{D}}L(X)(S)$ and $S \triangleleft X = R_{\mathcal{D}}(X)(S)$, the data of β is exactly the same as the natural isomorphism data of a balancing (notwithstanding the condition such a balancing must satisfy). Since the middle associativity isomorphism in $\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is the identity, we then see directly that the balancing condition (3.2.1) is exactly the half-braiding condition (1.5.1).

Hence in order to find that the objects of $\mathcal{Z}({}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}_{\mathcal{D}})$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ are precisely the same, we just must verify that compatibility of τ_Y and β_X in the sense of a bibalanced object corresponds to the statement that β_X is a morphism of $\mathcal{C}$-balanced objects for all $X \in \mathcal{D}$. Fortunately, this is exactly the case; we recognise the left-bottom and right-top corner composites of (3.2.2) as the action of $X \in \mathcal{D}$ on each side of the balancing of (S,τ_Y), hence giving a square equivalent to (3.2.4).

Finally, the data of a morphism in $\mathcal{Z}({}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}_{\mathcal{D}})$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ being identical, the morphism square (1.5.2) for $\mathcal{Z}$ corresponds exactly to the morphism square (3.2.4). □

Corollary 3.4.4.1. *There is an equivalence of categories $\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N} \simeq \mathrm{BiBal}_{(\mathsf{Vec},\mathcal{C})}(\mathcal{M},\mathcal{N})$.*

Proof. Every left or right $\mathcal{C}$-module category can be made a bimodule category by letting Vec act on the other side. By the proof of Proposition 1.6.2 the Drinfeld center of the bimodule category ${}_{\mathsf{Vec}}\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}_{\mathsf{Vec}}$ is equivalent to $\mathcal{M} \underset{\mathcal{C}}{\overset{\mathrm{bal}}{\otimes}} \mathcal{N}$, so the claim follows by Proposition 3.4.4. □

Note that the naïve tensor product $\mathcal{M} \otimes \mathcal{N}$ of module categories $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ is naturally a $(\mathcal{C}, \mathcal{C})$-bimodule category by "X-ray vision"; that is $X \triangleright (M \otimes N) := M \otimes (X \triangleright N)$ and $(M \otimes N) \triangleleft Y := (M \triangleleft Y) \otimes N$.

Proposition 3.4.5. *Let $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ be module categories. Then there is an equivalence of categories $\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N} \simeq \mathcal{Z}(\mathcal{M} \otimes \mathcal{N})$.*

Proof. Unfold the definition of both objects. □

Proposition 3.4.6. *When $\mathcal{C}$, $\mathcal{D}$, $\mathcal{M}$, and $\mathcal{N}$ are all finitely semisimple the category* $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$ *is finitely semisimple as well.*

Proof. By Proposition 3.4.3 the category $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$ is equivalent to the balanced tensor product $\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{D}^{\mathrm{mop}} \otimes \mathcal{C}}{\otimes}} \mathcal{N}$. The latter category is equivalent to the Drinfeld center of a particular $(\mathcal{D}^{\mathrm{mop}} \otimes \mathcal{C}, \mathcal{D}^{\mathrm{mop}} \otimes \mathcal{C})$-bimodule category by Proposition 3.4.5, namely the ordinary naïve tensor product $\mathcal{M} \otimes \mathcal{N}$. Since $\mathcal{C}$, $\mathcal{D}$, $\mathcal{M}$, and $\mathcal{N}$ are all finitely semisimple the categories $\mathcal{D}^{\mathrm{mop}} \otimes \mathcal{C}$ and $\mathcal{M} \otimes \mathcal{N}$ are as well, so the claim follows by Theorem 1.5.8. □

Theorem 3.4.7. *When $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{M}$, and $\mathcal{N}$ are all finitely semisimple there is a $(\mathcal{D}, \mathcal{E})$-bimodule equivalence between the balanced tensor product ${}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}_{\mathcal{E}}$ and Deligne's tensor product ${}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}_{\mathcal{E}}$.*

Proof. When ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{E}}$ are not merely respectively right and left $\mathcal{C}$-module categories then the functor category $[{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$ is made a $(\mathcal{D}, \mathcal{E})$-bimodule category by "acting pointwise". For $F : {}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}$ a module functor this just means that $(Y \triangleright F)(M) := F(M \triangleleft {}^*Y)$ and $(F \triangleleft Z)(M) := F(M) \triangleleft Z$. By Proposition 3.5 of [11] together with Remark 3.6 following it[1] (see also [8]) in this situation there is an equivalence $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N} \simeq [{}_{\mathcal{C}}\mathcal{M}^* \to {}_{\mathcal{C}}\mathcal{N}]$ of $(\mathcal{D}, \mathcal{E})$-bimodule categories. On the other hand, we observe an obvious right $\mathcal{E}$-module functor structure on the equivalence P of Proposition 3.3.2 as a consequence of elementary properties of direct sums. Fixing $Y \in \mathcal{D}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $K \in \mathcal{M}^*$ we also see a left $\mathcal{D}$-module functor structure as a consequence of Proposition 3.1.5, which provides an isomorphism

$$P({}_{\mathcal{D}}L(Y)(M \otimes N))(K) = \mathcal{M}(K \to Y \triangleright M)N \to \mathcal{M}(K \triangleleft Y^* \to M)N = (Y \triangleright P(M \otimes N))(K).$$

It is easy to see that these module functor structures are compatible and turn P into a bimodule functor (this is exactly asserted by Proposition 3.1.7). The claim follows by composing P with the functor to Deligne's tensor product above and appealing to the module category variant of Proposition 1.2.16. □

Of course by taking $\mathcal{D} = \mathsf{Vec}$ or $\mathcal{E} = \mathsf{Vec}$ (or both) we obtain analogous results for balanced tensor products involving module categories. We conclude by giving further consequences of the main result Theorem 3.4.7 of this section.

Corollary 3.4.7.1. *Let $\mathcal{M}_{\mathcal{C}}$ be a finitely semisimple module category over a finitely semisimple monoidal category $\mathcal{C}$. Then there is an equivalence of right $\mathcal{C}$-module categories $\mathcal{M}_{\mathcal{C}} \simeq \mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{C}_{\mathcal{C}}$. When $\mathcal{M}$ is a $(\mathcal{D}, \mathcal{C})$-bimodule category this is a $(\mathcal{D}, \mathcal{C})$-bimodule equivalence.*

Proof. The hypotheses ensure that Theorem 3.4.7 applies, and the same fact holds for the Deligne product [11]. □

[1] Remark 3.5 of [9] points out that as stated [11] contains an error—it is off by a twist of the double dual functor. Here we invoke the corrected result involving functors from the dual category ${}_{\mathcal{C}}\mathcal{M}^*$ (c.f. Proposition 3.1.4).

Corollary 3.4.7.2. *Let* ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ *be a bimodule category. Then there is an equivalence* $\mathrm{BiBal}_{(\mathcal{C},\mathcal{C})}(\mathcal{M},\mathcal{C}) \simeq \mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$.

Proof. By Corollary 3.4.7.1 we have that ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}} \simeq {}_{\mathcal{C}}\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{C}_{\mathcal{C}}$ as $(\mathcal{C},\mathcal{C})$-bimodule categories, and hence $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}) \simeq \mathcal{Z}(\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{C}))$. By Proposition 3.4.4 there is a second equivalence $\mathcal{Z}(\mathrm{Bal}_{\mathcal{C}}(\mathcal{M},\mathcal{C})) \simeq \mathrm{BiBal}_{(\mathcal{C},\mathcal{C})}(\mathcal{M},\mathcal{C})$, so we are done by composing the equivalences. □

Corollary 3.4.7.3. *Let* ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ *and* ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ *be bimodule categories. Then there is an equivalence* $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N}) \simeq \mathrm{BiBal}_{(\mathcal{D},\mathcal{D})}({}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}_{\mathcal{D}}, \mathcal{D})$.

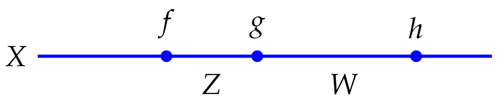
Proof. Combine Proposition 3.4.4 with Corollary 3.4.7.2. □

Remark 3.4.8. The equivalence of Corollary 3.4.7.1 can also be constructed directly. It turns out that when $\mathcal{C}$ is finitely semisimple there is a canonical bibalanced object $A \in \mathrm{BiBal}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C})$, which under the equivalence of Proposition 3.3.3 is the functor which takes left duals. The object A arises from the so-called *internal hom* in module categories, when $\mathcal{C}$ is considered via a flip as a left $(\mathcal{C} \otimes \mathcal{C}^{\mathrm{mop}})$-module category. This can be thought of as an extension of Section 9.2 of [12].

3.5 Marked diagram categories for bimodule categories

In Chapter 2 the annular category for a $\Bbbk$-linear pivotal category was defined using the general theory of 2-dimensional diagrams in pivotal categories. On the other hand, if $\mathcal{M}$ is just an ordinary $\Bbbk$-linear category there is nonetheless a good (though perhaps rather boring) theory of 1-dimensional diagrams in $\mathcal{M}$. This is essentially just because ordinary categories have an associative composition law.

To build the category $\int_\bullet \mathcal{M}$ of diagrams in $\mathcal{M}$ on the interval, we first let the objects be the same as those of $\mathcal{M}$. The vector space $(\int_\bullet \mathcal{M})(X \to Y)$ is then a quotient of the free vector space on diagrams of the form

e.g. 

with $f : X \to Z$, $g : Z \to W$, and $h : W \to Y$.

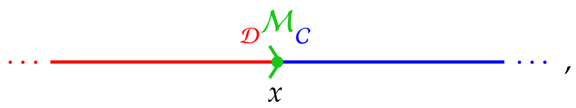
That is, with finitely many morphisms in $\mathcal{M}$ distinguishing points in the interior of the interval, and with the oriented open intervals separating each point being labelled with an object which is compatible with the morphisms which it connects. Diagrams $X \to Y$ can be "evaluated" by composing the morphisms associated to the distinguished points left-to-right, yielding an ordinary morphism in $\mathcal{M}$ between the same objects X and Y. In the quotient we identify diagrams which are isotopic in the obvious sense, where diagrams obey a linear relation, and where one diagram is obtained from another by replacing a sub-interval with another sub-interval which evaluates to the same ordinary morphism in $\mathcal{M}$ (this is the 1-dimensional variant of local replacement—which we are very familiar with by now). It is an easy exercise to verify that $\mathcal{M}$ and $\int_\bullet \mathcal{M}$ are equivalent (even isomorphic) categories.

We will leverage a slight variant of this construction to build an annular category given a collection of compatible bimodule categories. Precisely, the input data is the following.

Definition 3.5.1. An oriented 1-manifold Ω is *marked* if it is equipped with finitely many distinguished points $\Omega_0 \subset \Omega$, such that:

- Each distinguished point in the interior of Ω is labelled with bimodule category ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ for some pivotal categories $\mathcal{C}$ and $\mathcal{D}$ (with all of this data varying for each distinguished point).

- Each such labelling of a point $x \in \Omega_0 \setminus \partial\Omega$ is made with respect to the orientation, diagrammatically of the form

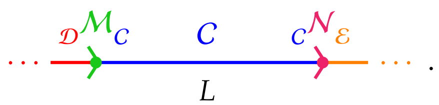


  i.e. so that both of the 1-manifolds in the complement $\Omega \setminus \Omega_0$ which are incident on x correspond to an opposite left- or right- side of the bimodule category ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$.

- Each point $x \in \Omega_0 \cap \partial\Omega$ is labelled by a module category, diagrammatically of the form

  in analogy with the bimodule case.

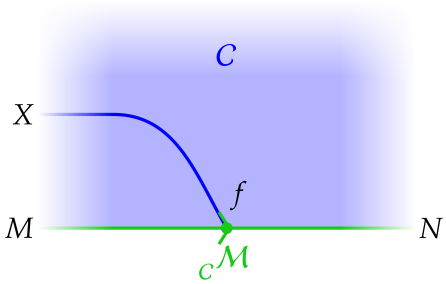
- Each path component $L \subset \Omega \setminus \Omega_0$ is assigned a pivotal category which is compatible with the labels of the distinguished points that meet L. Diagrammatically, we require



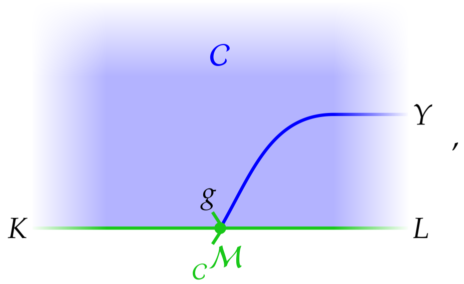
We use $\widehat{\Omega}$ to denote the 1-manifold Ω equipped with this data.

Upon taking the product of a marked 1-manifold $\widehat{\Omega}$ with I we obtain a 2-manifold $\Sigma_{\widehat{\Omega}}$ with 1-manifolds (*bulk boundaries*, always closed intervals) labelling bimodule categories, separating 2-manifolds (*bulks*, always disks) labelled by pivotal categories. There is an immediate theory of diagrams in such objects, which requires almost no additional explanation beyond what we have already seen. Schematic diagrams arising from marked intervals and circles are respectively depicted in Figure 3.5 and Figure 3.6.

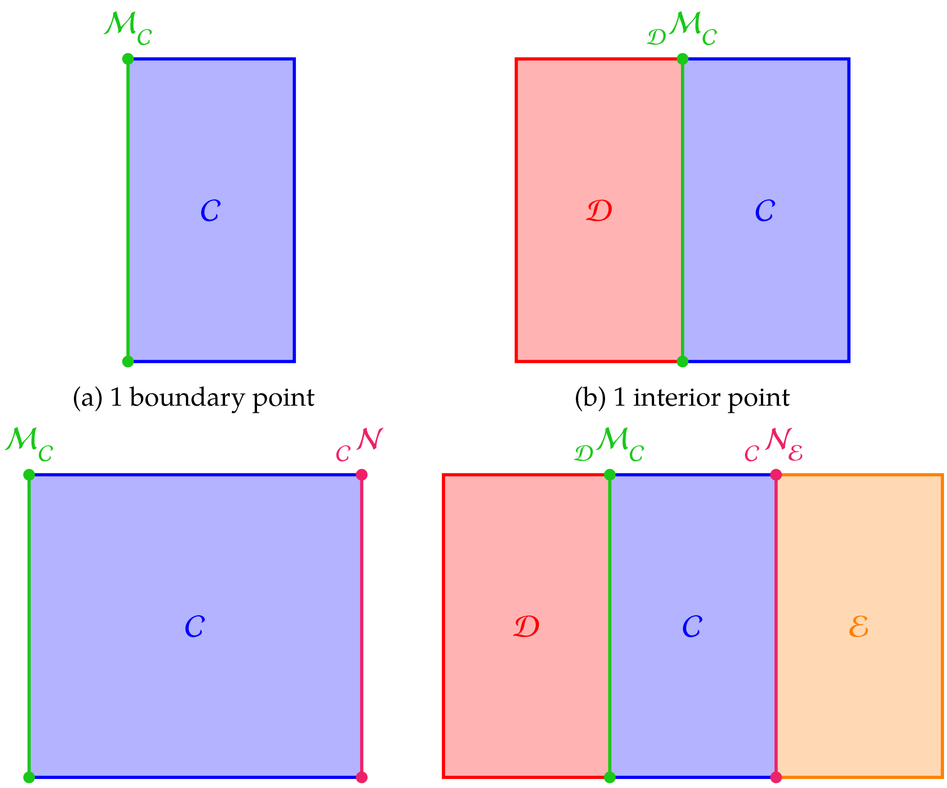
In particular, a diagram drawn in $\Sigma_{\widehat{\Omega}}$ is just a diagram in each 2-manifold bulk corresponding to a particular pivotal category, along with a diagram drawn on each 1-manifold bulk boundary in the respective (bi)module category to which it corresponds. The only caveat is that we permit diagrams in the pivotal categories (the bulks) to meet the diagrams drawn in the boundary 1-manifolds at labelled points, with this changing the meaning of the associated morphism label. For instance, in the diagrams[2]



and



[2]From now on we will use a single colour for the (bi)module category labels of bulk boundaries.



(a) 1 boundary point (b) 1 interior point

(c) 2 boundary points (d) 2 interior points

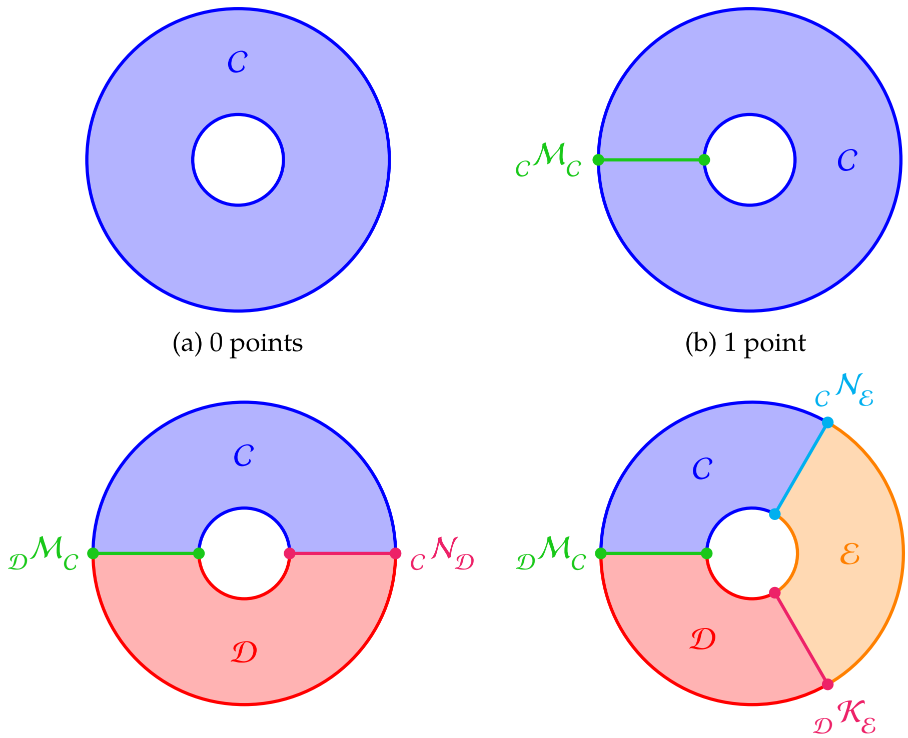
Figure 3.5: Diagram categories arising from a marked I with n distinguished points, $n \leq 2$.

the morphism label f should actually be a morphism $X \triangleright M \to N$, and g should be a morphism $K \to Y \triangleright L$. Of course we would like to continue to consider isotopies of such diagrams, but a problem arises when a string in (for example) the $\mathcal{C}$-bulk above is able to vary its angle of incidence with a point on the $\mathcal{M}_\mathcal{C}$-bulk boundary.

Consider for example a string incident on the $\mathcal{M}_\mathcal{C}$-bulk boundary depicted in the former figure changing its angle of incidence (via an isotopy) to the local configuration as depicted in the latter figure. The result is that the label for the distinguished point of intersection with $\mathcal{M}_\mathcal{C}$ is now incorrect with respect to the labelling scheme which we have just described. The remedy (as in the case of ordinary diagrams in a pivotal category) is to label distinguished points of the $\mathcal{M}_\mathcal{C}$-bulk boundary not with actual morphisms in $\mathcal{M}$, but to instead fashion an object which represents the morphism in an isotopy-invariant way and to use that as a label instead. In this case we use the isomorphisms provided by rigidity (in Proposition 1.3.10) and pivotality to freely pass between morphisms $M \to X \triangleright N$ and morphisms $X^* \triangleright M \to N$.

As in the case of diagrams in pivotal categories, when we draw diagrams we are perfectly content to suppress the technical construction of labels we have made and to instead work with ordinary morphism labels for most purposes. Finally, the bimodule associators permit us to allow string diagrams where a string from the $\mathcal{C}$-bulk and a string from the $\mathcal{D}$-bulk both intersect the same point on a given bulk boundary. In fact by Theorem 1.3.9 we can even assume up to equivalence that all bimodule associators are the identity.

Since all of these diagrams are formed by a product with I, there is again a natural composition law arising from juxtaposition and gluing diagrams with the same boundary labels. We also impose an analogue for local replacement in the bulk boundaries. To state it, observe that the right $\mathcal{C}$-module category structure on $\mathcal{M}_\mathcal{C}$ in Figure 3.5a gives a natural evaluation map from honest diagrams drawn in the figure to ordinary morphisms in $\mathcal{M}$ (this is a generalisation of



(a) 0 points (b) 1 point

(c) 2 points (d) 3 points

Figure 3.6: Diagram categories arising from a marked S^1 with n distinguished points, $n \leq 3$.

eval from Section 1.4). Essentially the only point of divergence from the ordinary evaluation of rectangular diagrams in pivotal categories is that we must apply the functor ◂ when evaluating the part of the diagram drawn in the $\mathcal{C}$-bulk. Similarly, if we instead have the $(\mathcal{D}, \mathcal{C})$-bimodule category ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ of Figure 3.5b there is again an evaluation map from diagrams drawn in the figure to ordinary morphisms in $\mathcal{M}$.

The objects of the category $\int_{\widehat{\Omega}}$ of diagrams for a marked 1-manifold $\widehat{\Omega}$ are just boundary label data ω for diagrams which arise from $\widehat{\Omega}$. The morphisms $\int_{\widehat{\Omega}}(\omega \to \chi)$ are then—as usual—the quotient of the free vector space on all diagrams with the same boundary data by isotopy, by local replacement inside bulks (with which are we familiar), and by local replacement on a bulk boundary (which we have just described).

The following theorem is a straightforward generalisation of Theorem 1.4.6, asserting that a pivotal category is equivalent to its category of diagrams.

Theorem 3.5.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be pivotal categories.*

1. *The category of diagrams of the form of Figure 3.5a associated to a module category $\mathcal{M}_{\mathcal{C}}$ is equivalent to $\mathcal{M}_{\mathcal{C}}$ itself.*

2. *The category of diagrams of the form of Figure 3.5b associated to a bimodule category ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ is equivalent to ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ itself.*

If $\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}$ are both module categories for a pivotal category $\mathcal{C}$, the category of diagrams of the form of Figure 3.5c is known as the associated *ladder category*[3] $\mathrm{Lad}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ [2]. If the module structures on $\mathcal{M}$ and $\mathcal{N}$ respectively extend to $(\mathcal{D}, \mathcal{C})$- and $(\mathcal{C}, \mathcal{E})$-bimodule category

[3]The notion of a ladder category for a pair of module categories over any monoidal category $\mathcal{C}$ also makes (purely algebraic) sense. However, when $\mathcal{C}$ is pivotal the typical algebraic construction is equivalent to the diagrammatic one we give here (and this is easy to check). Thus we identify the two notions here without any concern.

structures, then we can form the category of diagrams of the form of Figure 3.5d. This category is itself naturally a $(\mathcal{D}, \mathcal{E})$-bimodule category by juxtaposition of rectangular diagrams in $\mathcal{D}$ on the left and $\mathcal{E}$ on the right, and we denote it by ${}_{\mathcal{D}}\mathrm{Lad}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_{\mathcal{E}}$. One reason ladder categories are of interest is the following folklore result, which we will posses the tools to give a proof of in the next section.

Theorem 3.5.3. *If $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{M}$, and $\mathcal{N}$ are all finitely semisimple the Deligne product ${}_{\mathcal{D}}\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}_{\mathcal{E}}$ is equivalent to ${}_{\mathcal{D}}\mathrm{Lad}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_{\mathcal{E}}$ as a $(\mathcal{D}, \mathcal{E})$-bimodule category.*

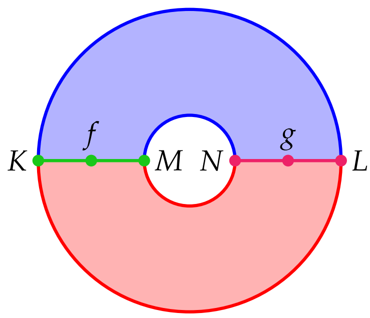
Now turning to marked copies of S^1, Figure 3.6 schematically represents so-called annular categories for various numbers of bimodule categories. The diagram category for Figure 3.6a is the case of zero bimodule category labels, and recovers precisely the ordinary annular category $\int_{S^1} \mathcal{C}$ which we studied in Chapter 2. We will be particularly interested in the diagram categories for Figures 3.6b and 3.6c, which we will denote respectively by $\int_{S^1} {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ and $\int_{S^1} ({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$. The "higher valence" punctures of the kind of Figure 3.6d and (their associated categories of representations) are directly relevant in the study of topological phases of matter (where they are closely related to so-called *point defects*[4]).

It will sometimes be convenient to replace $\mathcal{M} \otimes \mathcal{N}$ with its full subcategory $\mathcal{M} \otimes^{\mathrm{pure}} \mathcal{N}$ consisting only of those objects which are a *pure* product $M \otimes N$ with $M \in \mathcal{M}$ and $N \in \mathcal{N}$ (i.e. and not a formal direct sum of such objects). This is mainly because the category $\int_{S^1} ({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ does not have direct sums in general, which is intuitively for the same reason that $\mathcal{M} \otimes^{\mathrm{pure}} \mathcal{N}$ does not either. Of course there is also a corresponding full subcategory $\mathrm{BiBal}^{\mathrm{pure}}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$ of $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$ of the objects which forget to pure products in $M \otimes N$, but in general restoring direct sums with $\mathrm{Mat}\left(\mathrm{BiBal}^{\mathrm{pure}}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})\right)$ will yield a category smaller than $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$.

As in Chapter 2 we will assume that $\mathrm{End}(\mathbb{1}^{\mathcal{C}}) \cong \Bbbk \cong \mathrm{End}(\mathbb{1}^{\mathcal{D}})$ from now on.

Proposition 3.5.4. *There is an essentially surjective faithful inclusion functor $J : \mathcal{M} \otimes^{\mathrm{pure}} \mathcal{N} \to \int_{S^1} ({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$.*

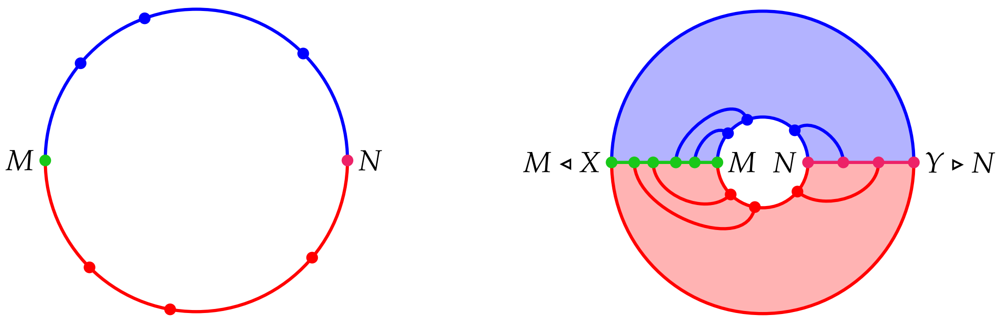
Proof. The construction is parallel to Proposition 2.2.2, and we will see later that we are actually giving a generalisation. First, we can view a morphism $f : M \otimes N \to K \otimes L$ as a morphism in $\int_{S^1} ({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ by taking a representative of f as a sum of morphisms in $\mathcal{M} \times \mathcal{N}$, and then using the equivalences between $\mathcal{M}$ and $\mathcal{N}$ and their respective categories of 1-dimensional diagrams $\int_{\bullet} \mathcal{M}$ and $\int_{\bullet} \mathcal{N}$. Diagrammatically, this is just the inclusion

$(f : M \to K) \otimes (g : N \to L) \quad \longmapsto$ 

with no diagram drawn in either the $\mathcal{C}$ or $\mathcal{D}$ bulks. We will omit the orientation of the edges labelled by objects from now on, as we did in the pivotal case. Note that while the category $\int_{S^1} ({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ does not have direct sums it nonetheless has vector space homsets and sums of morphisms, so we can freely extend this definition to formal sums of morphisms.

[4] See Definition 8 of [4] and the references therein.

In $\int_{S^1} \mathcal{C}$ the boundaries of arbitrary annular diagrams were isomorphic to boundaries labelled by only a single object, we can similarly regularise objects of $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$. Given an arbitrary boundary for example of the form depicted in Figure 3.7a (with some $\mathcal{C}$- and $\mathcal{D}$-object labels suppressed on the blue and red points) we always have an isomorphism to a boundary which only has labels on the bimodule category parts, i.e.



(a) (b)

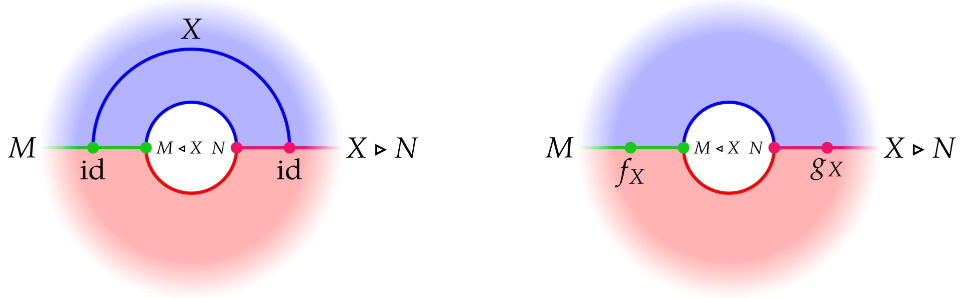
Figure 3.7: An arbitrary object in $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ and its isomorphism to the image of J.

Here each distinguished point on the bulk boundaries is labelled with an identity morphism, and X and Y are tensor products of objects respectively labelling the blue $\mathcal{C}$-line and red $\mathcal{D}$-line of the boundary of Figure 3.7a. It is easy to see that each such morphism of this kind is an isomorphism, having an inverse obtained by just "turning the annulus inside out". This establishes essential surjectivity of J. Finally, J is faithful by the direct extension of Proposition 2.3.3 to the marked case. □

Using the inclusion functor J, we can now build representations of $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ from bibalanced objects.

Proposition 3.5.5. *There is a functor* $B : \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N}) \longrightarrow \mathsf{Rep}^{\mathrm{op}} \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$.

Proof. We produce what amounts to a generalisation of the faithful functor from Proposition 2.3.1. So that we are not overwhelmed by indices, we first handle the case of pure objects; fix $(S, \tau_X, \sigma_Y) \in \mathrm{BiBal}^{\mathrm{pure}}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ and write $S = M \otimes N$ (by hypothesis S is of this form). For each $T \in \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ there is a vector space of diagrams $J(S) \longrightarrow T$, and we declare the vector space $V(T) = B(S, \tau_X, \sigma_Y)(T)$ to be a particular quotient. To specify the relation, suppose that in a neighbourhood of the inner puncture of some diagram $D : J(S) \longrightarrow T$ we locally encounter the configuration of Figure 3.8a. Then in this situation we permit a local replacement to the diagram depicted in Figure 3.8b for some morphisms f_X and g_X.

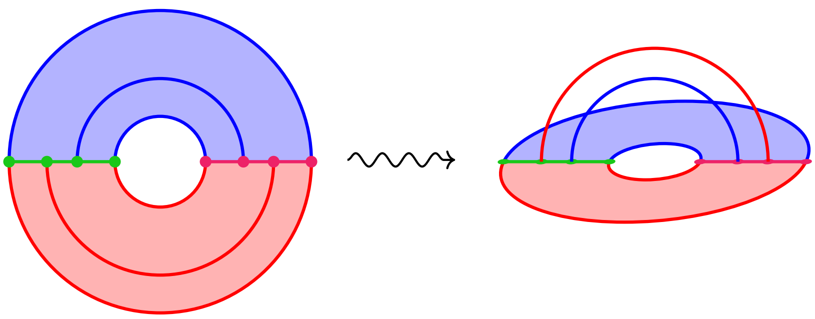


(a) (b)

Figure 3.8: The local replacement relation in $B(S, \tau_X, \sigma_Y)$ for the balancing τ_X.

We in turn produce the morphisms f_X and g_X by observing that the balancing τ_X is in this situation a morphism $(M \triangleleft X) \otimes N \to M \otimes (X \triangleright N)$, hence precisely a product $f_X \otimes g_X$. The balancing σ_Y for the $\mathcal{D}$-marked bulk similarly gives a relation for resolving identity strings in a neighbourhood of the annular puncture to morphisms in the image of the functor J. These are the two additional relations—one for each balancing—which we impose in order to define the representation $B(S, \tau_X, \sigma_Y)$ on each $T \in \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$.

Geometrically, one should imagine that the balancings τ_X and σ_Y together give a way to lift the arc of Figure 3.8a and the dual picture for the $\mathcal{D}$-marked region off the plane of the page, instead passing directly up and over the diagram, as depicted below.



(3.5.1)

From the perspective of the plane, only the two vertices where each 3-dimensional arc meets the plane (e.g. labelled by f_X and g_X for the blue arc) can actually be seen.

This defines a representation V of $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ on the objects. The definition of the representation on morphisms now follows Proposition 2.3.1 closely. That is, morphisms $h : T \to U$ in $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ induce maps by gluing around the outer boundary of diagrams $D : J(S) \to T$. Each such linear map descends to a map of quotients $V(h) : V(T) \to V(U)$ exactly because the balancings satisfy the associativity constraint (3.2.1) (in direct analogy with the half-braiding axiom invoked in the $\mathcal{Z}(\mathcal{C})$ case), in addition to the bibalancing compatibility condition we also require. In particular, since the two arcs of (3.5.1) can be slid past one another on the surface of the annulus, we must ensure that they may also be slid past one another after being lifted off the surface.

Morphisms $h : (S, \tau_X, \sigma_Y) \to (S', \kappa_X, \rho_Y)$ of bibalanced objects similarly give rise to representation intertwiners $B(S', \kappa_X, \rho_Y) \to B(S, \tau_X, \sigma_Y)$ by gluing on the inside of annuli (as in Proposition 2.3.1). The intertwiners are natural in each component for the same reason as the $\mathcal{Z}(\mathcal{C})$ case, and thus we obtain a definition of the functor B on $\mathrm{BiBal}^{\mathrm{pure}}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$.

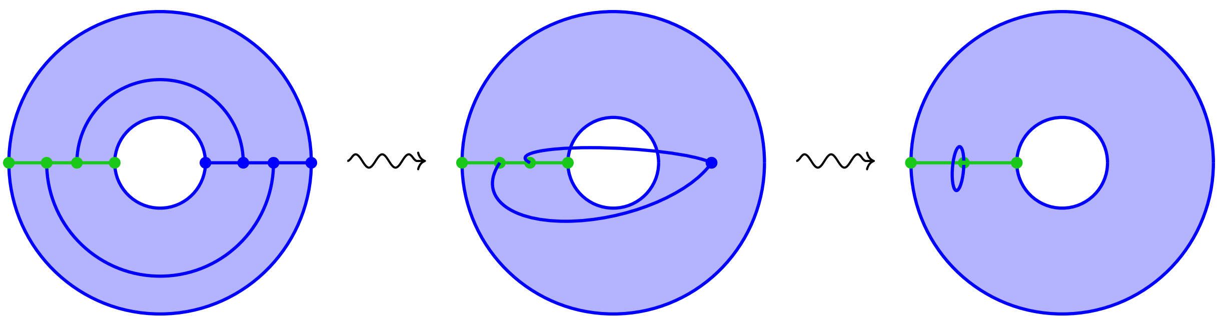
In the case of arbitrary $(S, \tau_X, \sigma_Y) \in \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$, there is a straightforward generalisation of this construction. For each $T \in \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ we let $B(S, \tau_X, \sigma_Y)(T)$ report a quotient of the direct sum of the vector spaces of morphisms $J(S_i) \to T$ as S_i ranges over the formal summands of S. The quotient relation of Figure 3.8 is now defined by asserting equalities of formal direct sums of diagrams arising from the balancings τ and σ (since the components of τ and σ are now morphisms of formal direct sums), but otherwise is the exactly the same.

Now let $h : \bigoplus_i S_i \to \bigoplus_j S'_j$ be a morphism between formal direct sums coming from a morphism $(S, \tau_X, \sigma_Y) \to (S', \kappa_X, \rho_Y)$ in $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$. Thus h splits into components $h_{i,j} : S_i \to S'_j$. The action of $B(h)$ is to map the summand of $B(S', \kappa_X, \rho_Y)$ corresponding to S'_j into the summand of $B(S, \tau_X, \sigma_Y)$ corresponding to S_i by gluing $f_{i,j}$ around the inner boundary of the annular diagrams. This construction is well defined in the quotient for the same reason as in the pure case above, so this completes the proof. □

In the next section it will be important to consider morphisms in $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ solely of the form depicted in Figure 3.8a, and dually with a single string in the red $\mathcal{D}$-bulk. Thus we let $r^{\mathcal{C}}_{X,M,N}$ denote the morphism depicted in Figure 3.8a for each $X \in \mathcal{C}$, $M \in \mathcal{M}$, and $N \in \mathcal{N}$, and similarly let $r^{\mathcal{D}}_{Y,M,N}$ be the dual morphism with $Y \in \mathcal{D}$.

Remark 3.5.6. There are a number of parallels between the construction of the functor B of Proposition 3.5.5 and the construction of the functor G of Proposition 2.3.1. This is due to an underlying geometrical relationship which we now briefly mention.

Let ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ be a $(\mathcal{C}, \mathcal{C})$-bimodule category. Then equipping $\mathcal{C}$ itself with its natural $(\mathcal{C}, \mathcal{C})$-bimodule category structure, we can form the diagram category $\int_{S^1}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}})$. But since in this diagram category $\mathcal{C}$ is straddled on both sides by $\mathcal{C}$-bulks and each $\mathcal{C}$ action on $\mathcal{C}$ is the natural one, there is a canonical map which just forgets about the distinguished ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-bulk boundary and regards it as just another string in the $\mathcal{C}$-bulk. This gives an equivalence between the categories $\int_{S^1}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}})$ and $\int_{S^1} {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$. Viewed under this correspondence, the construction of Proposition 3.5.5 can be interpreted as performing the following sequence of geometric moves, which we describe below.



First, each balancing in $\mathrm{BiBal}_{(\mathcal{C},\mathcal{C})}(\mathcal{M}, \mathcal{C})$ can be used to pull a blue arc into the space above the plane of the page. The two distinguished points on the interior of the ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-boundary can then be brought together and the ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-boundary can be forgotten about, resulting in the situation depicted in the middle figure. Next, the lack of a ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-boundary permits the single labelled blue point on the interior of the blue $\mathcal{C}$-bulk to isotoped around the bulk and coalesced with one of the green distinguished points on the ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$-line. Finally the two green points may be brought together, resulting in a single little loop emerging from one side of the lone green point, passing above the plane of the diagram and over the green ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$-line, and re-entering the green point on the other side.

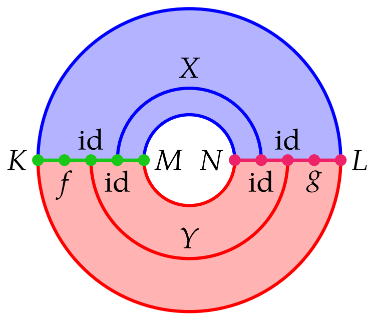
Further specialising to the case $\mathcal{M} = \mathcal{C}$, naturality of the balancing natural isomorphisms we have just described mean that we can interpret this move as exactly pulling a blue string through the annular puncture. That is, precisely the geometric move we permitted when defining the functor $G : \mathcal{Z}(\mathcal{C}) \to \int_{S^1} \mathcal{C}$ of Chapter 2. Thus we see that Proposition 3.5.5 is actually a generalisation of Proposition 2.3.1 in a strict sense.

The observation that the diagram categories $\int_{S^1}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}})$ and $\int_{S^1} {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ are equivalent implies that their categories of representations are also equivalent, so in fact we will be able obtain a correspondence between the diagram category $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ and the representations of $\int_{S^1} {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ as a corollary of the main result of the next section.

3.6 Diagrammatic consequences of finite semisimplicity

Let ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ be finitely semisimple bimodule categories for pivotal categories $\mathcal{C}$ and $\mathcal{D}$. In order to prove an equivalence between $\mathsf{Rep}^{\mathrm{op}} \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ and $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$ we will again require a key but geometrically intuitive factorisation lemma (c.f. Lemma 2.2.3).

Lemma 3.6.1 (Generalised annular factorisation)**.** *Every morphism $d : J(M \otimes N) \to J(K \otimes L)$ in $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ is represented by a diagram of the form*



(3.6.1)

with $X \in \mathcal{C}$, $Y \in \mathcal{D}$, and morphisms

$$f : {}^*Y \triangleright (M \triangleleft X^*) \to K \quad \text{and} \quad g : (X \triangleright N) \triangleleft Y \to L.$$

That is, the morphism d is a composite

$$d = J(f \otimes g) \circ r^{\mathcal{D}}_{Y, M \triangleleft X^*, X \triangleright N} \circ r^{\mathcal{C}}_{X,M,N}$$

Proof. The claim follows by performing successive geometric manipulations. To spell them out, fix a representative of a morphism $d : J(M \otimes N) \to J(K \otimes L)$. Then points where strings in the $\mathcal{C}$- or $\mathcal{D}$-bulk intersect the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary may be slid past one another because of the bimodule middle associativity constraint. The associator for the tensor products in $\mathcal{C}$ and $\mathcal{D}$ along with the module associators for $\mathcal{M}$ on each side then permit us to "pinch-off" bundles of $\mathcal{C}$- or $\mathcal{D}$-bulk strings which are incident on the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary and coalesce them into a single string. Thus we can arrange a representing diagram for d which agrees with (3.6.1) in a neighbourhood of the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary. The only potential difficulty is that a point where a string from the $\mathcal{C}$- or $\mathcal{D}$-bulk intersects the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary could be labelled with a non-identity morphism. However, we can always resolve this problem by writing such a morphism as an identity composed on one side with the morphism itself (thus pushing the non-identity morphism further down the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary, away from the point of intersection).

Of course we can then also do the same for the ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-boundary. The complement of these two neighbourhoods of the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$- and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-boundaries just consists of a string diagram in the $\mathcal{C}$-bulk and a string diagram in the $\mathcal{D}$-bulk, and hence both can be locally replaced with a single labelled point with some number of incoming strings. But we arranged that the $\mathcal{C}$-bulk and $\mathcal{D}$-bulk were each incident on the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$- and ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-boundaries at precisely one point, so by local replacement we can coalesce the $\mathcal{C}$- and $\mathcal{D}$-bulk diagrams each into a single point with only two incident strings each. The result is a diagram precisely of the form (3.6.1) with the exception of having a single point on the interiors of both the blue and red strings which is labelled with a potentially arbitrary morphism. However, we can again slide these points down their respective strings and into one of the bulk boundaries (making yet another arbitrary choice). There we perform a local replacement using pre-composition with the identity to coalesce each into a bulk boundary, while keeping the label of the point of intersection with each $\mathcal{C}$- and $\mathcal{D}$-bulk the identity. We arrive at a representative of d with the claimed form (3.6.1). □

While slightly cumbersome to state, Lemma 3.6.1 is especially powerful because the functor J is essentially surjective. In fact it follows immediately that every representation $V : \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}}) \to \mathsf{Vec}$ is completely determined by its values on $r^{\mathcal{C}}$, $r^{\mathcal{D}}$, and the image of the functor J.

We now reduce this data further by exploiting the hypothesis of finite semisimplicity of $\mathcal{M}$ and $\mathcal{N}$. In this situation the naïve product $\mathcal{M} \otimes \mathcal{N}$ is itself finitely semisimple as well (c.f. Proposition 1.1.13). We would then like the inclusion functor J to restrict V to a representation of $\mathcal{M} \otimes \mathcal{N}$, but this is not quite possible since J does not admit an extension from $\mathcal{M} \otimes^{\mathrm{pure}} \mathcal{N}$ to $\mathcal{M} \otimes \mathcal{N}$. The following lemma indirectly resolves this dilemma.

Lemma 3.6.2. *Let $\mathcal{A}$ be any ($\Bbbk$-linear) category. Then the categories* $\mathsf{Rep}\,\mathcal{A}$ *and* $\mathsf{Rep}\,\mathrm{Mat}\,\mathcal{A}$ *are equivalent.*

Proof. First note that every representation $V : \mathcal{A} \to \mathsf{Vec}$ extends to a representation $\overline{V}$ of $\mathrm{Mat}\,\mathcal{A}$ just because Vec has direct sums; one sets $\overline{V}(\bigoplus_i A_i) := \bigoplus_i V(A_i)$. Letting $D : \mathcal{A} \to \mathrm{Mat}(\mathcal{A})$ be the obvious inclusion, there is a natural isomorphism $V \cong \overline{V} \circ D$.

Now, it can be seen directly that the conditions defining a direct sum in any $\Bbbk$-linear category are preserved by every linear functor. Hence if $W : \mathrm{Mat}\,\mathcal{A} \to \mathsf{Vec}$ is a representation there is an isomorphism $W(\bigoplus_i A_i) \cong \bigoplus_i W(A_i)$ natural in all of the summands $A_i \in \mathcal{A}$. It follows that W is recovered up to natural isomorphism from its restriction $W \circ D$ upon taking the extension of $W \circ D$ to $\mathrm{Mat}\,\mathcal{A}$. It is easy to see that these families of natural isomorphisms assemble into an equivalence of categories. □

Of course J does admit an extension to a functor $\mathcal{M} \otimes \mathcal{N} \to \mathrm{Mat} \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$ into the direct sum completion (which we call by the same name J). Thus, suppressing the equivalence of Lemma 3.6.2 we obtain a representation $V \circ J$ of the semisimple category $\mathcal{M} \otimes \mathcal{N}$. Proposition 2.1.5 then produces an object $S \in \mathcal{M} \otimes \mathcal{N}$ which represents $V \circ J$. Up to natural isomorphism $V(J(T))$ is the vector space $(\mathcal{M} \otimes \mathcal{N})(J(S) \to J(T))$ for all $T \in \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}})$, so essential surjectivity of J means that V is completely determined on the objects by S. Also, the Yoneda lemma asserts that on morphisms in the image of J the representation V is completely fixed as well—V acts just by post-composition.

In order to speak in convenient terms about representations of the direct sum completion, for each $S = \bigoplus_i M_i \otimes N_i \in \mathcal{M} \otimes \mathcal{N}$ we define

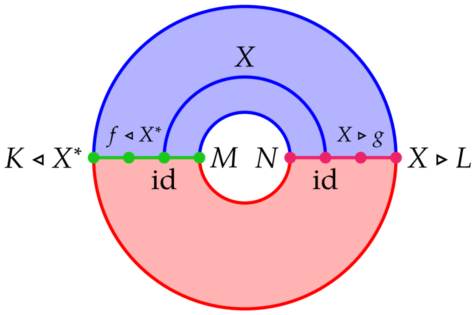
$$r^{\mathcal{C}}_{X,S} = \bigoplus_i r^{\mathcal{C}}_{X,M_i,N_i} \quad \text{and} \quad r^{\mathcal{D}}_{Y,S} = \bigoplus_i r^{\mathcal{D}}_{Y,M_i,N_i}.$$

Here the direct sum of morphisms is taken to specify a diagonal matrix in the direct sum completion.

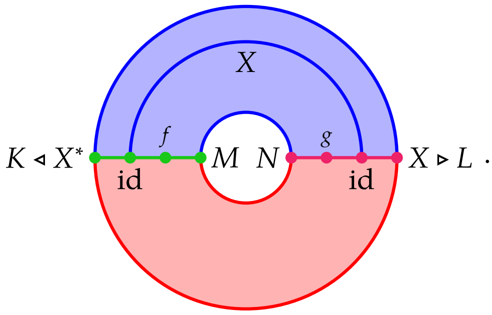
Lemma 3.6.3. *Let $V : \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}'\mathcal{C}}\mathcal{N}_{\mathcal{D}}) \to$* Vec *be any representation which when restricted to a representation of $\mathcal{M} \otimes \mathcal{N}$ is represented by $S \in \mathcal{M} \otimes \mathcal{N}$. Then given any morphism $h : S \to T$ and* *objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ we have*

$$V(r^{\mathcal{C}}_{X,T})(f) = ({}_{\mathcal{C}}R(X^*) \circ L_{\mathcal{C}}(X))(h) \circ V(r^{\mathcal{C}}_{X,S})(\mathrm{id}_S), \quad \textit{and}$$
$$V(r^{\mathcal{D}}_{Y,T})(f) = (R_{\mathcal{D}}({}^*Y) \circ {}_{\mathcal{D}}L(Y))(h) \circ V(r^{\mathcal{D}}_{Y,S})(\mathrm{id}_S).$$

Proof. For each pair of morphisms $f : M \to K$ and $g : N \to L$ there is an equality up to isotopy of the diagrams



and



.

Hence the claim follows by functoriality of V as in the proof of Lemma 2.2.4. □

Thus a representation $V : \mathrm{Mat}\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}'{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}) \to \mathsf{Vec}$ which is represented by $S \in \mathcal{M} \otimes \mathcal{N}$ when restricted to $\mathcal{M} \otimes \mathcal{N}$ is completely determined in general by the morphisms $V(r^{\mathcal{C}}_{X,S})(\mathrm{id}_S) : S \to ({}_{\mathcal{C}}R(X^*) \circ L_{\mathcal{C}}(X))(S)$ and $V(r^{\mathcal{D}}_{Y,S})(\mathrm{id}_S) : S \to (R_{\mathcal{D}}({}^*Y) \circ {}_{\mathcal{D}}L(Y))(S)$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. By Frobenius reciprocity for module categories (Proposition 1.3.10) these are families of morphisms

$$\begin{aligned} \tau_X &: L_{\mathcal{C}}(X)(S) \to {}_{\mathcal{C}}R(X)(S), \\ \sigma_Y &: {}_{\mathcal{D}}L(Y)(S) \to R_{\mathcal{D}}(Y)(S), \end{aligned} \tag{3.6.2}$$

which by analogy with Chapter 2 one should expect assemble into compatible balancings of the object S. Indeed, we have the following.

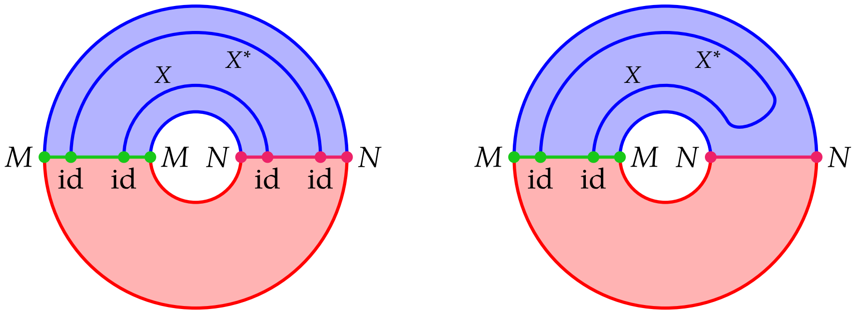
Proposition 3.6.4. *There is a faithful functor* $C : \mathsf{Rep}^{\mathrm{op}}\,\mathrm{Mat}\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}'{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}) \to \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M}, \mathcal{N})$.

Proof. Given a representation $V : \mathrm{Mat}\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}\,{}'{}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}) \to \mathsf{Vec}$, let $S \in \mathcal{M} \otimes \mathcal{N}$ represent V when restricted using J. Then procure associated families of morphisms $\tau_X : L_{\mathcal{C}}(X)(S) \to {}_{\mathcal{C}}R(X)(S)$ and $\sigma_Y : V(r^{\mathcal{D}}_{Y,S})(\mathrm{id}_S) : {}_{\mathcal{D}}L(Y)(S)$ respectively from $V(r^{\mathcal{C}}_{X,S})(\mathrm{id}_S)$ and $V(r^{\mathcal{D}}_{Y,S})(\mathrm{id}_S)$ using Frobenius reciprocity.

The verification that these families actually define compatible balancings now largely follows the strategy of Section 2.3, so we will just explain where we depart from the development given there. In particular, we are able to mirror the corresponding diagrammatic arguments via our generalisation Lemma 3.6.3 of Lemma 2.2.4.

First, the claim that each component τ_X is an isomorphism is verified by considering the diagram depicted in Figure 3.9a. Since $\mathcal{C}$-bulk strings intersect the ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-boundary at points labelled by the identity, we may bring the two points of intersection together and then separate the $\mathcal{C}$-bulk string from the ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-boundary altogether by a local replacement. The result is the diagram depicted in Figure 3.9b.

The free loop in the $\mathcal{C}$ bulk can now be retracted into the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary, and since its points of intersection are again labelled by identity morphisms the loop annihilates on contact (or really, is excised by a final local replacement). The result is the identity morphism (in the image of J), and witnesses that τ_X has a one-sided inverse. The natural corresponding diagram

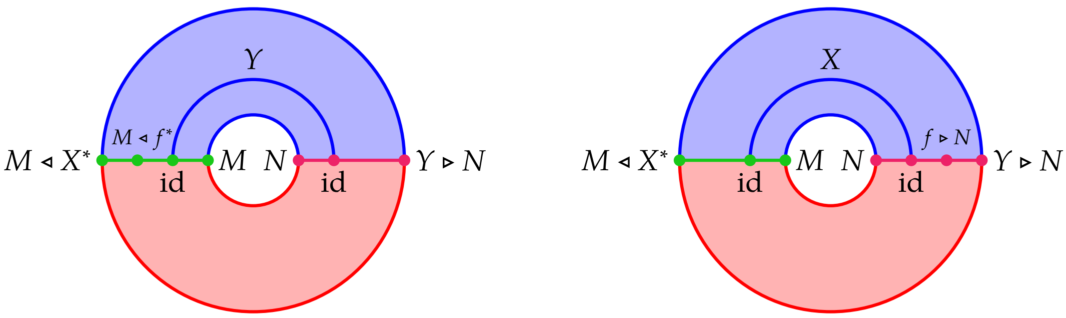


(a) (b)

Figure 3.9: The diagram manipulation asserting that each component τ_X is an isomorphism.

with the $\mathcal{C}$-bulk arcs interchanged and same argument also shows that τ_X has an inverse on the other side as well, as desired.

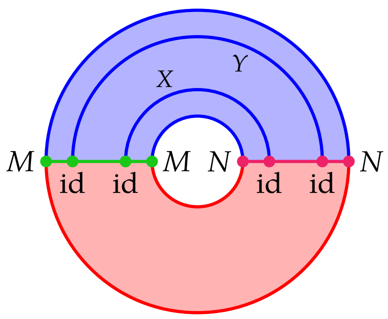
The fact that the components τ_X assemble into a natural transformation follows for each $f : X \to Y$ by comparing the diagrams of Figure 3.10. Just using local relations the $M \triangleleft f^*$ morphism on the ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$-boundary of Figure 3.10a may be "sucked-up" into the arc in the $\mathcal{C}$-bulk via a local replacement, whence it can be transferred to a neighbourhood of the ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-boundary via a second local replacement (as in Figure 3.10b). The result might not look like a naturality relation at first glance because f is "on the same side" of the $\mathcal{C}$-bulk arc in both cases, but this is precisely due to the application of Frobenius reciprocity used to define each component τ_X.



(a) (b)

Figure 3.10: The diagram manipulation asserting that the components τ_X are natural.

The natural transformation τ_X obeys the balancing associativity constraint (3.2.1) just because the morphism represented by the two diagrams



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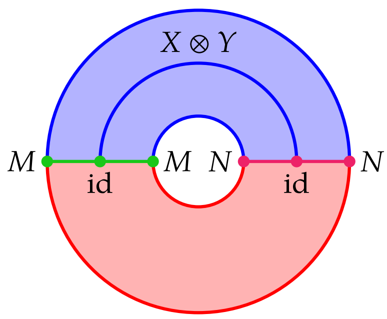


is the same.

Of course, a completely analogous argument shows that the components σ_Y assemble into a $\mathcal{D}^{\mathrm{mop}}$-balancing of the object S as well. The bibalancing compatibility constraint (3.2.2) for the balancings τ_X and σ_Y follows from sliding the identity-labelled arcs in the $\mathcal{C}$- and $\mathcal{D}$-bulks in

(3.5.1) past one another (a move which we are accustomed to making by now).

All of this defines a functor $C : \mathsf{Rep}^{\mathrm{op}}\,\mathrm{Mat}\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{C'C}\mathcal{N}_{\mathcal{D}}) \to \mathrm{BiBal}_{(\mathcal{D},C)}(\mathcal{M},\mathcal{N})$ on the objects. To define C on the morphisms, let $\eta : V \to W$ be a representation intertwiner with V and W represented by objects S and T of $\mathcal{M} \otimes \mathcal{N}$ respectively. By the Yoneda lemma η is just pre-composition with a fixed morphism $f : T \to S$ on every homset; we set $C(\eta) = f$, and faithfulness follows from Corollary 2.1.6.1. The claim that f is a morphism of bibalanced objects is verified by drawing the naturality square for η analogous to (2.2.6); the only material change is substitution of the natural isomorphism r with our generalisations r^C and $r^{\mathcal{D}}$. □

Theorem 3.6.5. *Let ${}_{\mathcal{D}}\mathcal{M}_C$ and ${}_C\mathcal{N}_{\mathcal{D}}$ be finitely semisimple bimodule categories over pivotal categories C and $\mathcal{D}$ with $\mathrm{End}(\mathbb{1}^C) \cong \Bbbk \cong \mathrm{End}(\mathbb{1}^{\mathcal{D}})$. Then the functors B and C together give an equivalence of the categories $\mathrm{BiBal}_{(\mathcal{D},C)}(\mathcal{M},\mathcal{N})$ and $\mathsf{Rep}^{\mathrm{op}}\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{C'C}\mathcal{N}_{\mathcal{D}})$.*

Proof. Most of the work is completed by performing the analogous technical reduction described at the beginning of the proof of Theorem 2.3.5. In particular, we get to assume that representations of $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{C'C}\mathcal{N}_{\mathcal{D}})$ all restrict to actual hom-functors. Together with our generalisation Lemma 3.6.3 of Lemma 2.2.4—which gives a formula to calculate with representations of this kind—we can build the corresponding chain of equalities (2.3.2). It then just remains to invoke another knot-theoretic result, which we now describe.

The situation is that we have a representation $V \in \mathsf{Rep}\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{C'C}\mathcal{N}_{\mathcal{D}})$ and an object $(S, \tau_X, \sigma_Y) := C(V)$ (in particular V is represented by S). We need to show that for all $T \in \int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{C'C}\mathcal{N}_{\mathcal{D}})$ the inclusion J induces an injective map from the vector space of morphisms $S \to T$ into the quotient defining $B(S, \tau_X, \sigma_Y)(T)$. We do this by explicitly constructing an inverse. Given a diagram in $\int_{S^1}({}_{\mathcal{D}}\mathcal{M}_{C'C}\mathcal{N}_{\mathcal{D}})$, fix a radial line segment L emanating from the inner puncture and passing through the interior of the C-bulk, before intersecting the outer boundary of the annulus. Assume that the strings drawn on the diagram intersect L at all points transversely, and note that the C-bulk is separated into two halves by L. The balancing τ_X then gives a way to cut all of the strings which intersect L and retract the resulting pair of diagrams drawn in each half of the C-bulk into the bulk boundaries (we visualised this before as lifting the strings off the surface of the plane).

Performing the analogous process for the $\mathcal{D}$-bulk using the balancing σ_Y and another line L', we obtain a morphism f in the image of the faithful functor J and hence a candidate for the definition of an inverse. Thus the only question is whether this map is well-defined, being independent of the choice of L and L' and invariant under isotopy and local replacement of the representing diagram. By Theorem 3.5.2 invariance under local replacement in the interior of the bulks is verified, as is invariance under isotopy away from a neighbourhood of the cut lines L and L'. It remains to see that we also have invariance of f whenever there is an isotopy of the marked annulus in which strings are permitted to move underneath the L and L' lines.

This final fact can be proved by appeal to a Reidemeister-type result similar to Theorem 2.3.6, which we do not give again (in this case there are two distinguished lines, but very little else has changed). We simply assert that the kinds of singularities which can arise in the C- and $\mathcal{D}$-bulks under a global isotopy of the marked annulus are resolved precisely because the balancings τ and σ are isomorphisms, are natural, each obey the associativity constraint, and (now in addition) obey the bibalancing compatibility constraint. The fashion in which singularities are resolved is the same as in Theorem 2.3.6. The bibalancing constraint can be interpreted here as ensuring that pairs of singularities which occur simultaneously on the cut lines L and L' can be resolved one-by-one in either order. This completes the proof. □

Corollary 3.6.5.1. *Let ${}_C\mathcal{M}_C$ be a finitely semisimple bimodule category over a pivotal category C with $\mathrm{End}(\mathbb{1}^C) \cong \Bbbk$. Then the categories $\mathcal{Z}({}_C\mathcal{M}_C)$ and $\mathsf{Rep}^{\mathrm{op}}\int_{S^1}{}_C\mathcal{M}_C$ are equivalent.*

Proof. We described an equivalence

$$\int_{S^1}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}'\,\mathcal{C}}\mathcal{C}_{\mathcal{C}}) \simeq \int_{S^1}{}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$$

in Remark 3.5.6. Taking $\mathsf{Rep}^{\mathrm{op}}$ of both sides, the claim follows by combining this result with Theorem 3.6.5 and appealing to Corollary 3.4.7.2. □

Finally, note the equivalence of diagram categories $\mathrm{Lad}_{\mathcal{C}}(\mathcal{M},\mathcal{N}) \simeq \int_{S^1}({}_{\mathsf{Vec}}\mathcal{M}_{\mathcal{C}'\,\mathcal{C}}\mathcal{N}_{\mathsf{Vec}})$ which arises from the fact that diagrams in a Vec bulk can be taken to be blank. Thus we obtain an equivalence by just cutting the annulus along the Vec bulk (see Figure 3.11 below). In fact, whenever $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{N}$ are all finitely semisimple, the category $\mathrm{Lad}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ of Theorem 3.5.3 is finitely semisimple as well. Thus $\mathrm{Lad}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is equivalent to its category of representations by Proposition 2.1.5, and so Theorem 3.4.7 implies that the ladder category and balanced tensor product are equivalent (when the theorem applies). Hence in fact we obtain a proof of Theorem 3.5.3 via Theorem 3.4.7. There is also a purely algebraic ladder category, and it is not difficult to see how to modify the constructions of this chapter to build an analogous equivalence with it, but this would take us too far afield.

3.7 Catalogue of diagram kirigami

Below we depict various diagram kirigami[5] and their algebraic consequences when Theorem 3.6.5 applies.

Figure 3.11 corresponds to the equivalence of Corollary 3.4.4.1 arising because every diagram in a Vec-bulk can be made blank. Proposition 3.4.5 gives the analogous result under the correspondence Proposition 3.4.5 between the balanced tensor product and the Drinfeld center. Figure 3.12 corresponds to Proposition 3.4.4 or, in reverse, Corollary 3.4.7.3. Figure 3.13 directly corresponds to Proposition 3.4.3 (one thinks of diagrams in $\mathcal{C} \otimes \mathcal{D}^{\mathrm{mop}}$ as a diagrams in $\mathcal{C}$ and $\mathcal{D}^{\mathrm{mop}}$ superimposed on one another). Figure 3.14 corresponds to Corollary 3.4.7.2 and is related to the observation we made in Remark 3.5.6.

As an illustration of the power of the diagrammatic viewpoint we also include Figure 3.15, which corresponds to the ordinary algebraic fact that the categories $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$ and $\mathrm{BiBal}_{(\mathcal{C},\mathcal{D})}(\mathcal{N},\mathcal{M})$ are equivalent (which we did not explicitly record above).

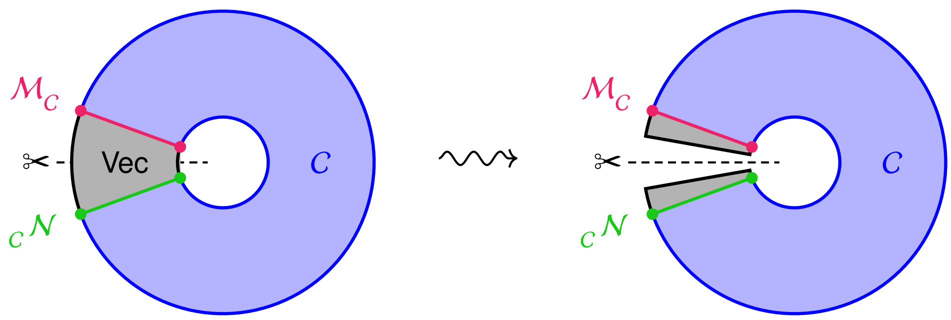


Figure 3.11: $\mathrm{BiBal}_{(\mathsf{Vec},\mathcal{C})}(\mathcal{M},\mathcal{N}) \simeq \mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}$

[5]Kirigami is a variation of origami where cutting and folding of the paper is allowed.

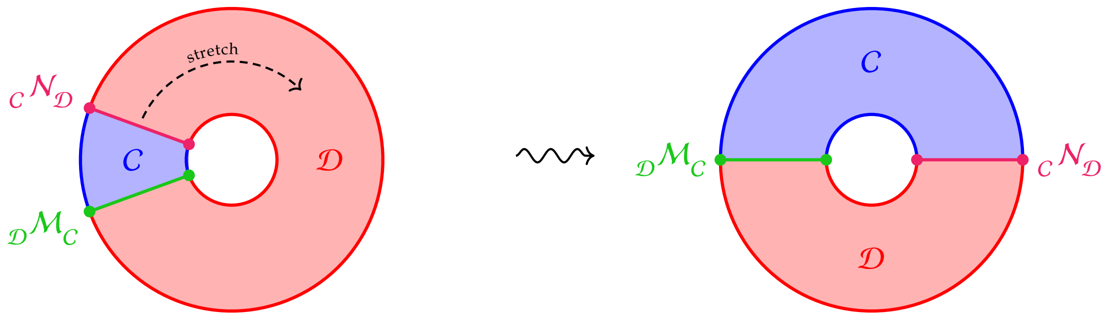


Figure 3.12: $\mathcal{Z}({}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}}{\otimes}} \mathcal{N}_{\mathcal{D}}) \simeq \mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N})$

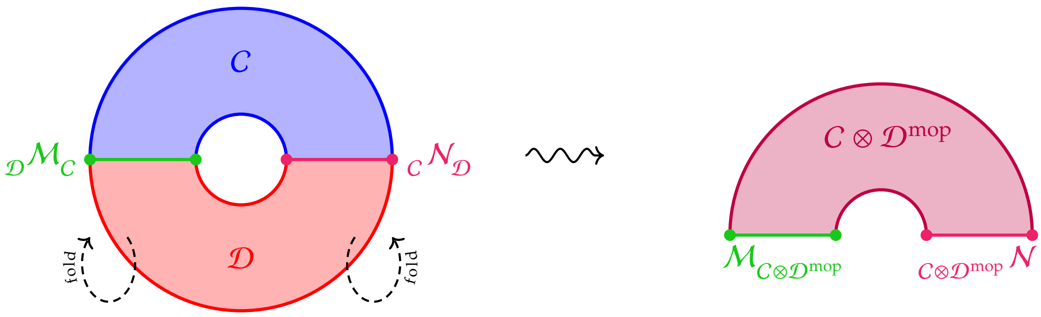


Figure 3.13: $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N}) \simeq \mathcal{M} \overset{\mathrm{bal}}{\underset{\mathcal{C}\otimes\mathcal{D}^{\mathrm{mop}}}{\otimes}} \mathcal{N}$

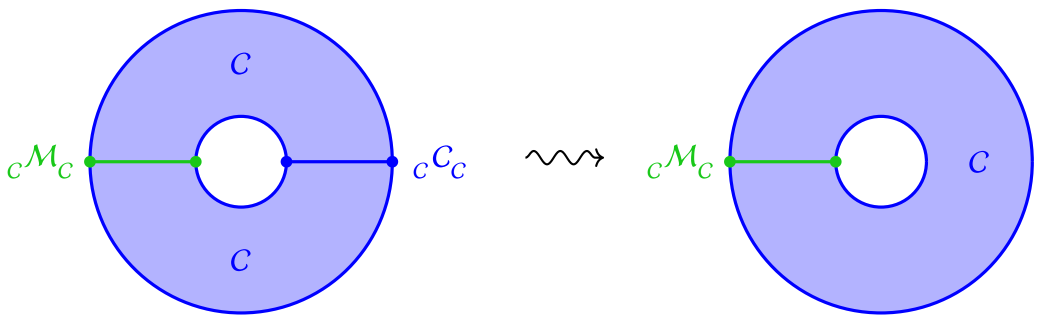


Figure 3.14: $\mathrm{BiBal}_{(\mathcal{C},\mathcal{C})}(\mathcal{M},\mathcal{C}) \simeq \mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$

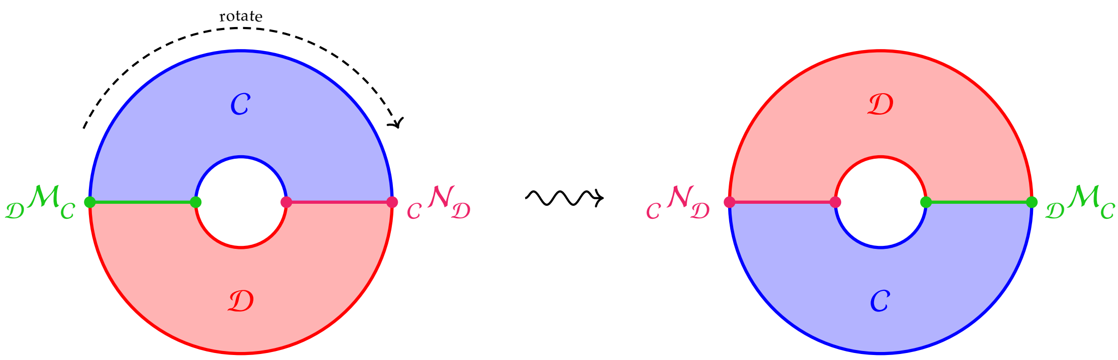


Figure 3.15: $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{M},\mathcal{N}) \simeq \mathrm{BiBal}_{(\mathcal{C},\mathcal{D})}(\mathcal{N},\mathcal{M})$

The bibalanced center of a bimodule category

In this chapter fix ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ a $(\mathcal{C}, \mathcal{D})$-bimodule category with $\mathcal{C}$ and $\mathcal{D}$ rigid. Using the BiBal construction of Chapter 3, we define a generalised Drinfeld center for bimodule categories.

Definition 4.0.1. The *bibalanced center* $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ of $\mathcal{M}$ is the category $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}({}^*\mathcal{M}, \mathcal{M})$, with the dual ${}^*\mathcal{M}$ defined as in Proposition 3.1.4.

It is the purpose of this chapter to explore the natural monoidal structure on $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$, which parallels the monoidal structure on the ordinary Drinfeld center $\mathcal{Z}(\mathcal{C})$. Indeed, the monoidal equivalence of $\mathcal{Z}(\mathcal{C})$ and $\mathrm{End}({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}})$ of Proposition 1.7.2 is analogous to the comparison functor $P : \mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}) \to [{}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}] = \mathrm{End}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ of Proposition 3.2.6 (note that $({}^*\mathcal{M})^* \simeq \mathcal{M}$).

We extend the analogy by showing that in this case P is naturally a monoidal functor, and hence that when $\mathcal{M}$ is finitely semisimple P is a monoidal equivalence as well (a consequence of Proposition 3.3.3). This is consistent with the Drinfeld center being a downward arrow (hence producing a monoidal category from a category) in Baez and Dolan's periodic table[1] of k-tuply monoidal n-categories.

We also built a functor $B : \mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}) \to \mathsf{Rep}^{\mathrm{op}} \int_{S^1}({}^*_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ in Proposition 3.5.5. The category $\int_{S^1}({}^*_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ can be interpreted as consisting of diagrams of the form depicted in Figure 4.1a, i.e. where the left bulk boundary is considered oriented opposite to our previous convention, being labelled by ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ as well (as opposed to the dual ${}^*\mathcal{M}$).

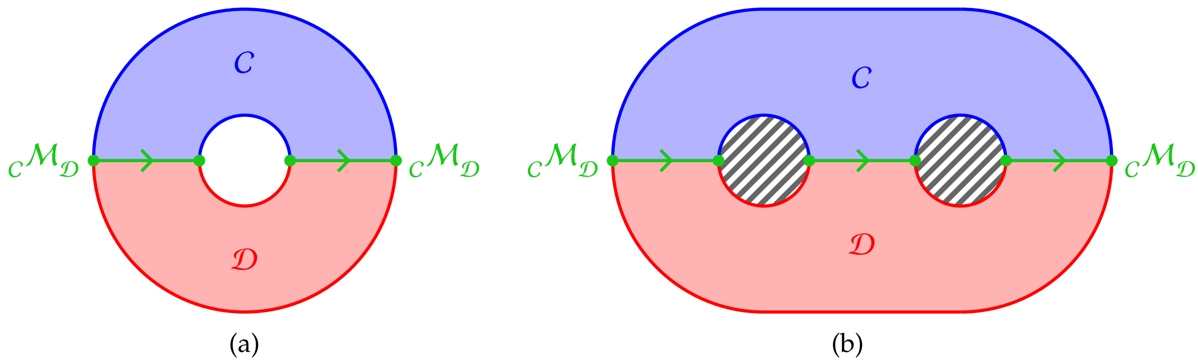


Figure 4.1: Morphisms of $\int_{S^1}({}^*_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ and the induced tensor product.

Just as we could interpret the tensor product of $\mathcal{Z}(\mathcal{C})$ as arising from inserting representations of $\int_{S^1} \mathcal{C}$ into doubly punctured disks (see Chapter 2), there is a similar interpretation of the

[1]This is Table 1 of [1].

monoidal product of $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ as inserting representations of $\int_{S^1}({}_{\mathcal{D}}^{*}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ into the striped regions of Figure 4.1b. We conclude by mentioning how additional algebraic structures on $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ come from similar geometric constructions.

4.1 A monoidal structure on the bibalanced center

The $\Bbbk$-linear structure of $\mathcal{M}$ together with the hom-pairing $\mathrm{Hom} : \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \to \mathsf{Vec}$ readily gives rise to a product on the category $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$; we simply assemble the tensor product (suppressing reassociation)

$$(\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}) \otimes (\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}) \xrightarrow{(\mathcal{M}^{\mathrm{op}}\,\mathrm{Hom}^{\mathrm{op}})\otimes\mathcal{M}} \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}, \tag{4.1.1}$$

and this gives a functorial way to multiply any $S, T \in \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$. The choice of tensor factor on which $\mathrm{Hom}^{\mathrm{op}}$ acts is arbitrary, but may be freely commuted up to natural isomorphism.

In general this product does not extend to a monoidal structure on $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ (the product need not be unital), but it does at least give rise to the structure of a semimonoidal category.

Definition 4.1.1. A category $\mathcal{C}$ equipped with an associator satisfying the associativity constraint of Definition 1.2.1 (but with no unit or unitors) is a *semimonoidal*[2] *category*.

Just as a monoidal category categorifies a monoid, a semimonoidal category categorifies a semigroup. The next proposition establishes that the relationship between monoids and semigroups—namely that a monoid is the data of a semigroup satisfying a property—is preserved in the categorical world.

Proposition 4.1.2. *The unit object of a monoidal category is defined by a universal property which makes the data of a unit object unique up to unique isomorphism.*

Proof. This follows from Proposition 2.2.6 of [12] and the subsequent remarks. □

We will build an associator for (4.1.1) from the following lemma.

Lemma 4.1.3. *There are canonical natural isomorphisms (with $M_1 \in \mathcal{M}^{\mathrm{op}}$, $M_2 \in \mathcal{M}$, and $V \in \mathsf{Vec}$)*

$$\delta_{M_1,M_2,V} : \mathcal{M}(M_1 \to M_2 \odot V) \xrightarrow{\sim} \mathcal{M}(M_1 \to M_2) \otimes V,$$

and (additionally with $N_1 \in \mathcal{M}^{\mathrm{op}}$ and $K, N_2 \in \mathcal{M}$)

$$\gamma_{K,N_1,N_2,M_1,M_2} : (K \odot \mathcal{M}(N_1 \to N_2)) \odot \mathcal{M}(M_1 \to M_2) \xrightarrow{\sim} K \odot \mathcal{M}(N_1 \to N_2 \odot \mathcal{M}(M_1 \to M_2)).$$

Proof. We observe that Vec is a symmetric pivotal category, and that $\mathcal{M}$ is made a $(\mathsf{Vec}, \mathsf{Vec})$-bimodule category by the copower operation of Proposition 2.1.4. The natural isomorphism δ can then be assembled from the composite of isomorphisms

$$\begin{aligned} \delta_{M_1,M_2,V} : \mathcal{M}(M_1 \to M_2 \odot V) &\xrightarrow{\sim} \mathcal{M}(M_1 \odot V^* \to M_2) \\ &\xrightarrow{\sim} \mathsf{Vec}(V^* \to \mathcal{M}(M_1 \to M_2)) \\ &\xrightarrow{\sim} \mathsf{Vec}(\mathbb{1} \to \mathcal{M}(M_1 \to M_2) \otimes V^{**}) \\ &\xrightarrow{\sim} \mathsf{Vec}(\mathbb{1} \to \mathcal{M}(M_1 \to M_2) \otimes V) \\ &\xrightarrow{\sim} \mathcal{M}(M_1 \to M_2) \otimes V. \end{aligned} \tag{4.1.2}$$

[2] We overlook the term *semigroup category* for this purpose, since corrupted phraseology such as *semigroupoidal functor* is confusing.

Then $\gamma_{K,N_1,N_2,M_1,M_2,V}$ may be constructed via (again using natural isomorphisms)

$$
\begin{aligned}
(K \odot \mathcal{M}(N_1 \to N_2)) \odot \mathcal{M}(M_1 \to M_2) &\xrightarrow{\qquad \sim \qquad} K \odot (\mathcal{M}(N_1 \to N_2) \otimes \mathcal{M}(M_1 \to M_2)) \\
&\xrightarrow{K \odot \delta^{-1}_{N_1,N_2,\mathcal{M}(M_1 \to M_2)}} J \odot \mathcal{M}(N_1 \to N_2 \odot \mathcal{M}(M_1 \to M_2)).
\end{aligned}
$$

□

Proposition 4.1.4. *The product* (4.1.1) *is naturally equipped with an associator isomorphism which turns $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ into a semimonoidal category.*

Proof. Denote the product by $\diamond : (\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}) \otimes (\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}) \to \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$; now fix objects $K_1, M_1, N_1 \in \mathcal{M}^{\mathrm{op}}$ and $K_2, M_2, N_2 \in \mathcal{M}$. We just directly calculate that (suppressing the notion for copowers and tensor products of vector spaces)

$$
\begin{aligned}
((K_1 \otimes K_2) \diamond (N_1 \otimes N_2)) \diamond (M_1 \otimes M_2) &= (K_1 \mathcal{M}(K_2 \to N_1) \otimes N_2) \diamond (M_1 \otimes M_2) \\
&= (K_1 \mathcal{M}(K_2 \to N_1)) \mathcal{M}(N_2 \to M_1) \otimes M_2
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(K_1 \otimes K_2) \diamond ((N_1 \otimes N_2) \diamond (M_1 \otimes M_2)) &= (K_1 \otimes K_2) \diamond (N_1 \mathcal{M}(N_2 \to M_1) \otimes M_2) \\
&= K_1 \mathcal{M}(K_2 \to N_1 \mathcal{M}(N_2 \to M_1)) \otimes M_2.
\end{aligned}
$$

Thus we may build a component $\alpha_{K_1 \otimes K_2, N_1 \otimes N_2, M_1 \otimes M_2}$ of the associator by tensoring the natural isomorphism γ of Lemma 4.1.3 on the left with M_2, obtaining[3]

$$
(K_1 \mathcal{M}(K_2 \to N_1)) \mathcal{M}(N_2 \to M_1) \otimes M_2 \xrightarrow{\gamma_{K_1,K_2,N_1,N_2,M} \otimes M_2} K_1 \mathcal{M}(K_2 \to N_1 \mathcal{M}(N_2 \to M_1)) \otimes M_2.
$$

The result is a natural isomorphism α.

In principle the pentagon associativity axiom for α can be verified by drawing a large commutative diagram, but this obscures the underlying pair of coherence results at play. The first is that all morphisms from $(\cdots(K_1 \odot V_1) \odot V_2) \odot \cdots) \odot V_n$ to a fixed reassociation, which were built from the associator in Vec and the associator for the Vec module action, are equal. This follows from the coherence theorem for bimodule categories Theorem 1.3.9. The second is that the analogous coherence result holds for all products

$$
(\cdots(\mathcal{M}(M_1 \to N_1) \otimes \mathcal{M}(M_2 \to N_2)) \otimes \cdots) \otimes \mathcal{M}(M_n \to N_n)
$$

and all reassociations made from the symmetric braiding and associator in Vec, possibly using the copower adjunction to move these products inside each other. For instance, we can build a morphism

$$
\begin{aligned}
&\Big(\mathcal{M}(M_1 \to N_1) \otimes \mathcal{M}(M_2 \to N_2)\Big) \otimes \mathcal{M}(M_3 \to N_3) \\
&\qquad \longrightarrow \mathcal{M}\Big(M_1 \to N_1 \odot \mathcal{M}\big(M_2 \to \mathcal{M}(M_3 \to N_3) \odot N_2\big)\Big),
\end{aligned}
$$

and any scheme to do so gives the same morphism. □

In fact, when $\mathcal{M}$ is finitely semisimple the product $\diamond$ on $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ is unital, and so $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$

[3] Certainly not every object of $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ is of the form $M \otimes N$, but every object $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ is isomorphic to a direct sum of such objects.

becomes a monoidal category. Thus henceforth again suppose that $\mathcal{M}$ is semisimple with finite collection of distinguished simple objects $\{X_i\}$.

Proposition 4.1.5. *The semimonoidal category $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ admits a unit with compatible unitor isomorphisms which turn $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ into a monoidal category.*

Proof. Define the monoidal unit by the direct sum

$$\mathbb{1} := \bigoplus_i X_i \otimes X_i.$$

Given a simple object $X_j \otimes X_k$ of $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ we can compute

$$\begin{aligned}
\mathbb{1} \diamond (X_j \otimes X_k) &= \left(\bigoplus_i X_i \otimes X_i \right) \diamond (X_j \otimes X_k) \\
&= \bigoplus_i X_i \mathcal{M}(X_i \to X_j) \otimes X_k
\end{aligned}$$

from which after choosing an isomorphism $\psi_j : \mathbb{1} \to \mathcal{M}(X_j \to X_j)$ we can build a left unitor component $l_{X_j \otimes X_k} : \mathbb{1} \diamond (X_j \otimes X_k) \to X_j \otimes X_k$. These components extend to the desired natural isomorphism by semisimplicity (explicitly, $l_{X_j \otimes X_k}$ is a composite of components of two natural isomorphisms), and the right unitor is constructed in completely analogous[4] fashion.

After unravelling the definition of the extension of the associator to direct sums and the natural isomorphism γ from which it was constructed, the triangle identity axiom amounts to verifying commutativity of the boundary of the following pentagon.

$$\begin{tikzcd}
(X_i \odot \mathcal{M}(X_j \to X_k)) \odot \mathcal{M}(X_k \to X_k) \arrow[rr, "\delta_{X_i, X_j, X_k, \mathcal{M}(X_k \to X_k)}"] & & X_i \odot \mathcal{M}(X_j \to X_k \odot \mathcal{M}(X_k \to X_k)) \\
(X_i \odot \mathcal{M}(X_j \to X_k)) \odot \mathbb{1} \arrow[u, "(X_i \mathcal{M}(X_j \to X_k)) \odot \psi_j"] \arrow[rr, "\delta_{X_i, X_j, X_k, \mathbb{1}}"] \arrow[dr, "r^{\mathsf{Vec}}_{X_i \odot \mathcal{M}(X_j \to X_k)}"'] & & X_i \odot \mathcal{M}(X_j \to X_k \odot \mathbb{1}) \arrow[u, "X_i \mathcal{M}(X_j \to X_k \odot \psi_j)"'] \arrow[dl, "X_i \odot \mathcal{M}(X_j \to l^{\mathsf{Vec}}_{X_k})"] \\
& X_i \odot \mathcal{M}(X_j \to X_k) &
\end{tikzcd}$$

The bottom triangle commutes since a similar triangle or square may be juxtaposed for each morphism in the composite (4.1.2) defining δ. Since the upper square also commutes by naturality of δ (given that the right and left unitors were both constructed using the same family of isomorphisms ψ_j), this completes the proof. □

Now suppose that $\mathcal{M}$ is a left $\mathcal{C}$-module category. In order to extend the monoidal product on $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ to a product on ${}^*\mathcal{M} \otimes \mathcal{M}$ and subsequently a category of (bi)balanced objects, we need to observe some basic interactions between the action functors $L_{\mathcal{C}}$ and ${}_{\mathcal{C}}R$ and the monoidal structure. Thus fix $N_1 \otimes N_2, M_1 \otimes M_2 \in \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$. For any $X \in \mathcal{C}$ we have

$$\begin{aligned}
L_{\mathcal{C}}(X)((N_1 \otimes N_2) \diamond (M_1 \otimes M_2)) &= L_{\mathcal{C}}(X)(N_1 \mathcal{M}(N_2 \to M_1) \otimes M_2)) \\
&= X \triangleright (N_1 \mathcal{M}(N_2 \to M_1)) \otimes M_2,
\end{aligned}$$

[4] So that the left and right unitors remain compatible with respect to the associator, we require that the same isomorphism $\psi_j : \mathbb{1} \to \mathcal{M}(X_j \to X_j)$ chosen for each X_j is used in the construction of each unitor.

while

$$L_{\mathcal{C}}(X)(N_1 \otimes N_2) \diamond (M_1 \otimes M_2) = (X \triangleright N_1)\mathcal{M}(N_2 \to M_1) \otimes M_2.$$

Thus the natural isomorphism $X \triangleright (M \odot V) \to (X \triangleright M) \odot V$ (with $V \in \mathsf{Vec}$) gives rise, after extending to direct sums, to a natural isomorphism

$$n^{L_{\mathcal{C}}}_{X,S,T} : L_{\mathcal{C}}(X)(S \diamond T) \to L_{\mathcal{C}}(X)(S) \diamond T. \tag{4.1.3}$$

Similarly, we have a natural isomorphism

$$m^{_{\mathcal{C}}R}_{X,S,T} : {}_{\mathcal{C}}R(X)(S \diamond T) \to S \diamond {}_{\mathcal{C}}R(X)(T). \tag{4.1.4}$$

Lemma 4.1.6. *We can always consider the monoidal category $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ as a bimodule category over itself. In this situation, the natural isomorphisms (4.1.3) and (4.1.4) for each $X \in \mathcal{C}$ turn $L_{\mathcal{C}}(X)$ and ${}_{\mathcal{C}}R(X)$ into right and left module functors, respectively.*

Proof. Observe that up to an interchange of direct sums and the module action $\triangleright$ or $\triangleleft$ the bimodule structure is strict. The claim follows. □

Although the functors $L_{\mathcal{C}}(X)$ and ${}_{\mathcal{C}}R(X)$ are not (in general) bimodule functors, we have a natural isomorphisms which compares an application of $L_{\mathcal{C}}(X)$ or ${}_{\mathcal{C}}R(X)$ in a tensor product "on the wrong side":

Lemma 4.1.7. *There is a natural isomorphism*

$$r_{X,S,T} : {}_{\mathcal{C}}R(X)(S) \diamond T \xrightarrow{\sim} S \diamond L_{\mathcal{C}}(X)(T).$$

Proof. Unwinding the definition of ${}^*\mathcal{M}$, for $X \in \mathcal{C}$, $N_1, M_1 \in \mathcal{M}^{\mathrm{op}}$, and $N_2, M_2 \in \mathcal{M}$ we have

$${}_{\mathcal{C}}R(X)(N_1 \otimes N_2) \diamond (M_1 \otimes M_2) = N_1\mathcal{M}(X \triangleright N_2 \to M_1) \otimes M_2$$

and similarly

$$\begin{aligned}(N_1 \otimes N_2) \diamond L_{\mathcal{C}}(X)(M_1 \otimes M_2) &= N_1\mathcal{M}(N_2 \to M_1 \triangleleft X) \otimes M_2\\ &= N_1\mathcal{M}(N_2 \to {}^*X \triangleright M_1) \otimes M_2.\end{aligned}$$

We obtain the desired natural isomorphism r by using the isomorphism $\mathcal{M}(X \triangleright N_2 \to M_1) \to \mathcal{M}(N_2 \to {}^*X \triangleright M_1)$ provided by Proposition 1.3.10 and extending to direct sums. □

Proposition 4.1.8. *The monoidal structure on $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ naturally extends to a monoidal structure on* $\mathrm{Bal}_{\mathcal{C}}({}^*\mathcal{M}, \mathcal{M})$ *and* $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$.

Proof. Given two balanced objects (S, τ) and (T, κ) of $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$, we can build a balancing on the tensor product $S \diamond T$ by forming the composite (for each $X \in \mathcal{C}$)

$$\begin{aligned}(\tau \diamond \kappa)_X : L_{\mathcal{C}}(X)(S \diamond T) &\xrightarrow{n^{L_{\mathcal{C}}}_{X,S,T}} L_{\mathcal{C}}(X)(S) \diamond T \xrightarrow{\tau_X \diamond T} {}_{\mathcal{C}}R(X)(S) \diamond T\\ &\xrightarrow{r_{X,S,T}} S \diamond L_{\mathcal{C}}(X)(T) \xrightarrow{S \diamond \kappa_X} S \diamond {}_{\mathcal{C}}R(X)(T) \xrightarrow{(m^{_{\mathcal{C}}R}_{X,S,T})^{-1}} {}_{\mathcal{C}}R(X)(S \diamond T).\end{aligned}$$

The resulting natural isomorphism $(\tau \diamond \kappa)_X : L_{\mathcal{C}}(X)(S \diamond T) \to {}_{\mathcal{C}}R(X)(S \diamond T)$ obeys the balancing condition (3.2.1) because both τ and κ do, and r obeys a bimodule associativity compatibility condition analogous to Proposition 3.1.8 for ϕ. We neglect to explicitly define the structure of a balancing on $\mathbb{1}$, since this involves a sum over a choice of bases for the vector spaces $\mathcal{M}(X_i \to X \triangleright X_i)$. Instead we appeal to the fact that P is an equivalence of ordinary categories in the finitely semisimple case, and thus we can pull back the modulators on the identity module functor $\mathcal{M}_C \to \mathcal{M}_C$.

Individual squares asserting that $f : S \to S'$ and $g : T \to T'$ are both morphisms of balanced objects directly assemble by superposition into a square asserting that $f \diamond g$ is a morphism of balanced objects. As in Proposition 1.5.5 the associativity condition (3.2.1) forces the associator to be a morphism of balanced objects, and the unitors pull back via P. In the situation of a pair of bibalanced objects (S, τ, σ) and (T, κ, χ), we obtain a pair of balancings on $S \diamond T$ in the way in which we have just seen. The unit is again the pull back of the identity bimodule functor ${}_C\mathcal{M}_\mathcal{D} \to {}_C\mathcal{M}_\mathcal{D}$ under P. The claimed extension to a functor from $\mathcal{Z}^{\mathrm{bibal}}({}_C\mathcal{M}_\mathcal{D})$ follows by assembling the respective bibalancing compatibility diagrams (3.2.2) for S and T into a single diagram for $S \diamond T$. □

Proposition 4.1.9. *The functor $P : \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \to \mathrm{End}(\mathcal{M})$ of Proposition 3.1.12 and its extensions to $\mathrm{Bal}_C({}^*\mathcal{M}, \mathcal{M})$ and $\mathcal{Z}^{\mathrm{bibal}}({}_C\mathcal{M}_\mathcal{D})$ are all naturally monoidal functors.*

Proof. A morphism $\iota : \mathbb{1}_{\mathrm{End}(\mathcal{M})} \to P(\mathbb{1}_{\mathcal{M}^{\mathrm{op}}\otimes\mathcal{M}})$ is the data for each simple X_i of a morphism

$$\iota_{X_i} : X_i \to P(\mathbb{1}_{\mathcal{M}^{\mathrm{op}}\otimes\mathcal{M}})(X_i) = \mathrm{Hom}_{\mathcal{M}^{\mathrm{op}}}\,\mathcal{M}\left(\bigoplus_i X_i \otimes X_i \otimes X_i\right) = \bigoplus_i \mathcal{M}(X_i \to X_i)X_i.$$

Recalling the isomorphisms $\psi_i : \mathbb{1} \to \mathcal{M}(X_i \to X_i)$ of Proposition 4.1.5 we directly construct each component ι_{X_i}. It is clear that the components ι_{X_i} are natural with respect to one another, and hence assemble into a natural transformation ι.

Given any pure objects $N_1 \otimes N_2$ and $M_1 \otimes M_2$ of $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$, a component of $J_{N_1\otimes N_2, M_1\otimes M_2}$ of a tensorator for P is specified by a natural isomorphism

$$J_{N_1\otimes N_2, M_1\otimes M_2} : P(N_1 \otimes N_2) \diamond P(M_1 \otimes M_2) \xrightarrow{\sim} P((N_1 \otimes N_2) \diamond (M_1 \otimes M_2)),$$

which is the data for each $K \in \mathcal{M}$ of a map

$$N_1\mathcal{M}(N_2 \to M_1\mathcal{M}(M_2 \to K)) \xrightarrow{\sim} (N_1\mathcal{M}(N_2 \to M_1))\mathcal{M}(M_2 \to K).$$

Such an isomorphism is exactly provided by the natural isomorphism component $\gamma_{N_1,N_2,M_1,M_2,K}$; extending to direct sums, it is then clear that the components of J and J itself are all natural isomorphisms. The associativity hexagon now follows from the fact that the associator of $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ was also built from γ. The unit constraint squares commute because ι was constructed using the same isomorphisms ψ_j used to define the unitors. In the situation of a category of balanced or bibalanced objects one sees that ι and the components of J are each morphisms of bimodule functors directly from our definitions. □

Thus when $\mathcal{M}$ is finitely semisimple we obtain the following (as usual by appeal to Proposition 1.2.16).

Theorem 4.1.10. *There are monoidal equivalences $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \overset{\otimes}{\simeq} \mathrm{End}(\mathcal{M})$, $\mathrm{Bal}_C({}^*\mathcal{M}, \mathcal{M}) \overset{\otimes}{\simeq} \mathrm{End}(\mathcal{M}_C)$, and $\mathcal{Z}^{\mathrm{bibal}}({}_C\mathcal{M}_\mathcal{D}) \overset{\otimes}{\simeq} \mathrm{End}({}_C\mathcal{M}_\mathcal{D})$.*

Remark 4.1.11. When $C = \mathcal{D}$ the Drinfeld center $\mathcal{Z}({}_C\mathcal{M}_C)$ makes sense, and when $\mathcal{M}$ is finitely semisimple we have $\mathcal{Z}({}_C\mathcal{M}_C) \simeq [{}_C C_C \to {}_C\mathcal{M}_C]$. If in addition the bimodule structure on $\mathcal{M}$ arises from a monoidal functor $H : \mathcal{C} \xrightarrow{\otimes} \mathcal{M}$ then precomposition with H gives a functor $\mathrm{End}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}) \to [{}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \to {}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}]$, and thus Theorem 4.1.10 gives a comparison between $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ and $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$.

4.2 A semimonoidal structure in the nonsemisimple case

The majority of the results of the previous section assume that $\mathcal{M}$ is finitely semisimple, but it is worth remarking that this was only partially necessary. Namely, all of our constructions and verifications which did not involve the tensor unit and unitors do not require the hypothesis of finite semisimplicity. We still obtain a semimonoidal category structure on $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ in any case, and the functor $P : \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M} \to \mathrm{End}(\mathcal{M})$—and even extensions of P to a functor from a category of (bi)balanced objects—is still a functor of semimonoidal categories with the provided tensorator.

Of course, this means that when P is an equivalence of semimonoidal categories, the category $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ (along with the associated categories of (bi)balancings) is actually monoidal. As we have just seen, this is certainly the situation when $\mathcal{M}$ is finitely semisimple and thus the functor Q exists, witnessing an explicit inverse. In fact, we have the following converse to this implication.

Proposition 4.2.1. *Suppose that $\mathcal{M}$ is abelian and locally finite. The category $\mathcal{M}$ is finitely semisimple if and only if $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ is unital (with respect to the semimonoidal structure afforded by Proposition 4.1.4).*

Proof. We have already seen that finite semisimplicity of $\mathcal{M}$ implies unitality of $\mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$ in Section 4.1, so we just handle the other implication.

Since the unit is determined up to arbitrary isomorphism, write

$$\mathbb{1} = \bigoplus_i M_i \otimes N_i \tag{4.2.1}$$

for some $M_i \in \mathcal{M}^{\mathrm{op}}$ and $N_i \in \mathcal{M}$. We can always arrange that in the sum (4.2.1) each summand is indecomposable (in that both M_i and N_i are indecomposable themselves), and we do this.

Then if X is a simple object of $\mathcal{M}$ there is necessarily an isomorphism

$$X \otimes X \cong \bigoplus_i X\mathcal{M}(X \to M_i) \otimes N_i \cong X \otimes \left(\bigoplus_i N_i \mathcal{M}(X \to M_i) \right).$$

Consequently, there exists a unique j such that $\mathcal{M}(X \to M_j)$ is nonzero (and in this case is one-dimensional), and it follows that then $N_j \cong X$ as well. Unitality on the other side similarly gives rise to an isomorphism

$$X \otimes X \cong \bigoplus_i M_i \mathcal{M}(N_i \to X) \otimes X \cong \left(\bigoplus_i M_i \mathcal{M}(N_i \to X) \right) \otimes X.$$

Since $N_j \cong X$ we have in particular that the vector space $\mathcal{M}(N_i \to X)$ is one-dimensional (and critically, nonzero). Together with simplicity of X we obtain that $\mathcal{M}(N_i \to X)$ is zero for all $i \neq j$, and therefore $M_j \cong X$ as well. Hence $X \otimes X$ appears (up to isomorphism) as a summand in (4.2.1) for X a representative of each isomorphism class of simple objects of $\mathcal{M}$. Since the direct sum (4.2.1) is necessarily finite, we conclude that $\mathcal{M}$ has finitely many simple objects.

Now let M be any indecomposable object of $\mathcal{M}$. Then there is again an isomorphism

$$M \otimes M \cong M \otimes \left(\bigoplus_i N_i \mathcal{M}(M \to M_i) \right).$$

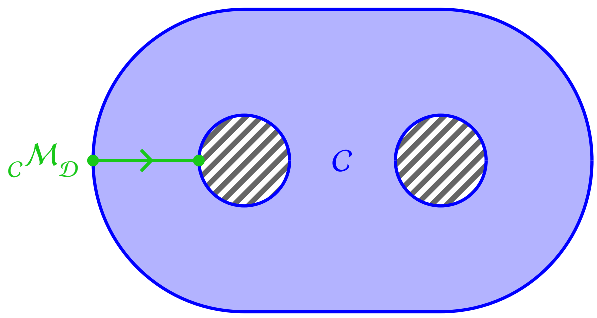
By the indecomposability hypothesis the only possibility is that there exists some j such that for all $i \neq j$ we have $\mathcal{M}(M \to M_i) \cong 0$, and for $i = j$ we have $\mathcal{M}(M \to M_j) \cong \Bbbk$ and $M \cong N_j$. But now since M has finite length (the category $\mathcal{M}$ is locally finite), M has a simple quotient Y (take the quotient of M by any maximal proper subobject, which exists again by the local finiteness of $\mathcal{M}$), and we have just seen that (up to isomorphism) $Y \otimes Y$ is a summand in (4.2.1). Hence $\mathcal{M}(M \to Y)$ is nonzero, so this establishes that $M_j \otimes N_j \cong Y \otimes Y$. Consequently $M \cong Y$, and therefore every indecomposable object of $\mathcal{M}$ is simple. This completes the proof. □

4.3 Monoidal structures on representations of annular categories

Let $\mathcal{C}$ and $\mathcal{D}$ be pivotal categories with $\mathrm{End}(\mathbb{1}) \cong \Bbbk$. In Section 2.4 we defined a braided monoidal structure on representations of the ordinary annular category $\int_{S^1} \mathcal{C}$ using diagrams in doubly punctured disks. The monoidal product did not generalise to the setting of representations of categories of marked annuli $\int_{S^1}({}_{\mathcal{D}}\mathcal{N}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ in Section 3.5, since the bulk boundaries labelled by module categories need not have matched-up when drawn in a doubly punctured disk.

Nonetheless, when we set $\mathcal{N} = {}^*\mathcal{M}$ the category $\int_{S^1}({}^*_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ consists of diagrams of the form of Figure 4.1a, which naturally fit into the punctures of the doubly punctured marked disk of Figure 4.1b. Following Section 2.4 we are once again able to construct a monoidal product, but there is now no natural braiding, since the punctures in Figure 4.1b cannot be twisted around one another while returning to a diagram of the same form (the bulk boundaries will become scrambled). This is directly analogous to the fact that $\mathrm{BiBal}_{(\mathcal{D},\mathcal{C})}(\mathcal{N}, \mathcal{M})$ is only naturally (semi)monoidal once we impose the same assumption $\mathcal{N} = {}^*\mathcal{M}$ (and not braided). In Theorem 3.6.5 we showed that finite semisimplicity of $\mathcal{M}$ implies that these two categories were equivalent (up to an opposite), and it is not difficult to see how to extend the results of Section 2.4 to establish that this is a monoidal equivalence.

We also saw from Corollary 3.6.5.1 that in the finitely semisimple case that $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ is equivalent to the opposite of the category representations of marked annuli of the form of Figure 3.6b. The representations of the associated diagram category do not inherit a monoidal product for the same reason as above; they cannot be made to fit compatibly in doubly punctured disks. However, by Proposition 1.5.5 the Drinfeld center $\mathcal{Z}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}})$ does have the structure of a $(\mathcal{Z}(\mathcal{C}), \mathcal{Z}(\mathcal{C}))$-bimodule category. Diagrammatically this corresponds to the fact that we are able to insert diagrams of the form of Figures 3.6a and 3.6b into the respective left and right punctures of the marked doubly punctured disk



. (4.3.1)

Of course there is a corresponding $(\mathcal{Z}(\mathcal{C}), \mathcal{Z}(\mathcal{D}))$-bimodule category structure on $\mathcal{Z}^{\mathrm{bibal}}({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})$ as well, but this is where our story ends.

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