# From the Temperley-Lieb Categories to Toric Code 

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May 2017

A thesis submitted for the degree of Bachelor of Philosophy - Science (Honours) in Mathematics of the Australian National University


For my family.

## Declaration

Except where otherwise stated, the work in this thesis is my own work with much helps from my supervisor, Scott Morrison.

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## Acknowledgements

Above all, I would like to thank my supervisor, Scott Morrison. Thank you for suggesting this interesting and enjoyable project. Without your extraordinary patience, nuanced proofreading and sublime feedback, none of the work would be possible.
Thanks also to David Smyth for introduce me to Scott Morrison. Your advice about honours year projects enlightened me a lot.
I'd like to thank David Powell for being my mentor. Talking with you is always pleasant.
Thanks to my parents for their remarkable amount of support these years. Although you are not in Australia, the weekly calling is one of my emotional supports.
I'd like to thank Albert, David, Zanbin and all my friends for your support throughout.
Last, thanks to Jiyue Ma for making unparalleled food. You are a remarkable confessor who make my day less stressful.

## Abstract

In this thesis our aim is to show the ground states of Hamiltonians on 2-dimensional surfaces, which is made up by putting together stablizer operators corresponding to edges(vertices) and faces, coincide with topology of surface [Kit03]. we first introduce the Temperley-Lieb categories and then define the skein module of $T L(\delta=1)$, which is the specialization of the Temperley-Lieb category at $\delta=1$. Then we introduce the Levin-Wen models, which is the generalization of the toric code, with general defined projections (operators in quantum physics) corresponding to edges and vertices. We then prove the kernel of Hamiltonian in the toric code is isomorphic to the skein module of $T L(\delta=1)$ and therefore only depends on the topology of surface it bases on.

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## Introduction

The toric code is a class of stabilizer quantum codes associated with lattices on 2 -dimensinal surfaces. There is a $1 / 2$-spin state, qubit, lives on each stage of the lattice while at each vertex and face there is a stabilizer operator [Kit03]. The sum of all these operators defines a Hamiltonian with local relations. The ground states of this Hamiltonian do not relate to the lattice the toric code model associated with but coincide with the topology of the surfaces. Specifically, it is $4^{g}$-fold degenerate, where $g$ is the genus of the surfaces.
The Levin-Wen models are lattice models build upon a unitary fusion category $\mathcal{C}$ to describe corresponding topological orders [Wen90]. The Levin-Wen models for higher dimensional show that their ground states naturally gives rise to both emergent gauge bosons and emergent fermions. Thus, the ground states provides a mechanism for unifying gauge bosons and fermions in dimensions higher than 3. For more details please refer [LW05] but we are not going to address this any further in this thesis.

The aim of this thesis is to describe the Levin-Wen model and show that the ground states of Hamiltonians in the toric code, the simplest example which illustrate the general properties of Levin-Wen models [KK], are isomorphic to the skein module of the quotient category $T L(\delta=1)$.
In Chapter 1, we introduce some preliminary definitions including fusion category and associated concepts. In Chapter 2, we define the Temperley-Lieb category based on the Temperley-Lieb algebra and introduce the Jones-Wenzl idempotents which is vitally important throughout this thesis. In Chapter 3, the Karoubi envelope for categories is defined and we then prove that the Karoubi envelope of the Temperley-Lieb category is semisimple.
In Chapter 4 we consider evaluating the Karoubi envelope of the Temperley-Lieb category at a root of unity and then define the negligible ideal. This gives a semisimple category with finitely many simple objects after taking the quotient by the negligible ideal. A special case when $\delta=1$ is introduced and we show that
the skein module of any surface associated to this category is isomorphic to the $\mathbb{C}$-linear span of first homology group $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$.
In Chapter 5, we introduce the Levin-Wen models for surface $\Sigma$, which for any cellulation $\Gamma$ on surface $\Sigma$, gives a Hilbert space $\mathcal{H}_{\Gamma}$ and an associated Hamiltonian $H$ which is the sum of all Hamiltonians $H_{E}$ for edges and $H_{F}$ for faces. We prove that the kernel, or physically the ground states, of the Hamiltonian $H$ is the intersection of kernels of all components $H_{E}$ and $H_{F}$.
Then in Chapter 6, we shown in the toric code models, the ground states of Hamiltonian $H$ for any cellulation $\Gamma \subset \Sigma$ is isomorphic to, surprisingly, the $\mathbb{C}$ linear span of first homology group $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$. So the kernel only depends on the topology of $\Sigma$. Combined with Section 4, the kernel of the Hamiltonian $H$ in the toric code model is isomorphic to the skein module of the quotient category $T L(\delta=1)$.

## Chapter 1

## Preliminaries

Here we introduce the basic notions in monoidal (tensor) category and fusion categories which will be frequently used in later chapters.
We start from the definition of monoidal category since it is a basic concept in category theory and widely used to define other complexer categories.

### 1.1 Monoidal category

Definition 1.1. A monoidal category is a sextuplet $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ where $\mathcal{C}$ is a category, tensor product $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a bifunctor, $1 \in \operatorname{Obj}(\mathcal{C})$ is a distinguished object and $\alpha, \lambda, \rho$ are natural isomorphisms with the following properties:

1. (Identity). There are natural isomorphisms $\lambda$ and $\rho$ with components such that for each $X \in \operatorname{Obj}(\mathcal{C})$,

$$
\begin{aligned}
& \lambda_{X}: 1 \otimes X \cong X \\
& \rho_{X}: X \otimes 1 \cong X
\end{aligned}
$$

2. (Associativity). There is a natural isomorphism $\alpha$ with components such that for all $X, Y, Z \in \operatorname{Obj}(\mathcal{C})$,

$$
\alpha_{X, Y, Z}:(X \otimes Y) \otimes Z \cong X \otimes(Y \otimes Z)
$$

3. (Coherence). For all $W, X, Y, Z \in \operatorname{Obj}(\mathcal{C})$, we have the triangle diagram

and the pentagon diagram

commutes.
The commuting diagrams show the order in which we parenthesize a tensor product of $a_{1}, \ldots, a_{n}$ with arbitrary insertions of the tensor identity 1 does not matter. Through a sequence of morphisms $\rho, \lambda, \alpha$ and their inverses, any two parenthesized tensor products $x$ and $y$ are isomorphic.

The "exchange relation" $(f \otimes g) \circ(h \otimes k)=(f \circ h) \otimes(g \circ k)$ holds for all morphisms $f, g, h, k \in \operatorname{Mor}(\mathcal{C})$ as a consequence of $\otimes$ being a bifunctor.

Remark 1.2. The structure which a monoidal category equipped with is call the monoidal structure.

A monoidal category is strict if the natural isomorphisms $\rho, \sigma$ and $\alpha$ are identities.

Consider the definition of monoidal categories and natural transformations, representation of groups are actually functors. We then define the category of such representations.

Definition 1.3. The category of representations $\operatorname{Rep}(G)$ is a category with $\operatorname{Obj}(\mathcal{R e p}(G))$ is the collection of representations of group $G$. For any two objects $\rho: G \rightarrow \operatorname{Aut}(V)$ for vector space $V$ and $\sigma: G \rightarrow \operatorname{Aut}(W)$ for vector sapce $W$, the morphisms $\operatorname{Hom}_{\mathcal{R e p}(G)}(\rho \rightarrow \sigma)=\left\{f: \operatorname{Hom}_{V e c(k)}(V \rightarrow W): f(g v)=\right.$ $g f(v)$ for any $\forall g \in G$ and $\forall v \in V\}$ where $\operatorname{Vec}(k)$ is the category of vector space $k$.

### 1.2 Duality

The definition of rigid category is based on duality. Specifically, we need to define left dual and right dual.

Definition 1.4. Left dual and right dual. Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category. Suppose $X \in \operatorname{Obj}(\mathcal{C})$ then the left dual of $X$ is an object $Y \in \operatorname{Obj}(\mathcal{C})$ with two morphisms $\epsilon_{X}: Y \otimes X \rightarrow 1$ and $\operatorname{co\epsilon }_{X}: 1 \rightarrow X \otimes Y$ called the evaluation and coevaluation morphisms. The morphisms satisfies:

$$
\mathbb{1}_{X} \otimes \epsilon_{X} \circ \operatorname{co\epsilon _{X}} \otimes \mathbb{1}_{X}=\mathbb{1}_{X}
$$

and

$$
\operatorname{co\epsilon _{X}} \otimes \mathbb{1}_{Y} \circ \mathbb{1}_{Y} \otimes \epsilon_{X}=\mathbb{1}_{Y}
$$

If we represent two morphisms $\epsilon_{X}: Y \otimes X \rightarrow 1$ and $\epsilon_{X}: 1 \rightarrow X \otimes Y$ by $\bigcap$ and $\bigcup$ respectively, the properties above becomes

and


Then $X$ is called the right dual of $Y$ at the same time.
Remark 1.5. If duals (left or right) exists for some object $X$, suppose they are $Y$ and $Y^{\prime}$, then we have $Y \cong Y^{\prime}$. The details of proof is shown in [Müg08].

From this remark we have the duals for an object, if exists, is unique up to isomorphism. Therefore we define a left duality of a monoidal category. $(\mathcal{C}, \otimes, 1)$ to be a map that gives each object $X$ in $\mathcal{C}$ a left dual $X^{*}$ and morphisms $\epsilon(X): X^{*} \otimes X \rightarrow 1$ and $\operatorname{co\epsilon }(X): 1 \rightarrow X \otimes X^{*}$ with properties in definition. Then right duality is defined correspondingly.

Definition 1.6. A monoidal category $(\mathcal{C}, \otimes, 1)$ is called rigid if it is equipped with both left duality and right duality. Alternatively, every object $X \in \mathcal{C}$ has both left dual and right dual.

Definition 1.7. In a rigid monoidal category $\mathcal{C}$ [Müg08], we define a functor, $*$ for morphisms $f: X \rightarrow Y$ as:

$$
f^{*}=\epsilon(Y) \otimes \mathbb{1}_{X^{*}} \circ \mathbb{1}_{Y^{*}} \otimes f \otimes \mathbb{1}_{X^{*}} \circ \mathbb{1}_{Y^{*}} \otimes \operatorname{co\epsilon }_{X}
$$

It can be equipped with a natural isomorphism $\gamma:\left(Y^{*}\right) \otimes\left(X^{*}\right) \rightarrow(X \otimes Y)^{*}$ and $1^{*} \rightarrow 1$.

Remark 1.8. If the category $\mathcal{C}$ has left duality only then it is left rigid. Similarly, if $\mathcal{C}$ has right duality, it is right rigid.

### 1.3 Fusion category

It is easier for introducing fusion categories to start from a so-called additive category.

Definition 1.9. An additive category $\mathcal{C}$ is a category, such that:

1. The additive category $\mathcal{C}$ has a zero object $0 \in \operatorname{Obj}(\mathcal{C})$ (for any object $X \in \operatorname{Obj}(\mathcal{C})$ there exists precisely one morphism $\operatorname{Hom}_{\mathcal{C}}(0 \rightarrow X)$ and one $\left.\operatorname{Hom}_{\mathcal{C}}(X \rightarrow 0)\right)$. Then for two arbitrary objects $X, Y \in \operatorname{Obj}(\mathcal{C})$, there is a unique morphism: $X \rightarrow 0 \rightarrow Y$.
2. For all $X, Y \in \operatorname{Obj}(\mathcal{C})$, $\operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ is an abelian group. The zero map gives the identity and the composition is bilinear.
3. $\mathcal{C}$ has finite biproducts.

We can then use this definition to define abelian categories. Before that, we need to define kernel/cokernel and image/coimage.

Definition 1.10. Kernel and cokernel. Suppose $\mathcal{C}$ is a category with zero morphisms. Let $X, Y \in \operatorname{Obj}(\mathcal{C})$ and $f \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$.
The kernel $\operatorname{Ker}(f)$ of $f$ is an object $K \in \operatorname{Obj}(\mathcal{C})$ together with a morphism $k \in \operatorname{Hom}_{\mathcal{C}}(K \rightarrow X)$ such that $f \circ k=0$. If there is a object $A$ with morphism $g \in \operatorname{Hom}_{\mathcal{C}}(A \rightarrow X)$ such that $f \circ g=0$, then there exists a unique $h \in \operatorname{Hom}_{\mathcal{C}}(A \rightarrow K)$ with $g=k \circ h$. and the pentagon diagram


The cokernel $\operatorname{Coker}(f)$ of $f$ is an object $C \in \operatorname{Obj}(\mathcal{C})$ together with a morphism $c \in \operatorname{Hom}_{\mathcal{C}}(C \rightarrow X)$ such that $c \circ f=0$. If there is a object $B$ with morphism $j \circ c \in \operatorname{Hom}_{\mathcal{C}}(B \rightarrow Y)$ such that $j \circ f=0$, then there exists a unique $l \in \operatorname{Hom}_{\mathcal{C}}(C \rightarrow B)$ with $j=l \circ c$. and the pentagon diagram


Definition 1.11. Image and coimage. Then the image is defined by, for $f$ the image $\operatorname{Im}(f)$ is the kernel of its cokernel. Then coimage $\operatorname{Coim}(f)$ of $f$ is just the cokernel of its kernel.

Then we have enough technology to define an abelian category
Definition 1.12. An abelian category is an additive category $\mathcal{C}$ with properties:

1. Every morphism $f \in \operatorname{Mor}(\mathcal{C})$ has a kernel and cokernel.
2. For an arbitrary morphism $f \in \operatorname{Mor}(\mathcal{C})$. the natural transformation $\eta$ : $\operatorname{Coim}(f) \rightarrow \operatorname{Im}(f)$ is a natural isomorphism.

Using the defintion of abelian category combined with semisimple, it is easy to define semisimple category.

Definition 1.13. For an abelian category $\mathcal{C}$, it is called semisimple if admits a family of simple objects $X_{i}, i \in I$ such that for any $X \in \operatorname{Obj}(\mathcal{C}), X$ is a finite direct sum of $X_{i}$.

To define the fusion category we also need to recall $k$-linear category.
Definition 1.14. An abelian category $\mathcal{C}$ is said to be $\mathbb{K}$-linear if for any objects $X, Y \in \operatorname{Obj}(\mathcal{C})$ the hom-set $\operatorname{Hom}(X \rightarrow Y)$ is a $\mathbb{K}$-vector space and the composition of morphisms $\circ$ (and $\otimes$ in the monoidal category) is bilinear.

Remark 1.15. Then the functors between two $k$-linear ctegory must be $\mathbb{K}$-linear.
Combine the definition of semisimple, rigid and $\mathbb{K}$-linear, the fusion category is easy to be defined.

Definition 1.16. A fusion category is a rigid semisimple $\mathbb{K}$-linear monoidal category with finite isomorphism classes of simple objects, such that the endomorphisms of the unit object 1 is $\operatorname{End}=\mathbb{K}$.

## Chapter 2

## The Temperley-Lieb category

### 2.1 The Temperley-Lieb algebra

In the following chapters, the ground field we take is $\mathbb{K}=\mathbb{C}(q)$ if there is no further statement, where $q$ is a parameter.

Definition 2.1. The quantum integer $[n]$ is given by

$$
[n]=q^{n-1}+q^{n-3}+\cdots+q^{-n+3}+q^{-n+1}=\frac{q^{n}-q^{-n}}{q-q^{-1}}
$$

where $q$ is the quantum parameter The n -th quantum integer satisfies many relations. There is an important one which we are going to use.

Lemma 2.2. If $n \geq 0$, then

$$
[n+1]=[2][n]-[n-1]
$$

This lemma can be easily proved from the definition.
Definition 2.3. Let $m, n$ be non-negative integers. Consider a rectangle with $m$ marked points on upper side and $n$ points on lower side. If $m+n$ is even, connect each pair of points by an arc inside the rectangle such that each point is connected by an arc and no points are connected by two arcs while there are no arcs intersect with each other. Such diagram is called a simple Temperley-Lieb diagram or TL diagram for convenience. Isotopic diagrams are considered equivalent. If $m+n$ is odd, there is no simple TL diagram from $m$ to $n$ points.

A through string is a string connecting a point on the upper side to a point on the lower side. A cup connects two points on the upper side while a cap connects two points on the lower side.


Figure 2.1: A Temperley-Lieb diagram showing terminologies

Remark 2.4. Temperley-Lieb diagrams with $m=n$, which means the rectangle has same number of points on both the upper and lower edges, are called $n$ strand Temperley-Lieb diagrams.

The composition for TL diagrams is given by stacking. Let $f: n \rightarrow m$ denotes a TL diagram $f$ from $n$ points to $m$ points. For any two TL diagrams $f: n \rightarrow m$ and $g: m \rightarrow l$, the composition is defined as $g \circ f: n \rightarrow k$ which is stacking $g$ over $f$ with joined end points on attached edges. This stacking is "well-defined" as both the top edge of $f$ and bottom edge of $g$ have $m$ points. Since isotopic diagrams are equivalent, we can rescaling the diagram into a new diagram with same height as $f$ and $g$.


Figure 2.2: Composition of TL diagrams

### 2.2 The Temperley-Lieb Category

Before we define the Temperley-Lieb category, we first specify the ground field $\mathbb{K}=\mathbb{C}(q)$, where $q$ is a quantum parameter. That is $\mathbb{K}$ is the fraction field of complex polynomials over a quantum paramenter $q$.

Definition 2.5. The Temperley-Lieb category TL. The objects of TL are $n \in \mathbb{N}$. The morphisms are defined that for each $m, n \in \mathbb{N}$, $\operatorname{Hom}(n \rightarrow m)$ is the $\mathbb{K}$-linear span of simple TL diagrams with $n$ and $m$ points on bottom and top edges respectively. If there are closed loops formed after the composition of diagrams, replace each loop by inserting a multiplicative factor $[2]=q+q^{-1}$. The composition of morphisms is derived by extending the composition of TL diagrams bilinearly over $\mathbb{K}$.


Figure 2.3: Composition of two 5 strand TL diagrams

Proposition 2.6. The Temperley-Lieb category is a strict linear monoidal category equipped with tensor product $\otimes$. For objects $n$ and $m, n \otimes m=n+m$. For simple TL diagrams $f: n \rightarrow m$ and $g: k \rightarrow l$, the tensor product $f \otimes g$ : $n+k \rightarrow m+l$ is placing $f$ to the left of $g$. Tensor product for formal TL diagrams obtained by bilinearly extending tensor product for simple diagrams.

Proof. The tensor product $\otimes$ defined on objects is the same as addition. Then the tensor product is strictly associative with identity $i d: 0 \rightarrow 0$. So $\rho, \lambda, \alpha$ are all equalities. It is then a strict monoidal category. Since the TemperleyLieb category is base on vector space $\mathbb{K}$, the tensor product and composition are linear.


Figure 2.4: Tensor product of TL diagrams

According to the definition, any simple closed loop is removed. Therefore all morphisms $f \in \operatorname{Hom}(n \rightarrow m)$ are $\mathbb{K}$-linear combinations of simple TL diagrams without simple closed loops. The set of all such simple TL diagrams without simple closed loops from n to m points is then a basis of $\operatorname{Hom}(n \rightarrow m)$. The total number of simple TL diagrams without simple closed loops from $n$ to $m$ points is $c_{n, m}=\frac{1}{n+m+1}\binom{2(n+m)}{n+m}$, which can be easily proved according to the Definition 2.3. It then gives that for any $n, m \in \mathbb{N}$ such that $n+m$ is even, $\operatorname{Hom}(n \rightarrow m)$ is finite dimentional.

Remark 2.7. The identity morphism $\mathbb{1}_{n} \in \operatorname{Hom}(n \rightarrow n)$ is defined as the TL diagram with $n$ through strings. It is the identity in $\operatorname{Hom}(n \rightarrow n)$.

$\mathbb{1}$
Figure 2.5: Identity TL diagram in $\operatorname{Hom}(n \rightarrow n)$

There is a special subalgebra of Temperley-Lieb algebra called $n$ strand TemperleyLieb algebra, which will be used to define idempotents in TL.

Definition 2.8. The $\mathbf{n}$ strand Temperley-Lieb algebra $T L_{n}$ is the algebra over $\mathbb{C}(q)$ spanned by TL diagrams with $n$ points on both bottom and top edges. It is the morphism space $\operatorname{Hom}(n \rightarrow n)$ together with a multiplication defined as that for any morphisms $f, g \in \operatorname{Hom}(n \rightarrow n), f g=f \circ g$ defined by stacking f over g.

Definition 2.9. The multiplicative generator $e_{i}(i \in\{1,2, \ldots, n-1\}$ is the $n$ strand TL diagram with a cap connecting the $i$-th and the $i+1$-th points on the bottom edge and a cup connecting the $i$-th and the $i+1$-th points on the top edge while the rest points are connected by $n-2$ through-strings.


Figure 2.6: Multiplicative generators

Since any $\operatorname{Hom}(n \rightarrow m)$ is finite dimensional for $n+m$ even, so $\operatorname{Hom}(n \rightarrow n)$ is finite dimensional. In particular, the n strand Temperley-Lieb algebra $T L_{n}$ is generated by $\left\{e_{1}, e_{2}, \cdots, e_{n-1}\right\}$ with the following relations:

$$
\begin{gather*}
e_{i}^{2}=[2] e_{i} \\
e_{i} e_{j}=e_{j} e_{i}, \quad|i-j| \leq 2  \tag{2.1}\\
e_{i} e_{i+1} e_{i}=e_{i} \text { and } e_{i+1} e_{i} e_{i+1}=e_{i+1}
\end{gather*}
$$

Proposition 2.10. The multiplicative generators $\left\{e_{1}, e_{2}, \cdots, e_{n-1}\right\}$ generate $T L_{n}$.
Proof. To show $\left\{e_{1}, e_{2}, \cdots, e_{n-1}\right\}$ generates $T L_{n}$ it is enough to show any simple TL diagram without simple closed loops is a composition of generators. Any formal TL diagram can be obtained by taking a $\mathbb{K}$-linear combination.
The key steps are to show that the generators can make caps/caps within caps/cups and move caps/cups left or right. Making cups within cups can be achieved by $e_{i} e_{i+1} e_{i-1}$, which is:


$$
\begin{gathered}
e_{i} \\
e_{i+1} \\
e_{i-1}
\end{gathered}
$$

To move cups left, use $e_{i} e_{i+1}$ :


The details please refer to [Kau90].

The figure 2.7 is an example for obtaining a simple TL diagram by a composition of generators. It shows that the provided diagram $f$ satisfies $f=$ $e_{2} e_{3} e_{1} e_{4} e_{2} e_{3}$. In fact the relations 2.1 give that there are different ways to obtain TL diagrams by composition of generators.


Figure 2.7: Decomposition of a simple diagram

For convenience of further discussion about the TL category, we introduce the following operation on TL morphisms.

Definition 2.11. Anti-involution. For any TL diagram $f \in \operatorname{Hom}(n \rightarrow m)$, the anti-involution of $f$ is $\bar{f} \in \operatorname{Hom}(m \rightarrow n)$ with complex conjugated coefficients of $f$, replaced quantum parameter $q \rightarrow q^{-1}$ and reflected every simple TL diagram about the horizontal line at the middle.

Definition 2.12. Dual. For any TL diagram $f \in \operatorname{Hom}(n \rightarrow m)$, the dual of $f$ is $f^{*} \in \operatorname{Hom}(m \rightarrow n)$ given by rotating every simple diagrams in $f$ around its center by $\pi$.


Figure 2.8: Anti-involution


Figure 2.9: Dual

Definition 2.13. Markov trace. The Markov trace of $f$ is denoted as $\operatorname{tr}(f)$, where $f \in T L_{n}$. It is defined by a tracial closure: for any simple TL diagram, connecting the bottom $n$ points with their corresponding top points with $n$ disjoint arcs outside the TL diagram. Then every traced simple TL diagram becomes a collection of disjoint simple closed loops in the plane. If the total number of loops is $n$, then $\operatorname{tr}(f)=[2]^{n}$. Then formal diagrams are obtained by taking a $\mathbb{K}$-linear combination. Trace of $f$ is used instead of Markov trace of $f$ for convenience.

Remark 2.14. We define the k-strand (right) trace $\operatorname{tr}_{n}(f)$ for $f \in T L_{n}$ : for any simple TL diagram $f$, connecting the right $k$ points on the bottom edge with their corresponding top points by $k$ disjoint arcs outside the TL diagram. Let all arcs on the right hand side of the TL diagram. Formal diagrams are obtained by taking a $\mathbb{K}$-linear combination of partially traced simple TL diagrams.



Figure 2.10: Markov trace

Definition 2.15. Ideal and tensor ideal. The definition of idea in a category follows canonically from the definition of an ideal in a field. For an arbitrary ategory $\mathcal{C}$, an ideal of $\mathcal{C}$ is a collection of morphisms $\mathcal{I}$ such that for any morphsim $f \in \mathcal{I}$ and $g \in \operatorname{Mor}(\mathcal{C})$, the composition of $f$ and $g$ is still in $\mathcal{I}$. whenever such a composition is defined. Additionally, if the category $\mathcal{C}$ is a linear monoidal category, then an ideal $\mathcal{I}$ is a tensor ideal if whenever $f \in \mathcal{I}$ and $g \in \operatorname{Mor}(\mathcal{C})$, $f \otimes g \in \mathcal{I}$ and $g \otimes f \in \mathcal{I}$. For any pair of objects $X, Y \in \operatorname{Obj}(\mathcal{C}), \mathcal{I} \cap \operatorname{Hom}(n \rightarrow$ $m) \forall \mathcal{I}$ is a linear subspace of $\operatorname{Hom}(n \rightarrow m)$.

Here we introduce a proposition and corollary which will be useful in proofs of later theorems.

Proposition 2.16. Suppose $f, g$ are two simple TL daigrams such that $f \in$ $\operatorname{Hom}(n \rightarrow m)$ with $\alpha$ through strings and $g \in \operatorname{Hom}(m \rightarrow l)$ with $\beta$ through strings. If the composition $g f$ has $\gamma$ through strings, then $\gamma \leq \min (\alpha, \beta)$.

Proof. We have $f \in \operatorname{Hom}(n \rightarrow m)$ with $\alpha$ through strings. So there are $n-\alpha$ points at the bottom of $f$ connected by caps. Similarly $g \in \operatorname{Hom}(m \rightarrow l)$ with $\beta$ through strings gives that there are $l-\beta$ points at the top of $g$ connected by cups. It is obvious that after the composition $g f$, the $n-\alpha$ points at the bottom connected by caps are still connected by caps and so are the $l-\beta$ points at the top. Thus the points connected by strings at the bottom of $g f$ are less than $n-(n-\alpha)=\alpha$ while such points at the top of $g f$ are also less than $l-(l-\beta)=\beta$.

Therefore the total number of through strings $\gamma$ satisfies $\gamma \leq \alpha$ and $\gamma \leq \beta$, which gives $\gamma \leq \min (\alpha, \beta)$.

Then we consider the formal TL diagrams. For convenience we are going to define the total number of strings in formal TL diagram.

Definition 2.17. An arbitrary formal TL diagram $f$ is a $\mathbb{K}$-linear combination of simple TL diagrams. Denote all such simple diagrams $f_{1}, f_{2}, \cdots$. If there are $\alpha_{i}$ through strings in $f_{i}$, then the total number of through strings in $f$ is $\alpha_{f}=\max \left(\alpha_{1}, \alpha_{2}, \cdots\right)$.

Then combining the definition of ideals with proposition 2.16 gives the following corollary.

Corollary 2.18. The set of all formal TL diagrams $\mathcal{I}_{\beta}:=\left\{f: f \in \operatorname{Mor}(T L), \alpha_{f} \leq\right.$ $\beta\}$ forms an ideal in TL. The proof is obvious since the composition of any $f \in \mathcal{I}_{\beta}$ and arbitrary $g \in \operatorname{Mor}(T L)$ will not increase the total number of strings by Proposition 2.16.

Remark 2.19. $\mathcal{I}_{\beta}$ is not a tensor ideal. The tensor composition of any $f \in \mathcal{I}_{\beta}$ and $\mathbb{1}_{n}$ will increase the total number of through strings in $f$ by $n$.

### 2.3 Jones-Wenzl idempotents

First we are going to introduce a special set of simple diagrams.
Definition 2.20. Suppose $n \geq 2$ and $1 \leq i<n$, define $C_{i, n} \in \operatorname{Hom}(n \rightarrow n-2)$ to be the simple TL diagram that is the combination of $i-1$ through strings, a cap at the bottom edge and $n-i-3$ through strings after that. Similarly define $C^{i, n} \in \operatorname{Hom}(n-2 \rightarrow n)$ to be the simple TL diagram that is the combination of $i-1$ through strings, a cup at the top edge and then another $n-i-3$ through strings.

Theorem 2.21. In the $n$ strand Temperley-Lieb algebra $T L_{n}$ there is a special endomorphism $j_{n} \in \operatorname{Hom}(n \rightarrow n)$, satisfying

$$
\begin{gather*}
j_{n} \neq 0 \\
j_{n} j_{n}=j_{n}  \tag{2.2}\\
C_{i, n} j_{n}=j_{n} C^{i, n}=0
\end{gather*}
$$



Figure 2.11: $C_{i, n}$ and $C^{i, n}$
called the Jones-Wenzl idempotent. Such an idempotent can be constructed uniquely by a recursive formula known as Wenzl's formula:

$$
j_{0}=\text { empty simple TL diagram }
$$



The equation 2.3 is therefore called Wenzl's recurrence formula.
Before proving the theorem, some lemmas and propositions will be introduced and proved first.

Lemma 2.22. The Jones-Wenzl idempotent $j_{n}$ always has an identity component $\mathbb{1}_{n}$ with coefficient 1 . Then the Jones-Wenzl idempotent $j_{n}=\mathbb{1}_{n}+g_{n}$ where $g_{n}$ is a linear combination of non-identity simple diagrams.

Proof. Suppose $j_{n}=a \mathbb{1}_{n}+g_{n}$ where $a \in \mathbb{C}$ and $g_{n}$ is a formal TL diagram composed by non-identity simple diagrams linearly. Then

$$
\begin{aligned}
j_{n} j_{n} & =j_{n}\left(a \mathbb{1}_{n}+g_{n}\right) \\
& =a j_{n} \mathbb{1}_{n}+j_{n} g_{n} \\
& =a j_{n}
\end{aligned}
$$

since $g_{n}$ is a sum of non-identity simple diagrams and relation 2.2 we then have $a=1$ as we want.

Proposition 2.23. (Absorption law) For Jones-Wenzl idempotent $j_{n}$,

$$
\left(j_{n} \otimes \mathbb{1}_{1}\right) \circ j_{n+1}=j_{n+1} \circ\left(j_{n} \otimes \mathbb{1}_{1}\right)=j_{n+1}
$$

Proof. By Lemma 2.22, $j_{n}=\mathbb{1}_{n}+g_{n}$, where $g_{n}$ is a linear combination of nonidentity simple diagrams. Let $g_{n}=\sum_{i} a_{i} h_{i}$ be a linear combination where each $h_{i}$ is non-identity simple diagram. Then we have:


Then we observe that for any non-identity simple diagram $h_{i}$, it must contain at least one cap at the bottom. By relation (2.2), $C_{i, n+1} j_{n+1}=0$, then $\left(h_{i} \otimes \mathbb{1}_{1}\right) \circ$ $j_{n+1}=0$.

Then,


Use a similar method we have $j_{n+1} \circ\left(j_{n} \otimes \mathbb{1}_{1}\right)=0$
We are going to prove the Jones-Wenzl idempotents theorem by induction.
Proof. The Jones-Wenzl idempotents $j_{0}$ and $j_{1}$ obviously agree with the Wenzl recurrence formula. They also satisfy $j_{n} \neq 0$ and $j_{n} j_{n}=j_{n}$ while $C_{i, n}$ and $C^{i, n}$ do not exist for $n<2$. For arbitrary $n \geq 1$, assume $j_{n-1}$ and $j_{n}$ satisfy relations (2.2) while $j_{n-1}, j_{n}$ and $j_{n+1}$ all derived from the Wenzl recurrence formula 2.3.

It is enough to show $j_{n+1}$ also satisfies relations (2.2).
(i) To show $j_{n+1} \neq 0$ :

By the induction hypothesis, $j_{n}$ satisfies relations (2.2). Then by Lemma 2.22, $j_{n}$ has an identity component $\mathbb{1}_{n+1}$ with coefficient 1 , so is the first term in the Wenzl recurrence formula 2.3. For the second term, observe that there is a multiplicative generator $e_{n}$ between the dashed lines:

where $e_{n}$ has $n-1$ through strings. The second term is a composition including $e_{n}$. By Corollary 2.18, this term has at most $n-1$ through strings which cannot contain the identity component $\mathbb{1}_{n+1}$. Thus the coefficient of $\mathbb{1}$ in $j_{n+1}$ is exactly 1 and $j_{n+1}$ cannot be zero.
(ii) To show $j_{n+1}^{2}=j_{n+1}$ :

According to Wenzl's formula, we have



Then by the induction hypothesis, $j_{n}^{2}=j_{n}$


The TL diagram in dashed box is a 1 -strand trace $\operatorname{tr}_{1}\left(j_{n}\right)$ by definition. Using the induction hypothesis, $j_{n}$ follows from Wenzl's formula.



Then $j_{n-1}^{2}=j_{n-1}$ according to induction hypothesis gives

$$
\begin{aligned}
& \operatorname{tr}_{1}\left(j_{n}\right)=[2] \begin{array}{|c|}
\hline \begin{array}{|c|}
\hline \cdots \\
\left\lvert\, \begin{array}{l}
j_{n-1} \\
j_{n}
\end{array}\right. \\
\hline \cdots \\
\hline
\end{array} \\
\hline \begin{array}{|c|c|}
\hline
\end{array} \\
j_{n-1} \\
\hline
\end{array} \\
& =\left([2]-\frac{[n-1]}{[n]}\right) j_{n-1} \\
& =\left(\frac{[2][n]-[n-1]}{[n]}\right) j_{n-1} \\
& =\left(\frac{[n+1]}{[n]}\right) j_{n-1}
\end{aligned}
$$

by lemma 2.2. Then substitute this into the equation of $j_{n+1}$ we get


Then by the absorption law,


Then by Wenzl's formula, the equation gives

$$
j_{n+1}^{2}=j_{n+1}
$$

(iii) We now need to show $C_{i, n+1} j_{n+1}=j_{n+1} C^{i, n+1}=0$ :

After $C_{i, n+1} j_{n+1}=0$ has been proved, it is easy to prove $j_{n+1} C^{i, n+1}=0$. To show $C_{i, n+1} j_{n+1}=0$, two different cases need to be considered.
For $C_{i, n+1}$, if $i=n, C_{i, n+1} j_{n+1}=0$ by Wenzl's formula:


In (ii) we calculated the 1-strand trace $\operatorname{tr}_{1}\left(j_{n}\right)$. Substituting it into $C_{n, n+1} j_{n+1}$ we then have


By the absorption law


If $i \neq n$, then $1 \leq i \leq n-1$ which gives


By the induction hypothesis, $C_{i, n} j_{n}=0$. Both the first term and the second term are the zero diagram. Therefore we conclude that $C_{i, n+1} j_{n+1}=0$. The same method then proves that $j_{n+1} C_{i, n+1}=0$.
At the end, we need to show the uniqueness of idempotents.
Suppose that there are two Jones-Wenzl idempotents $j_{n}$ and $j_{n}^{\prime}$ in $\operatorname{Hom}(n \rightarrow n)$ satisfies relations (2.2). By Lemma 2.22, let $j_{n}=\mathbb{1}_{n}+g_{n}$ and $j_{n}^{\prime}=\mathbb{1}_{n}+g_{n}^{\prime}$. Then we have $j_{n}^{\prime} g_{n}=j_{n} g_{n}^{\prime}=0$ as $g_{n}$ and $g_{n}^{\prime}$ are both linear combinations of non-identity simple diagrams. Thus

$$
\begin{aligned}
j_{n} & =j_{n} \mathbb{1}_{n} \\
& =j_{n} \mathbb{1}_{n}+j_{n} g_{n}^{\prime} \\
& =j_{n}\left(\mathbb{1}_{n}+g_{n}^{\prime}\right) \\
& =\left(\mathbb{1}_{n}+g_{n}\right) j_{n}^{\prime} \\
& =\mathbb{1}_{n} j_{n}^{\prime} \\
& =j_{n}^{\prime}
\end{aligned}
$$

Therefore Jones-Wenzl idempotents $j_{n}$ is unique for each $n$.

Remark 2.24. In the proof of the Jones-Wenzl idempotent theorem, we proved and used the proposition

$$
\operatorname{tr}_{1}\left(j_{n}\right)=\frac{[n+1]}{[n]} j_{n-1}
$$

This proposition can be extend to the complete Markov trace of $j_{n}$.

$$
\operatorname{tr}\left(j_{n}\right)=[n+1] .
$$

This theorem is easily proved by induction. For the empty TL diagram $j_{0}$, $\operatorname{tr}\left(j_{0}\right)=[2]^{0}=[1]$.
Then assume $\operatorname{tr}\left(j_{n}\right)=[n+1]$ is true for $n \leq k$, Then for $k+1$


The simple diagram in dashed box is $\operatorname{tr}_{1}\left(j_{k+1}\right)$ which is substituted by $\frac{[n+1]}{[n]} j_{n-1}$.

## Chapter 3

## Karoubi envelope and its semisimplicity

### 3.1 Karoubi envelope

After proving the existence of Jones-Wenzl idempotents, we would like to construct a new category in which the Jones-Wenzl idempotents are included as objects. So we need to define and take the Karoubi envelope.
One important property for the Temperley-Lieb idempotents $j_{n}$ is $j_{n}^{2}=j_{n}$. Recall that if an endomorphism $p$ satisfies $p^{2}=p, p$ is an idempotent. Thus TemperleyLieb idempotents are idempotents. If $p$ itself is an idempotent then $1-p$ would be idempotent at the same time.
For any category, the definition of idempotent $p$ always make sense. Then in any additive category $1-p$ also make sense. However, the image of $p$ and $1-p$ does not make sense in some category, even the category is additive. The Karoubi envelope of a category $\mathcal{C}$ ensures that no matter whether the image of $p$ in the original category $\mathcal{C}$ exists or not, every idempotent in the Karoubi envelop $p: \in \operatorname{Hom}(X \rightarrow X)$ where $X \in \operatorname{Obj}(\mathcal{C})$ has an image and if $\mathcal{C}$ is additive then $X \cong \operatorname{Im}(p) \oplus \operatorname{Im}(1-p)$. Thus by using the Karoubi envelope, we can decompose objects which are not able to be decomposed in the original category $\mathcal{C}$.

Definition 3.1. Given an arbitrary category $\mathcal{C}$, the Karoubi envelope $\mathcal{K} \operatorname{ar}(\mathcal{C})$ is constructed with properties:

1. The objects in Karoubi envelope $\operatorname{Kar}(\mathcal{C})$ are ordered pairs $(X, p)$ where $X$ is an arbitrary object in category $\mathcal{C}$ with $p \in \operatorname{Hom}(X \rightarrow X)$ an idempotent.
2. Let $(X, p)$ and $(Y, q)$ be objects in $\mathcal{K} \operatorname{ar}(\mathcal{C})$ then the set of morphisms

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$\operatorname{Hom}((X, p) \rightarrow(Y, q))$ is a collection of all morphisms $f \in \operatorname{Hom}(X \rightarrow Y)$ such that $f p=f=q f$. The composition is the same as in the category $\mathcal{C}$.

The identity morphism $\mathbb{1}_{p}$ for an object $(X, p)$ is $p$ itself. Consider an embedding functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{K} \operatorname{ar}(\mathcal{C})$ such that for any object $X$ in $\mathcal{C}, \mathcal{F}(X)=\left(X, \mathbb{1}_{X}\right)$. Recall that identity morphism is obviously an idempotent. Then for any pair of objects $X, Y \in \operatorname{Obj}(\mathcal{C})$, the corresponding morphism in Karoubi envelope is $\operatorname{Hom}_{\mathcal{K a r}(\mathcal{C})}\left(\left(X, \mathbb{1}_{X}\right) \rightarrow\left(Y, \mathbb{1}_{Y}\right)\right)$. By definition $\operatorname{Hom}_{\mathcal{K a r}(\mathcal{C})}\left(\left(X, \mathbb{1}_{X}\right) \rightarrow\left(Y, \mathbb{1}_{Y}\right)\right)$ is the collection of $f \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ such that $f \mathbb{1}_{X}=f=\mathbb{1}_{Y} f$, which is equivalent to every $f$ in $\operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$. Therefore $\operatorname{Hom}_{\mathcal{K a r}(\mathcal{C})}\left(\left(X, \mathbb{1}_{X}\right) \rightarrow\left(Y, \mathbb{1}_{Y}\right)\right)=$ $\operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ and the functor $\mathcal{F}$ is faithful.

Lemma 3.2. Suppose we have an arbitrary category $\mathcal{C}$ and the corresponding Karoubi envelope $\mathcal{K} \operatorname{ar}(\mathcal{C})$, then for any pair of idempotents $p, q$ with $p: X \rightarrow X$ and $q: Y \rightarrow Y$ where $X, Y \in \operatorname{Obj}(\mathcal{C})$, there is a surjective map between the morphisms

$$
g: \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y) \rightarrow \operatorname{Hom}_{\mathcal{K a r}(\mathcal{C})}((X, p) \rightarrow(Y, q))
$$

defined by for each $f \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$,

$$
g(f)=q f p
$$

Proof. Let $f$ be an arbitrary morphism in $\operatorname{Hom}_{\mathcal{K} \operatorname{ar}(\mathcal{C})}((X, p) \rightarrow(Y, q))$. Then by definition, $f \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ and $f p=f=q f$. So $f=q f p$, so $g$ is surjective. Then we need to show such $g(f)$ defines a map from $\operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ to $\operatorname{Hom}_{\mathcal{K a r}(\mathcal{C})}((X, p) \rightarrow$ $(Y, q))$.
Let $f$ be an arbitrary morphism in $\operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$. Then $g(f)=q f p$ is still a morphism from $X$ to $Y$, so $g(f) \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$. Then by calculation

$$
g(f) p=q f p p=q f p=g(f)
$$

and

$$
q g(f)=q q f p=q f p=g(f)
$$

Therefore $g(f) \in \operatorname{Hom}_{\mathcal{K} \operatorname{ar}(\mathcal{C})}((X, p) \rightarrow(Y, q))$ as we need.
So combining with this lemma, the category $\mathcal{C}$ is embedded fully and faithfully into $\operatorname{Kar}(\mathcal{C})$.
The Karoubi envelope of a category $\mathcal{C}$ ensure that for such category, all idempotent morphisms split in $\operatorname{Kar}(\mathcal{C})$. Now we assume that the category $\mathcal{C}$ is an
additive category and the direct sum is defined in $\mathcal{C}$, then we have the following properties.

Proposition 3.3. Suppose we have an additive category $\mathcal{C}$ and $p: X \rightarrow X$ is an endomorphism for some object $X$ in $\mathcal{C}$.

1. Suppose $p$ is an idempotent on $X$ then $1-p$ is also an idempotent.
2. In $\operatorname{Kar}(\mathcal{C})$, we have $\left(X, \mathbb{1}_{X}\right) \cong(X, p) \oplus(X, 1-p)$.

Proof. (1) In general to show $1-p$ is an idempotent, we just need to show $(1-p)^{2}=(1-p)$. Then

$$
\begin{aligned}
(1-p)^{2} & =1-2 p+p^{2} \\
& =1-2 p+p \\
& =1-p
\end{aligned}
$$

(2) To show there is an isomorphism between $\mathcal{K a r}(\mathcal{C}),\left(X, \mathbb{1}_{X}\right)$ and $(X, p) \oplus(X, 1-$ $p)$, we just need to define $f: \mathcal{K} \operatorname{ar}(\mathcal{C}),\left(X, \mathbb{1}_{X}\right) \rightarrow(X, p) \oplus(X, 1-p)$ and its inverse $g:(X, p) \oplus(X, 1-p) \rightarrow\left(X, \mathbb{1}_{X}\right)$ such that both $f g$ and $g f$ give identity maps. Define $f=\left[\begin{array}{c}p \\ 1-p\end{array}\right]$ and $g=\left[\begin{array}{ll}p & 1-p\end{array}\right]$ Then

$$
\begin{aligned}
f g & =\left[\begin{array}{c}
p \\
1-p
\end{array}\right]\left[\begin{array}{ll}
p & 1-p
\end{array}\right] \\
& =\left[\begin{array}{cc}
p^{2} & p(1-p) \\
(1-p) p & (1-p)^{2}
\end{array}\right] \\
& =\left[\begin{array}{cc}
p & 0 \\
0 & 1-p
\end{array}\right]
\end{aligned}
$$

As $p$ and $1-p$ are identity morphisms on objects $(X, p)$ and $(X, 1-p), f g$ is then identity morphism on $(X, p) \oplus(X, 1-p)$.

$$
\begin{aligned}
g f & =\left[\begin{array}{ll}
p & 1-p
\end{array}\right]\left[\begin{array}{c}
p \\
1-p
\end{array}\right] \\
& =p^{2}+(1-p)^{2} \\
& =p+1-p \\
& =1
\end{aligned}
$$

which is the identity morphism on $\left(X, \mathbb{1}_{X}\right)$. Therefore $\left(X, \mathbb{1}_{X}\right) \cong(X, p) \oplus(X, 1-$ p).

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Then consider the Karoubi envelope of the Temperley-Lieb category $\operatorname{Kar}(\mathcal{T} \mathcal{L})$. By definitions shown above, the objects in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$ are of the form $(n, p)$ where $n$ is an object in $T L$ and $p$ is an idempotent (not necessary to be Jones-Wenzl) for $n$ points. The morphisms between a pair of objects ( $n, p$ ) and ( $m, q$ ) are formal $T L$ diagrams $f: n \rightarrow m$ with property $f p=f=q f$.

Proposition 3.4. The Karoubi envelope is a strict $\mathbb{K}$-linear monoidal category with tensor product $\otimes$.

Proof. From property 2.6, the Temperley-Lieb category is a strict monoidal category and it equips a tensor product $\otimes$. Such tensor product then can be extended to Karoubi envelop $\operatorname{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$. Tensor product defined in $T L$ preserves idempotent, so for objects $(n, p)$ and $(m, q)$ in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$, we have $(n, p) \otimes(m, q)=$ $(n+m, p \otimes q)$ which gives that the tensor product of two objects in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$ is still an object in $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$. The zero TL diagram is the identity for tensor product and $\otimes$ is strictly associative for TL diagrams, and so the coherence conditions of tensor product hold.
For any morphisms $f, g: n \rightarrow m$ in $\operatorname{Mor}(\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L}))$ and $a, b \in \mathbb{K}, a f+b g: n \rightarrow m$ still satisfies $(a f+b g) p=(a f+b g)=q(a f+b g)$. Since $\otimes$ is bilinear, we have the Karoubi envelope is a strict $\mathbb{K}$-linear monoidal category.

### 3.2 Semisimplicity of $\operatorname{Kar}(\mathcal{T} \mathcal{L})$

As we already shown that all idempotent morphisms split in the Karoubi envelope $\mathcal{K} \operatorname{ar}(\mathcal{C})$, all idempotent morphism in $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ also split. To find such decomposition as a direct sum, let us consider Theorem 2.21 of Jones-Wenzl idempotents.

Theorem 3.5. In the Karoubi envelope $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$, let $j_{n}$ be the $n$-th Jones-Wenzl idempotent. There is an isomorphism $\left(n, j_{n}\right) \otimes\left(1, j_{1}\right) \cong\left(n+1, j_{n+1}\right) \oplus\left(n-1, j_{n-1}\right)$ which can be representatively shown as

with

such that $f: j_{n} \otimes j_{1} \rightarrow j_{n+1} \oplus j_{n-1}$ and $g: j_{n+1} \oplus j_{n-1} \rightarrow j_{n} \otimes j_{1}$ are isomorphisms

Remark 3.6. For objects in Karoubi envelope $\operatorname{Kar}(\mathcal{T} \mathcal{L}), j_{n}$ is used instead of $\left(n, j_{n}\right)$ for convenience. This is permitted since the morphisms in the Karoubi envelope $\operatorname{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ are actually all formal TL diagrams.

Proof. To show $\left(n, j_{n}\right) \otimes\left(1, j_{1}\right) \cong\left(n+1, j_{n+1}\right) \oplus\left(n-1, j_{n-1}\right)$ with defined $f$ and $g$, it is enough to show both $f g$ and $g f$ are identity morphisms.

Then by given definition


It is easy to simplify the first matrix entries by using the property of idempotent that $j_{n}^{2}=j_{n}$ for every $j_{n}$ and the absorption law 2.23. For the last matrix entry, after applying property $j_{n}^{2}=j_{n}$ and absorption law, it is then $\frac{[n]}{[n+1]} \operatorname{tr}_{1}\left(j_{n}\right)$ which
still is not simple enough. Recall that $\operatorname{tr}_{1}\left(j_{n}\right)=\frac{[n+1]}{[n]} j_{n-1}$. Therefore $f g$ becomes

which is the identity map on $\left(n+1, j_{n+1}\right) \oplus\left(n-1, j_{n-1}\right)$. For $g f$,


Then according to the Jones-Wenzl idempotent theorem 2.21,

which is the identity map on $\left(n, j_{n}\right) \otimes\left(1, j_{1}\right)$. Therefore $f$ and $g$ are isomorphisms and $\left(n, j_{n}\right) \otimes\left(1, j_{1}\right) \cong\left(n+1, j_{n+1}\right) \oplus\left(n-1, j_{n-1}\right)$.

We are going to talk about the morphisms $f: j_{a} \otimes j_{b} \rightarrow j_{c}$ in general. New notations will be introduced.

Definition 3.7. If ( $a, b, c$ ) with $a, b, c \in \mathbb{N}$ satisfies triangle inequalities $a+b \geq c$, $b+c \geq a+c \geq b$ and $a+b+c \equiv 0 \bmod 2$, then the triple is called admissible.

The admissible triples in $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ have good properties which shows $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ is semisimple. Before showing the semisimplicity property, we need to define simple object.

Definition 3.8. A simple object $X$ in a $\mathbb{K}$-linear category $\mathcal{C}$ is an object in $\operatorname{Obj}(\mathcal{C})$ with $\operatorname{Hom}_{\mathbb{K}}(X \rightarrow X)=\operatorname{span}_{\mathbb{K}}\left(\mathbb{1}_{X}\right)$. If for any pair of objects $X$ and $Y$ in $S \subset \mathcal{C}, \operatorname{Hom}_{\mathbb{K}}(X \rightarrow Y)=0$, then all objects in $S$ are called disjoint simple objects.

Theorem 3.9. In Karoubi envelope $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$, if $(a, b, c)$ is an admissible triple, the dimension of the morphism space $\operatorname{dim}\left(\operatorname{Hom}_{\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})}\left(j_{a} \otimes j_{b} \rightarrow j_{c}\right)\right)=1$, otherwise $\operatorname{dim}\left(\operatorname{Hom}_{\mathcal{K a r}(\mathcal{T} \mathcal{L})}\left(j_{a} \otimes j_{b} \rightarrow j_{c}\right)\right)=0$

Proof. Recall that a formal TL diagram is just a linear combination of simple TL diagrams. From lemma 3.2, without loss of generality define $f=j_{c} \circ g \circ\left(j_{a} \otimes j_{b}\right)$ with morphisms $g \in \operatorname{Hom}_{T L}(a+b \rightarrow c)$ as simple TL diagrams without loops. we need to show that $f$ is zero when triple ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ) is not admissible for all TL diagrams as the linear combination of simple TL diagrams $g$ and when triple ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ) is admissible $f$ is non zero for one unique $g$.

1. Suppose $a+b+c$ is odd, by definition of Temperley-Lieb category there is no such $g$. Then $f$ as the composition $f=j_{c} \circ g \circ\left(j_{a} \otimes j_{b}\right)$ does not exists and $\operatorname{Hom}_{\mathcal{K a r ( \mathcal { T } )}}\left(j_{a} \otimes j_{b}, j_{c}\right)$ only contains the zero map.
2. Suppose $a+b+c$ is even and $a+b<c$. Then the simple diagram $g$ has at most $a+b$ through strings while there are $c$ points at the top. It then has at least one cup at the top edge. Then $j_{c} \circ f=0$ by property of the Jones-Wenzl idempotents. So $f=j_{c} \circ g \circ\left(j_{a} \otimes j_{b}\right)=0$ is the only permissible morphism.
3. Suppose $a+b+c$ is even and $a+b \geq c$. If the simple diagram $g$ has fewer than $c$ strings, there will be cups at the top edge again, then $j_{c} \circ g=0$ and consequently $f=j_{c} \circ g \circ\left(j_{a} \otimes j_{b}\right)=0$. Then $g$ must have exactly $c$ through strings as it has at most $c$ strings. There will be $\frac{a+b-c}{2}$ caps at the bottom edge. If any cap connects two strand of $j_{a}$, then $f \circ\left(j_{a} \otimes j_{b}\right)=0$, so is for $j_{b}$. Therefore, for $\frac{a+b-c}{2}$ caps at the bottom edge of $g$, they must connect exactly the $\frac{a+b-c}{2}$ right strands of $j_{a}$ and the $\frac{a+b-c}{2}$ left strands of $j_{b}$ without intersection between any pairs of caps. Then $g$ is unique up to isotopy, with $\frac{a+b-c}{2}$ caps, $\frac{a+c-b}{2}$ through strings connecting the $\frac{a+c-b}{2}$ left strands of both $j_{a}$ and $j_{c}$ and $\frac{b+c-a}{2}$ through strings connecting the $\frac{b+c-a}{2}$ right strands of both $j_{b}$ and $j_{c}$.


Figure 3.1: Nonzero morphism $f$ as the composition $f=j_{c} g\left(j_{a} \otimes j_{b}\right)$.

The permissible $f$ is shown in Fig. 3.1 where the simple TL diagram between dashed lines is corresponding $g$. From the figure, $g$ exists requires that

$$
\begin{array}{ll}
a \geq \frac{a+c-b}{2} & a \geq \frac{a+b-c}{2} \\
b \geq \frac{a+b-c}{2} & b \geq \frac{b+c-a}{2} \\
c \geq \frac{a+c-b}{2} & c \geq \frac{b+c-a}{2}
\end{array}
$$

and there are equivalent to

$$
a+b \geq c, \quad b+c \geq a \quad \text { and } \quad a+c \geq b .
$$

Thus $(a, b, c)$ is a admissible triple. After define the unique $g$ as above, the rest is to test that $f=j_{c} g\left(j_{a} \otimes j_{b}\right)$ is non-zero.
Recall lemma 2.22, every Jones-Wenzl idepotent $j_{n}=\mathbb{1}_{n}+p_{n}$. Thus we have $j_{a}=\mathbb{1}_{a}+p_{a}, j_{b}=\mathbb{1}_{b}+p_{b}$ and $j_{c}=\mathbb{1}_{c}+p_{c}$ and $f$ then becomes

$$
\begin{align*}
f & =j_{c} g\left(j_{a} \otimes j_{b}\right)  \tag{3.1}\\
& =g+g\left(p_{a} \otimes \mathbb{1}_{b}\right)+g\left(\mathbb{1}_{a} \otimes p_{b}\right)+g\left(\mathbb{1}_{a} \otimes \mathbb{1}_{b}\right)+p_{c} f\left(j_{a} \otimes j_{b}\right)
\end{align*}
$$

To show $f$ is non-zero, it is enough to show the coefficient of $g$ in $f$ is non-zero. In face the coefficient is 1 here. Define $A$ to be the group of $a$ left points at bottom edge of $g, B$ be the group of $b$ right points at bottom edge and $C$ the $c$ points at top edge. The figure 3.1 shows that every string in $g$ connects points in different group. However, all simple diagram components in $p_{a}, p_{b}$ and $p_{c}$ have caps and cups. Then their composition with $g$ connects points in same group $A$, $B$ or $C$ which do not contain components as $g$. Since every term beside the first one in (3.1) contains $p_{a}, p_{b}$ or $p_{c}$, so there is no component $g$ in these terms. The coefficient of $g$ in $f$ is 1 .
Since permissible $g$ is unique, $\operatorname{dim}\left(\operatorname{Hom}_{\mathcal{K} \operatorname{ar}(\mathcal{T \mathcal { L }})}\left(j_{a} \otimes j_{b}, j_{c}\right)\right)=1$.

In order to show the Karoubi envelop $\operatorname{Kar}(\mathcal{T} \mathcal{L})$ is a semisimple category, we need to find simple objects first, which are Jones-Wenzl idempotents. By using Theorem 3.9, we can prove Jones-Wenzl idempotents are simple.

Corollary 3.10. The Jones-Wenzl idempotents are simple objects and the collection of $\left(n, j_{n}\right)$ is a collection of disjoint simple objects in $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$.

Proof. For any Jones-Wenzl idempotent $j_{n}, j_{n} \otimes j_{0}=j_{0}$ since $j_{0}$ is identity in tensor product. Then $\operatorname{Hom}_{\mathcal{K a r}(\mathcal{T L})}\left(j_{n}, j_{n}\right)=\operatorname{Hom}_{\mathcal{K a r}(\mathcal{T \mathcal { L }})}\left(j_{n} \otimes j_{0}, j_{n}\right)$, where by theorem 3.9 and the triple ( $\mathrm{n}, 0, \mathrm{n}$ ) is always admissible for any $n \in \mathbb{N} \operatorname{Hom}_{\mathcal{K a r}(\mathcal{T \mathcal { L }})}\left(j_{n} \otimes\right.$ $j_{0}, j_{n}$ is 1 dimensional. Then $\operatorname{Hom}_{\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})}\left(j_{n}, j_{n}\right)$ is the space spanned by $j_{n} \mathbb{1}_{n} j_{n}=$ $j_{n}$ which is the identity in $\operatorname{Hom}_{\mathcal{K} \operatorname{ar}(\mathcal{T})}\left(j_{n}, j_{n}\right)$. Jones-Wenzl idempotents $j_{n}$ are simple objects.
For $n \neq m, \operatorname{Hom}_{\mathcal{K a r}(\mathcal{T} \mathcal{L})}\left(j_{n}, j_{m}\right)=\operatorname{Hom}_{\mathcal{K a r}(\mathcal{T L})}\left(j_{n} \otimes j_{0}, j_{m}\right)$. The triple $(n, 0, m)$ is not admissible, so $\operatorname{Hom}_{\mathcal{K a r}(\mathcal{T} \mathcal{L})}\left(j_{n}, j_{m}\right)=\{0\}$ and the collection of $\left(n, j_{n}\right)$ is a collection of disjoint simple objects in $\operatorname{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$.

After we find enough simple objects in $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ to form a collection of disjoint simple objects, to show the Karoubi envelop $\operatorname{Kar}(\mathcal{T} \mathcal{K})$ is (object) semisiple, it is equivalent to show every object in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$ is isomorphic to a direct sum of simple objects.

Theorem 3.11. Any object $(n, p) \in \mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ with $p \in \operatorname{Hom}_{T L}(n \rightarrow n)$ is isomorphic to some direct sum of Jones-Wenzl idempotents:

$$
(n, p) \cong \bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)
$$

Proof. We start from showing for an arbitrary natural number $n,\left(n, \mathbb{1}_{n}\right)$ is isomorphic to a direct sum of $\left(i, j_{i}\right)$ 's, that is:

$$
\left(n, \mathbb{1}_{n}\right) \cong \bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right), \text { for some } m_{i} \in \mathbb{N}
$$

Using induction:

1. we have that $\left(0, \mathbb{1}_{0}\right) \cong\left(0, j_{0}\right)$ where $j_{0}=\mathbb{1}_{0}$ by definition.
2. Suppose that $\left(k-1, \mathbb{1}_{k-1}\right) \cong \bigoplus_{i=0}^{k-1} m_{i}\left(i, j_{i}\right)$ is true. Then for $k$, by distributive property of $\otimes$ and $\oplus$ and theorem 3.5

$$
\begin{aligned}
\left(k, \mathbb{1}_{k}\right) & \cong\left(k-1, \mathbb{1}_{k-1}\right) \otimes\left(1, j_{1}\right) \\
& \cong\left(\bigoplus_{i=0}^{k-1} m_{i}\left(i, j_{i}\right)\right) \otimes\left(1, j_{1}\right) \\
& \cong \bigoplus_{i=0}^{k-1} m_{i}\left(\left(i, j_{i}\right) \otimes\left(1, j_{1}\right)\right) \\
& \cong m_{0}\left(1, j_{1}\right) \oplus \bigoplus_{i=1}^{k-1} m_{i}\left(\left(i+1, j_{i+1}\right) \oplus\left(i-1, j_{i-1}\right)\right)
\end{aligned}
$$

Then by induction $\left(n, \mathbb{1}_{n}\right) \cong \bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)$. is proved. Therefore we can define such function for each $n \in \mathbb{N}$ :

$$
\phi_{n}:\left(n, \mathbb{1}_{n}\right) \rightarrow \bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)
$$

the function $\phi_{n}$ is an isomorphism. Now consider a new category $\operatorname{K} \operatorname{ar}(\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L}))$. Then we claim:

$$
\begin{equation*}
\left(\left(n, \mathbb{1}_{n}\right), p\right) \cong\left(\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right), \phi_{n} p \phi_{n}^{-1}\right) \tag{3.2}
\end{equation*}
$$

where both $\left(\left(n, \mathbb{1}_{n}\right), p\right)$ and $\left(\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right), \phi_{n} p \phi_{n}^{-1}\right)$ are objects in the double Karoubi envelop $\mathcal{K} \operatorname{ar}(\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L}))$.

For $\left(\left(n, \mathbb{1}_{n}\right), p\right)$, it is required that $\left(n, \mathbb{1}_{n}\right)$ is an object in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$ and $p \in$ $\operatorname{Hom}\left(\left(n, \mathbb{1}_{n}\right) \rightarrow\left(n, \mathbb{1}_{n}\right)\right)$ is an idempotent. Since $\mathbb{1}_{n}$ is an idempotent, we have $\left(n, \mathbb{1}_{n}\right) \in \operatorname{Kar}(\mathcal{T} \mathcal{L})$. By definition, the morphism $p \in \operatorname{Hom}\left(\left(n, \mathbb{1}_{n}\right) \rightarrow\left(n, \mathbb{1}_{n}\right)\right)$ is just $p \in \operatorname{Hom}(n \rightarrow n)$ and $p \mathbb{1}_{n}=p=\mathbb{1}_{n} p$, which is true for $p$.
Similarly, for $\left(\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right), \phi_{n} p \phi_{n}^{-1}\right)$, we have $\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)$ so it is an object in $\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$. Then by definition of $\phi_{n}$, the morphism is $\phi_{n} p \phi_{n}^{-1}: \bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right) \rightarrow$ $\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)$ and $\left(\phi_{n} p \phi_{n}^{-1}\right)^{2}=\phi_{n} p \phi_{n}^{-1} \phi_{n} p \phi_{n}^{-1}=\phi_{n} p \phi_{n}^{-1}$ as $p$ is an idempotent. Then as $\phi_{n}\left(p\left(n, \mathbb{1}_{n}\right)\right)=\phi_{n} p \phi_{n}^{-1}\left(\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)\right)$ we have the isomorphic relation 3.2.

For $\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)$, the endomorphism $\phi_{n} p \phi_{n}^{-1}$ may be represented as a matrix of morphisms between the summands of its source and target. Observe that $\phi_{n} p \phi_{n}^{-1}$ maps $\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)$ to itself, then $\phi_{n} p \phi_{n}^{-1}$ can be represented as a block diagonal matrix. Each $i$-th block is a $m_{i} \times m_{i}$ matrix and it represents a morphism $\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}, m_{i}$ is the coefficient of $\left(i, j_{i}\right)$ and each entry in the $i$-th block is a map $\left(i, j_{i}\right) \rightarrow\left(i, j_{i}\right)$ which is just a scalar. The block diagonal matrix is shown as below:

$$
\left[\begin{array}{cccc}
{\left[M_{0} \in M_{m_{0} \times m_{0}}(\mathbb{C})\right]} & 0 & 0 & 0 \\
0 & {\left[M_{1} \in M_{m_{1} \times m_{1}}(\mathbb{C})\right]} & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & {\left[M_{n} \in M_{m_{n} \times m_{n}}(\mathbb{C})\right]}
\end{array}\right]
$$

Since $\phi_{n} p \phi_{n}^{-1}$ is an idempotent, the block diagonal matrix and every block in it are also idempotents, which is $M_{i} M_{i}=M_{i}$. Since the trace of an idempotent matrix is the same as its rank, then we claim

$$
\begin{equation*}
\left(\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right), \phi_{n} p \phi_{n}^{-1}\right) \cong\left(\bigoplus_{i=0}^{n} \operatorname{tr}\left(M_{i}\right)\left(i, j_{i}\right), \mathbb{1}\right) \tag{3.3}
\end{equation*}
$$

The identity maps are always idempotents. To show such isomorphism exists, we need to build up suitable functions. Since $\phi_{n} p \phi_{n}^{-1}: \bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right) \rightarrow$ $\bigoplus_{i=0}^{n} m_{i}\left(i, j_{i}\right)$ is an idempotent, we can choose a basis $\left\{b_{i, j}\right\}$ for the image of $\phi_{n} p \phi_{n}^{-1}$, where $i$ is corresponding to $i$-th block matrix $M_{i}$ which represents the $\operatorname{map}\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}$. Let $J_{i}$ be the dimension of $\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}$, so $j \in\left(1,2, \cdots, J_{i}\right)$. If we can show that for each fixed $i$,

$$
\left(m_{i}\left(i, j_{i}\right),\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right) \cong\left(\operatorname{tr}\left(M_{i}\right)\left(i, j_{i}\right), \mathbb{1}_{i}\right)
$$

then equation 3.3 is derived.
After $i$ is fixed, let $\left\{e_{i, j}\right\}_{j=1}^{\operatorname{tr}(M i)}$ be the basis for $\operatorname{Im}\left(\mathbb{1}_{i}\right)$. The basis for $\operatorname{Im}\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}$
is $\left\{b_{i, j}\right\}_{j=1}^{J_{i}}$. Then $J_{i}=\operatorname{tr}\left(M_{i}\right)$ since $M_{i}$ is idempotent. Define function $f$ such that $f\left(e_{i, j}\right)=f_{i, j}$ where $i \leq \operatorname{tr}\left(M_{i}\right)$.
Recall prop 3.3, the morphism $\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}$ is an idempotent, and it gives:

$$
m_{i}\left(i, j_{i}\right) \cong\left(\phi_{n} p \phi_{n}^{-1}\right)_{i} m_{i}\left(i, j_{i}\right) \oplus\left(1-\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right) m_{i}\left(i, j_{i}\right) .
$$

For any element $x \in \operatorname{Im}\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}$, it can be represented as a linear sum of basis elements $x=\sum_{j=1}^{\operatorname{tr}\left(M_{i}\right)} a_{i} b_{i}$. Then we define function $g$ such that

$$
g(x)= \begin{cases}g\left(\sum_{j=1}^{\operatorname{tr}\left(M_{i}\right)} a_{i} b_{i}\right)=\sum_{j=1}^{\operatorname{tr}\left(M_{i}\right)} a_{i} e_{i} & x \in \operatorname{Im}\left(\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right) \\ 0 & x \in \operatorname{Im}\left(1-\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right)\end{cases}
$$

Then we have $f g$ and $g f$ are identity maps for $x \in \operatorname{Im}\left(\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right)$ and $x \in \operatorname{Im}\left(\mathbb{1}_{i}\right)$, then we need to check whether they are well-defined morphisms. It is equivalent to show

$$
\left(\phi_{n} p \phi_{n}^{-1}\right)_{i} g=g=g \mathbb{1}_{i}
$$

and

$$
\mathbb{1}_{i} f=f=f\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}
$$

It is true that $g=g \mathbb{1}$. Then for $x \in \operatorname{Im}\left(\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right), \phi_{n} p \phi_{n}^{-1}$ is identity morphism and for $x \in \operatorname{Im}\left(1-\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}\right), \phi_{n} p \phi_{n}^{-1}$ is zero morphism. Thus $\left(\phi_{n} p \phi_{n}^{-1}\right)_{i} g=g=$ $g \mathbb{1}_{i}$. The proof for $\mathbb{1}_{i} f=f=f\left(\phi_{n} p \phi_{n}^{-1}\right)_{i}$ is similar. Therefore relation 3.3 holds. For any idempotent $p$ such that $(n, p)$ is an object in $\operatorname{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$, then $\left(\left(n, \mathbb{1}_{n}\right), p\right)$ is an object in $\mathcal{K} \operatorname{ar}(\mathcal{K} \operatorname{ar}(\mathcal{T} \mathcal{L}))$ since $p \in \operatorname{Hom}(X \rightarrow X)$ is an idempotent and $p \mathbb{1}_{n}=p=\mathbb{1}_{n} p$. Similarly, if $\left(\left(n, \mathbb{1}_{n}\right), p\right)$ is an object in $\operatorname{Kar}(\mathcal{K} \operatorname{Kar}(\mathcal{T} \mathcal{L}))$, then we get $(n, p)$ is an object in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$. We then define an isomorphism:

$$
\psi:\left(\left(n, \mathbb{1}_{n}\right), p\right) \rightarrow(n, p)
$$

. Combining with relation 3.2 and 3.3, we have:

$$
(n, p) \cong\left(\bigoplus_{i=0}^{n} \operatorname{tr}\left(M_{i}\right)\left(i, j_{i}\right), \mathbb{1}\right)
$$

It gives an isomorphism to a direct sum of Jones-Wenzl idempotents.

40CHAPTER 3. KAROUBI ENVELOPE AND ITS SEMISIMPLICITY

## Chapter 4

## At root of unity

The Karoubi envelope $\operatorname{Kar}(\mathcal{T} \mathcal{L})$ has infinitely many simple objects $\left(n, j_{n}\right)$ in general. If we label diagrams in toric code with simple objects in $\operatorname{Kar}(\mathcal{T} \mathcal{L})$, it becomes infinite-dimensional.
To find a suitable category derived from $\operatorname{K} \operatorname{ar}(\mathcal{T} \mathcal{L})$ such that there are only finitely many simple objects, we consider the Karoubi envelops $\operatorname{K} \operatorname{Kar}(\mathcal{T} \mathcal{L})$ at a root of unity and quotient out negligible elements (introduced later).

### 4.1 Pivotal category

To analyse some special properties of the Temperley-Lieb category, we are going to introduce some new definitions here.

Definition 4.1. A pivotal category [Müg08] $\mathcal{C}$ is a rigid category with a monoidal structure on the functor $\nu: X \rightarrow X^{*}$ and a natural isomorphism $\tau: X \rightarrow X^{* *}$. A strict pivotal category is a strict rigid category with a monoidal structure on the functor $\nu: X \rightarrow X^{*}$ and a natural isomorphism $\tau: X \rightarrow X^{* *}$ which is identity.

Remark 4.2. In fact there is a theorem states that every pivotal category is equivalent to a strict pivotal category. For proof in details please refer to [BNRW13].

Theorem 4.3. The Temperley-Lieb category TL is a strict pivotal category.
Proof. The Temperley-Lieb category TL is already proved to be strict monoidal. For any object $n \in \operatorname{Obj}(T L)$ the dual of $n$ is $n^{*}=n$. For any morphism $f$, the
definition of the dual $f^{*}$ is given by 2.12. It is not hard to check $\tau: n \rightarrow n^{* *}$ and the associated natural isomorphisms $\gamma:(m *) \otimes\left(n^{*}\right) \rightarrow(n \otimes m)^{*}$, where $n, m \in \mathbb{N}$, are identities. The coevaluation morphism $\operatorname{co\epsilon }(n): 1 \rightarrow n \otimes n^{*}$ is then just a $n$-fold cups and so $\operatorname{co\epsilon }(n) \in \operatorname{Hom}(1 \rightarrow 2 n)$. Correspondingly, the evaluation morphism $\epsilon(n): n \otimes n^{*} \rightarrow 1$ is a $n$-fold caps and $\epsilon(n) \in \operatorname{Hom}(2 n \rightarrow 1)$, then we have the monoidal structure.

In the Temperley-Lieb category trace of endomorphisms $f$ is defined. Generally, any strict pivotal category has trace maps constructed by composing with evaluation and coevaluation morphisms.

Definition 4.4. The right trace of a endomorphism $f: X \rightarrow X$ in a strict pivotal category $\mathcal{C}$ is a composition:

$$
\operatorname{tr}_{R}(f)=\operatorname{co\epsilon }(X)(f \otimes \mathbb{1}) \operatorname{co\epsilon }(X)^{*}
$$

and the left trace is a composition:

$$
\operatorname{tr}_{L}(f)=\operatorname{co\epsilon }\left(X^{*}\right)(\mathbb{1} \otimes f) \operatorname{co\epsilon }\left(X^{*}\right)^{*}
$$

For trace map in Temperley-Lieb category, the only simple TL diagram in $\operatorname{Hom}(0 \rightarrow 0)$ without loops is $\mathbb{1}_{0}$, so $\operatorname{Hom}(0 \rightarrow 0) \cong \mathbb{K}$ where $\mathcal{K}$ is the ground field. Then we canonically consider the trace as a map $\operatorname{tr}(f): \operatorname{Hom}(n \rightarrow n) \rightarrow \mathcal{K}$ given by $[2]^{d}$ where $d$ is the number of loops in $\operatorname{tr}(f)$. By isotopy, the number of loops does not depend on which side we trace off. Thus the right trace and left trace for $f$ are the same.

Lemma 4.5. For any TL morphisms $f: n \rightarrow m$ and $g: m \rightarrow n$ :

$$
\operatorname{tr}(g f)=\operatorname{tr}(f g)
$$

Proof. The trace of TL diagrams are preseved by isotopies.


Therefore $\operatorname{tr}(g f)=\operatorname{tr}(f g)$.
Definition 4.6. A Spherical category $\mathcal{C}$ is a category such that for any morphism $f$,

$$
\operatorname{tr}_{r}(f)=\operatorname{tr}(g)
$$

Using spherical category, we can then define a collection of morphisms called negligible morphisms.

Definition 4.7. A morphism $f: X \rightarrow Y$ in a spherical category is negligible if for any morphism $g: Y \rightarrow X$,

$$
\operatorname{tr}(g f)=0 .
$$

A good property of negligible morphisms is that they actually form an tensor ideal which enable us to make a quotient over negligible morphisms.

Lemma 4.8. Let $\mathcal{N e}(T L)$ be the set of all negligible morphisms in TemperleyLieb category. Then $\mathcal{N} e g(T L)$ forms a tensor ideal and is called the negligible ideal.

The proof is not hard. By using Lemma 4.5, it is easy to show that if $f \in$ $\mathcal{N} e g(T L)$ for $f: n \rightarrow m$ then $g f \in \mathcal{N} e g(T L)$ and $f h \in \mathcal{N} e g(T L)$. For tensor multiplication we just use the property that trace of TL diagrams is invariant under isotopies.

### 4.2 Quantum parameter

Let's consider the quantum integer $[n]$. Recall the Wenzl's recurrence formula 2.3, it gives that for Jones-Wenzl idempotent $j_{n+1}$ the coefficient of the second term is $\frac{[n]}{[n+1]}$. If $[k]=0$ then the coefficient $\frac{[k]}{[k+1]}$ is not defined, so $j_{k}$ does not exist and so are the all Jones-Wenzl idempotents $j_{n}$ for $n \geq k$. It then gives a finite many Jones-Wenzl idempotents $j_{i}$ where $0 \leq i \leq k-1$ which also form the collection of simple objects. Finite simple objects make the category simpler and there will be some special properties we would expect.
For $[n]=0$, we have by definition:

$$
\frac{q^{n}-q^{-n}}{q-q^{-1}}=0
$$

To ensure such fraction make sense, it is then required that $q-q^{-1} \neq 0$ which is $q \neq 0 \& \pm 1$ and then:

$$
q^{n}-q^{-n}=0 \Leftrightarrow q^{2 n}=1
$$

where $q$ is then a $2 n$-th root of unity. The ground field is $\mathbb{K}=\mathbb{C}(q)$. After the parameter becomes a fixed value, $\mathbb{K}=\mathbb{C}$ so the Temperley-Lieb category is a $\mathbb{C}$-linear category.

Definition 4.9. For any $a \in \mathbb{C}$ and a given Temperley-Lieb category $T L$, the category $T L(q=a)$ is the specialization of the Temperley-Lieb category at $q=a$. It is obtained by replacing the quantum parameter $q$ by $a$. The objects are the same as TL. The set of morphisms is the $\mathbb{C}$-linear space with simple TL diagrams as basis. Composition and tensor product are induced from TL. Specifically, if $a$ is a $2 n$-th root of unity, such category is called specialization of the Temperley-Lieb category evaluated at a $2 n$-th root of unity $q=a$.

Observe that if we evaluate $T L$ at a $2 n$-th root of unity, the Jones-Wenzl idempotent $j_{n-1}$ becomes negligible. Then we have the following lemma.

Lemma 4.10. If $q=e^{k \pi i / n}$ where $1 \leq k<n$ is a $2 n$-th root of unity. Then $T L(q)$ has only finite many Jones-Wenzl idempotents $j_{i}$ where $0 \leq i \leq n-1$ and $j_{n-1}$ is the unique negligible Jones-Wenzl idempotent.

Proof. From Wenzl's recurrence formula 2.3, if $q=e^{k \pi i / n}$ then $q^{2 n}-1=0$, so $[k]=0$. We then have $\frac{[k-1]}{[k]}$ does not exists, so $j_{n}$ is not defined. Thus $j_{i}$ are not defined for $i \geq n$ recursively.
To show $j_{n}$ is the unique negligible Jones-Wenzl idempotent, we recall the definition of negligible morphism that for a negligible morphism $f: X \rightarrow Y$, the $\operatorname{trace} \operatorname{tr}(g f)=0$ for arbitrary morphism $g: Y \rightarrow X$. For $j_{i}$, let $f: i \rightarrow i$ be an arbitrary morphism. Then $f$ can be represented as a composition $f=a \mathbb{1}_{i}+g$ where $g$ is a sum of non-identity morphisms. Then by relations 2.2 :

$$
f j_{i}=a j_{i}
$$

By Remark 2.24:

$$
\begin{aligned}
\operatorname{tr}\left(f j_{i}\right) & =\operatorname{tr}\left(a j_{i}\right) \\
& =a[i+1]
\end{aligned}
$$

If $\operatorname{tr}\left(f j_{i}\right)=0$ for arbitrary $f$, it is required that $[i+1]=0$. As $0 \leq i \leq(n-1)$, $j_{n-1}$ is the unique Jones-Wenzl idempotent satisfies the conditions.

From Lemma 4.8, the collection of negligible morphisms for Temperley-Lieb category evaluated at $q=e^{k \pi i / n}$ is $\mathcal{N} e g\left(T L\left(q=e^{k \pi i / n}\right)\right)$, which forms an ideal.

From the work of Goodman and Wenzl [GW93], the ideal of negligible morphisms in $T L\left(q=e^{k \pi i / n}\right)$ is exactly generated by the idempotent $j_{n-1}$. Therefore we produce a quotient Temperley-Lieb category.

Definition 4.11. For a specialization of the Temperley-Lieb category at a $2 n$-th root of unity $T L\left(q=e^{k \pi i / n}\right)$ with negligible ideal $\mathcal{N} \operatorname{eg}\left(T L\left(q=e^{k \pi i / n}\right)\right)$ generated by $j_{n-1}\left(q=e^{k \pi i / n}\right)$, we define a new pivotal $\mathbb{C}$ category $T L^{N}\left(q=e^{k \pi i / n}\right)$. The object set of new category is $\mathbb{N}$, the set of morphisms from $n$ to $m$ where $n, m \in \mathbb{N}$ is

$$
\operatorname{Hom}_{T L^{N}}(n \rightarrow m)=\operatorname{Hom}_{T L}(n \rightarrow m) / \mathcal{N} e g(n \rightarrow m)
$$

where $\mathcal{N} \operatorname{eg}(n \rightarrow m)$ is the set of negligible morphisms from $n$ to $m$. The composition, tensor product and pivotal structure are induced from $T L\left(q=e^{k \pi i / n}\right)$.

Remark 4.12. Given that the generator of $\mathcal{N} \operatorname{eg}\left(T L\left(q=e^{k \pi i / n}\right)\right)$ is $j_{n-1}(q=$ $\left.e^{k \pi i / n}\right)$, then the set of morphisms $\operatorname{Mor}\left(T L^{N}\left(q=e^{k \pi i / n}\right)\right.$ is just the morphism set of $T L\left(q=e^{k \pi i / n}\right)$ with $j_{n-1}=0$.

### 4.3 Skein module

Consider a special case of evaluated Temperley-Lieb category. Let us define a parameter $\delta:=q+q^{-1}$. If we set $\delta=1$, then $q=e^{ \pm \pi i / 3}$. Thus in this case, $q$ is a root of $q^{6}=1$ and the corresponding specialization of the Temperley-Lieb category $T L(\delta=1)$ only has Jones-Wenzl idempontents $j_{0}, j_{1}$ and $j_{2}$ by Lemma 4.10 .

For Jones-Wenzl idempotent $j_{2}$, from Wenzl's recurrence formula we then have


In the quotient category $T L^{N}(\delta=1)$, the Jones-Wenzl idempotent $j_{2}=0$ give the relation:

in category $T L^{N}(\delta=1)$.
Then let us consider the skein module of the category $T L^{N}(\delta=1)$. The definition of skein module for a 3 -manifold based on bradied tensor category is introduced in Chapter 12 of [Tur16]. The general picture, of a skein module for a n-manifold based on a disklike n-category is described in [MW10]. The generalized skein module $\mathcal{S}(X)$ for $n$-manifold $X$ is $\mathcal{S}(X):=\mathcal{C}(X) / U(X)$ where $\mathcal{C}$ is a system of fields, $U(X)$ is the space of local relations in $\mathcal{C}(X)$. These terms are not introduce in this thesis. In this chapter, we only deal with a simple case $T L(\delta=1)$. For more details please see [MW10]. Refer to the generalized skein module, we then define a simple and specific skein module of $T L$.

Definition 4.13. The skein module of any surface $\Sigma$ associated to a specialization of the Temperley-Lieb category $T L(\delta=1)$ and is just a linear combination of closed loops drawn on $\Sigma$ modulo the local relations in $T L(\delta=1)$ :

$$
\begin{equation*}
\mathcal{S}_{\Sigma}(T L(\delta=1))=\mathbb{C}\{\text { closed loops on } \Sigma \text { over } \mathbb{Z} / 2 \mathbb{Z}\} / U(T L(\delta=1)) \tag{4.2}
\end{equation*}
$$

Where $U(T L(\delta=1))$ is the space of local relation in $T L(\delta=1)$ which is \{isotopy, closed loop replaced by coefficient $1, \delta=1$ and $\left.j_{2}=0\right\}$.

From the definition we can then explicitly calculate the skein modules for any surface $\Sigma$ and the specialization of the Temperley-Lieb $T L(\delta=1)$.

Theorem 4.14. The skein module of any surface $\Sigma$ associated to specialization of the Temperley-Lieb category $T L(\delta=1)$ is the same as the linear expansion of the first homology group of surface $\Sigma$ over $\mathbb{Z} / 2 \mathbb{Z}$ :

$$
\begin{equation*}
\mathcal{S}_{\Sigma}(T L(\delta=1))=\mathbb{C}\left\{H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right\} \tag{4.3}
\end{equation*}
$$

Proof. By definition of homology group,

$$
H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})=\frac{\operatorname{Ker}\left(\partial C_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right)}{\operatorname{Im}\left(\partial C_{2}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right)}
$$

For $\operatorname{Ker}\left(\partial C_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right)$, it is just 1 -cycle of $\Sigma$ over $\mathbb{Z} / 2 \mathbb{Z}$, which is a sum of 1 -chains. Recall any surface is triangularble, so the basis for the 1 -chains is 1 simplexes. Then a basis for the 1 -cycles is the set of sums of 1 -simplexes which form oriented closed loops with vertexes. In general, a 1-simplex has orientations in $H_{1}$ with relation:
$\rightarrow=-\ldots$
In $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$, we have $-1=1$. Then:
$\rightarrow \quad \rightarrow$
Thus orientations do not matter and we won't include orientations in following graphs.
For vertices in 1-chains, consider the relation in general $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ :

which is true since by overlapping right hand side to left hand side we then get a 2 -simplex whose image of function $\partial$ is zero in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$. Thus we have all 1-chains with different vertices (non-zero) only are in the same homology class in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ and such equivalence class can be described by a closed loop without vertices even though this loop itself is not in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$.
Let us then consider $\operatorname{Im}\left(\partial C_{2}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right.$, which is the collection of all boundaries of 2-chains in $\Sigma$. Therefore quotient by $\operatorname{Im}\left(\partial C_{2}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right.$ gives:

where the third term is the equivalence class.
For any $n$-simplex $S$ over $\mathbb{Z} / 2 \mathbb{Z}$, we have $S+S=(1+1) S=0$. Now consider two curves:

which is given by cancellation of repeat elements over $\mathbb{Z} / 2 \mathbb{Z}$ and quotient by $\operatorname{Im}\left(\partial C_{2}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right.$. This relation is equivalent to relations $\delta=1$ and $j_{2}=0$ by 4.1 in Temperley-Lieb category.

Inversely, in the skein module $\mathcal{S}_{\Sigma}(T L(\delta=1))$, we have the relation that every closed loop is replaced by an coefficient [2] and $[2]=1$ when $\delta=1$ :

which is equivalent to quotient out 1-boundaries in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$.
Now we can define an injective map $f: \mathbb{C}\left\{H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right\} \rightarrow \mathcal{S}_{\Sigma}(T L(\delta=1))$ by mapping each equivalence class element in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ to closed loops in the skein module, and extending this linearly. As we prove that each equivalence class in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ can be represented by a closed loop, such map is well defined
and it is injective since all local relations in skein module are equivalent to some equivalence relations in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ and in $\mathbb{Z} / 2 \mathbb{Z}$ the orientation does not matter. We can also define an injective map $g: \mathcal{S}_{\Sigma}(T L(\delta=1)) \rightarrow \mathbb{C}\left\{H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right\}$ by mapping each element in skein module to an equivalence class in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$. By what we have shown, this map is well defined and injective. The theorem just follows.

## Chapter 5

## Levin-Wen models

### 5.1 General introduction to Levin-Wen models

Before introduce Levin-wen model, we first introduce the idea of cellulation which will be used later.

Definition 5.1. Let $\Sigma$ be a surface, a cellulation $\Gamma \subset \Sigma$ is a graph embedded in the surface so every connected component of $\Sigma \backslash \Gamma$ is a topological disc.

Remark 5.2. The cellulation is usually set to be a square lattice for convenience.
The Levin-Wen model provides an explicit Hamiltonian formulation of TuraevViro TQFTs.In this thesis we only study the 2-dimensional part of the TuraevViro invariant, that is, the skein module for a surface. we do not address the 3-dimensional numerical invariant, or how it relates to the Levin-Wen model.
For a semisimple spherical monoidal category $\mathcal{C}, \operatorname{IrrC}$ is just a collection of simple objects in $\mathcal{C}$. Then for each cellulation $\Gamma$, we often refer to $\Gamma$ as a graph in $\Sigma$ by thinking about the 1-skeleton of $\Gamma$.
We then assign each edge $E$ in cellulation a vector space:

$$
\mathcal{H}_{E}=\mathbb{C} \operatorname{IrrC}
$$

which is just labelling each edge a simple object in $\operatorname{IrrC}$.
Each $n$-valent vertex $V$ is a vector space:

$$
\mathcal{H}_{n(V)}=\bigoplus_{\begin{array}{c}
\text { ways to labelling the } \\
\text { adjacent elges to } \\
\text { V by } X_{i} \in \operatorname{IrrC}
\end{array}} \operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{i=1}^{n} X_{i}\right)
$$



Figure 5.1: A cellulation of surface $\Sigma$. The edge $V$ and face $F$ are highlighted. There are many cellulations for surfaces, but they share some properties which can help us make analysis. In the following sections, this cellulation will be used as it is easy to draw.

For each edge $E$, we will define a projection (introduce in details later):

$$
H_{E}: \mathcal{H}_{n(V)} \otimes \mathcal{H}_{E} \otimes \mathcal{H}_{n\left(V^{\prime}\right)} \rightarrow \mathcal{H}_{n(V)} \otimes \mathcal{H}_{E} \otimes \mathcal{H}_{n\left(V^{\prime}\right)}
$$

where $V$ and $V^{\prime}$ are two vertices adjacent to edge $E$. For each face $F$, we will define a projection:

$$
H_{F}: \bigotimes_{\substack{V^{\prime} \text { sadjacent } \\
\text { to } F}} \mathcal{H}_{n(v)} \otimes \bigotimes_{\begin{array}{c}
E^{\prime} \text { s adjacent } \\
\text { to } F
\end{array}} \mathcal{H}_{E} \rightarrow \bigotimes_{\begin{array}{c}
V^{\prime} \text { sadjacent } \\
\text { to } F
\end{array}} \mathcal{H}_{n(v)} \otimes \bigotimes_{\begin{array}{c}
E^{\prime} \text { sadjacent } \\
\text { to } F
\end{array}} \mathcal{H}_{E}
$$

Then for any surface $\Sigma$, with a cellulation $\Gamma$ on it we have a Hilbert space:

$$
\mathcal{H}_{\Gamma}=\bigotimes_{\text {edges } E \in \Gamma} \otimes \bigotimes_{\text {vertices } V \in \Gamma} \mathcal{H}_{n(V)}
$$

with a Hamiltonian:

$$
H: \mathcal{H}_{\Gamma} \rightarrow \mathcal{H}_{\Gamma}
$$

given by:

$$
H=\sum_{\text {edges } E} H_{E}+\sum_{f a c e s ~}^{F} \text { } H_{F}
$$

In physics, the ground state for a Hamiltonian is the lowest eigenstate of such Hamiltonian. The Levin-Wen model then gives such Hilbert space $\mathcal{H}_{\Gamma}$ and Hamiltonian $H$ satisfy the following theorem:

Theorem 5.3. The ground state of the Hamiltonian $H_{\Gamma}$ does not depends on the cellulation $\Gamma$ but just depends on the surface $\Sigma$. The ground state is the skein module $\mathcal{S}_{\Sigma}(\mathcal{C})$ where $\mathcal{C}$ is a spherical fusion cateogry which the Hilbert space $\mathcal{H}_{\Gamma}$ based on.

Remark 5.4. We can also define a Hamiltonian $H_{\Gamma}=H=\sum_{\text {vertices } V} H_{V}+$ $\sum_{\text {faces } F} H_{F}$ over $\mathcal{H}_{\Gamma}$ where $H_{V}$ is a projection over vertex $V$ and edges adjacent to $V$. In fact, both projections $\sum H_{E}$ and $\sum H_{V}$ have the same kernel and therefore the kernel of $H$ and $H_{\Gamma}$ are same. The details won't be introduced in this thesis.

However we are not going to prove this theorem in general. The theorem in a specific case called toric code will be proved in next chapter. Our aim in the rest of this chapter is to show:

$$
\operatorname{Ker}(H)=\left(\bigcap_{E} \operatorname{Ker}\left(H_{E}\right)\right) \cap\left(\bigcap_{F} \operatorname{Ker}\left(H_{F}\right)\right)
$$

for all Levin-Wen models. Before the proof, we will analysis the properties of projections $H_{E}$ for edges and $H_{F}$ for faces, which will help us derive the ground state of $H$.

### 5.2 Projection $H_{E}$

Let us define the projection $H_{E}$ for edges $E \in \Gamma$. Suppose $f \otimes X \otimes g$ is an element in $\mathcal{H}_{n(V)} \otimes \mathcal{H}_{E} \otimes \mathcal{H}_{n\left(V^{\prime}\right)}$ where we set:

$$
\begin{equation*}
H_{E}(f \otimes X \otimes g)=\left(f-\Pi_{X}(f)\right) \otimes X \otimes\left(g-\Pi_{X}(g)\right) \tag{5.1}
\end{equation*}
$$

where $\Pi_{X}$ is the projection to summands with the edge is labelled by $X$.
Remark 5.5. The reason for using $I-\Pi_{X}$ instead of $\Pi_{X}$ as the projection is that we wish the corresponding eigenvalue to ground state is 0 .

The morphism $H_{E}$ is a projection since both $\Pi_{X}$ and identity morphism are projections. The eigenvalue of $H_{E}$ is 0 or 1 by definition. Since we are intending to find the ground state, the kernel of $H_{E}$ is what we interest.

Proposition 5.6. For any pair of edges $E$ and $E^{\prime}$ in cellulation $\Gamma$, the projection $H_{E}$ and $H_{E^{\prime}}$ commutes.

Proof. If edges $E$ and $E^{\prime}$ are different and do not share any vertices, then each projection acts on different Hilbert space, so they commute.
Now suppose the Hilbert space where the projections $H_{E}$ and $H_{E^{\prime}}$ act is:

$$
\mathcal{H}_{n\left(V_{1}\right)} \otimes \mathcal{H}_{E} \otimes \mathcal{H}_{n\left(V_{2}\right)} \otimes \mathcal{H}_{E^{\prime}} \otimes \mathcal{H}_{n\left(V_{3}\right)}
$$

Every element of this Hilbert space is a sum of elements of the form:

$$
\begin{equation*}
f \otimes X \otimes g \otimes X^{\prime} \otimes h \tag{5.2}
\end{equation*}
$$

where $f \in \mathcal{H}_{n\left(V_{1}\right)}, f \in \mathcal{H}_{n\left(V_{2}\right)}, h \in \mathcal{H}_{n\left(V_{3}\right)}, X$ is the labelled edge adjacent to $V_{1}$ and $V_{2}, X^{\prime}$ is the labelled edge adjacent to both $V_{2}$ and $V_{3}$.
Apply $H_{E} H_{E^{\prime}}$ to 5.2 , we then have:

$$
\begin{aligned}
& H_{E} H_{E^{\prime}}\left(f \otimes X \otimes g \otimes X^{\prime} \otimes h\right) \\
= & H_{E}\left(f \otimes X \otimes\left(g-\Pi_{X^{\prime}}(g)\right) \otimes X^{\prime} \otimes\left(h-\Pi_{X^{\prime}}(h)\right)\right) \\
= & \left(f-\Pi_{X}(f)\right) \otimes X \otimes\left(g-\Pi_{X}(g)\right)\left(g-\Pi_{X^{\prime}}(g)\right) \otimes X^{\prime} \otimes\left(h-\Pi_{X^{\prime}}(h)\right) \\
= & H_{E^{\prime}}\left(f-\Pi_{X}(f)\right) \otimes X \otimes\left(g-\Pi_{X}(g)\right) \otimes X^{\prime} \otimes \\
= & H_{E^{\prime}} H_{E}\left(f \otimes X \otimes g \otimes X^{\prime} \otimes h\right)
\end{aligned}
$$

Therefore they commute.
By Proposition above, the sum $\sum H_{E}$ is diagonalisable with eigenvalues in $\mathbb{N}$.
Proposition 5.7. The kernel of $\Sigma_{E \in \Gamma} H_{E}$ satisfies:

$$
\operatorname{Ker}\left(\Sigma_{E \in \Gamma} H_{E}\right)=\bigoplus_{\begin{array}{c}
\text { ways to label }  \tag{5.3}\\
\text { all edges by IrrC }
\end{array}} \bigotimes_{V \in \Gamma} \operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{\begin{array}{c}
\text { labelled edges } \\
X_{i} \text { adjacent } \\
\text { to } V
\end{array}} X_{i}\right)
$$

By the definition of $H_{E}, f \otimes X \otimes g$ is in its kernel if and only if both $f$ and $g$ are supported in the summands where the edge labels agree with $X$. Then in the kernel $\operatorname{Ker}\left(\Sigma_{E \in \Gamma} H_{E}\right)$, when the edges of $\Gamma$ are labelled, the morphism $f$ at each vertex can only be in $\operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{\substack{\text { labelled edges } \\ X_{i} \text { ajda acent } \\ \text { to } V}} X_{i}\right)$, which gives the formula above.

### 5.3 Grothendieck group and graphical calculus

### 5.3.1 Grothendieck group

We need the following concepts to define the Hamiltonian for faces $H_{F}$.

Definition 5.8. Let $\mathcal{C}$ be an additive category. The split Grothendieck group $K_{0}(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by isomorphism classes $[A]$ of objects in $\mathcal{C}$ modulo the relations $[A \oplus B]=[A]+[B]$ for any $A, B \in \operatorname{Obj}(\mathcal{C})$.

Remark 5.9. In general the Grothendieck group (without split) of an abelian category $\mathcal{C}$ is the same abelian group defined above but modulo the relation $[C]=[A]+[B]$ for every exact sequence:

$$
0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0
$$

For semisimple category, its Grothendieck group can be used to represent its objects.

Theorem 5.10. Suppose $\mathcal{C}$ is a semisimple category with collection of all isomorphism classes of simple objects $\left\{X_{i}\right\}$, then in Grothendieck group $K_{0}(\mathcal{C})$, since the objects in Grothendieck group are isomorphism classes we have:

$$
A \cong \bigoplus_{i} n_{i} X_{i} \quad \Leftrightarrow \quad[A]=\sum_{i} n_{i}\left[X_{i}\right]
$$

where $n_{i} \in \mathbb{N}$.
Proof. " $\Rightarrow$ ":
For $A \cong \bigoplus_{i} n_{i} X_{i} \cong n_{1} X_{1} \oplus n_{2} X_{2} \oplus n_{3} X_{3} \oplus \cdots$ where $n_{i} \in \mathbb{N}$, we get:

$$
\begin{aligned}
{[A]=} & {\left[n_{1} X_{1} \oplus\left(n_{2} X_{2} \oplus n_{3} X_{3} \oplus \cdots\right)\right] } \\
= & {\left[n_{1} X_{1}\right]+\left[n_{2} X_{2} \oplus\left(n_{3} X_{3} \oplus \cdots\right)\right] } \\
& \cdots \\
= & \sum_{i}\left[n_{i} X_{i}\right] \\
= & \sum_{i} n_{i}\left[X_{i}\right]
\end{aligned}
$$

in $K_{0}(\mathcal{C})$ inductively.
" $\Leftarrow$ ":
On the Grothendieck group $K_{0}(\mathcal{C})$, for any pair of classes $[X],[Y]$ let us define an inner product:

$$
\begin{equation*}
\langle[X],[Y]\rangle:=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y) . \tag{5.4}
\end{equation*}
$$

Firstly we want to show this inner product is well-defined:

1. For $X \cong X^{\prime}$, they are both in the same isomorphism class $[X]$ in $K_{0}(\mathcal{C})$, we need $\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}\left(X^{\prime} \rightarrow Y\right)$ for any $Y$.
Since $X \cong X^{\prime}$, there is an isomorphism $g: X \rightarrow X^{\prime}$ such that for any $f \in$ $\operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ there is a unique morphism $f^{\prime}=f \circ g \in \operatorname{Hom}_{\mathcal{C}}\left(X^{\prime} \rightarrow Y\right)$. Similarly, for any $f^{\prime} \in \operatorname{Hom}_{\mathcal{C}}\left(X^{\prime} \rightarrow Y\right), f=f^{\prime} \circ g^{-1}$ is uniquely defined. Therefore $\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}\left(X^{\prime} \rightarrow Y\right)$.
For $\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}\left(X \rightarrow Y^{\prime}\right)$ since there is also an isomorphism $j: Y \rightarrow Y^{\prime}$ such that for any $i \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ there is a unique $i^{\prime}=j \circ i$ in $\operatorname{Hom}_{\mathcal{C}}\left(X \rightarrow Y^{\prime}\right)$.
2. We need to show $\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \oplus Y \rightarrow Z)=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Z)+\operatorname{dim}_{\operatorname{Hom}}^{\mathcal{C}}(Y \rightarrow$ $Z)$ for any $X, Y, Z \in K_{0}(\mathcal{C})$.
By definition:

$$
\begin{aligned}
\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \oplus Y \rightarrow Z) & =\langle[X \oplus Y],[Z]\rangle \\
& =\langle[X]+[Y],[Z]\rangle
\end{aligned}
$$

by linearlity of inner product

$$
\begin{aligned}
& =\langle[X],[Z]\rangle+\langle[Y],[Z]\rangle \\
& =\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Z)+\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}(Y \rightarrow Z)
\end{aligned}
$$

Then we have $\langle\cdot, \cdot\rangle$ is a well-defined inner production. Since $\mathcal{C}$ is semisimple with the collection of all simple object $\left\{X_{i}\right\},\left\{\left[X_{i}\right]\right\}$ forms an orthonormal basis w.r.t the inner product $\langle\cdot, \cdot\rangle$. Suppose $[A]=\sum_{i} n_{i}\left[X_{i}\right]$ in $K_{0}(\mathcal{C})$, as $\mathcal{C}$ is semisimple then $A \cong \bigoplus_{i} m_{i} X_{i}$ for some $m_{i} \in \mathbb{N}$. Then $\left\langle[A],\left[X_{i}\right]\right\rangle=n_{i}$ is uniquely defined for each $\left[X_{i}\right]$ by the inner product. Therefore $m_{i}=n_{i}$ for each $i$.

### 5.3.2 Graphic calculus

We are going to use string diagram calculus and it is valid in fusion category which the Levin-Wen model based on [HP16].

Definition 5.11. In graphical calculus, objects are denoted by strands and morphisms are denoted by vertices or discs. The dimension of a dualizable object $X \in \mathcal{C}$ is given by:

$$
d_{X}:=\operatorname{co\epsilon }^{*}(X) \circ \operatorname{co\epsilon }(X)=\epsilon(X) \circ \epsilon^{*}(X)
$$

where cot : $1 \rightarrow X \otimes X^{*}$ is the coevaluation function and $\epsilon: X^{*} \otimes X \rightarrow 1$ is the evaluation function and $d_{X} \in \mathbb{R}_{\geq 0}$. The term $N_{X, Y}^{Z}$ is the dimension of $\operatorname{Hom}_{\mathcal{C}}(X \otimes Y \rightarrow Z)$.

Then Henriques and Penneys give some important lemmas in their paper, which will be used to define the Hamiltonian $H_{V}$. Proof in details is in the chapter 2 of [HP16].
In this lemma, same colour vertex in each graph does not represent same morphisms. They are actually the conjugation of each other. Recall the antiinvolution of a morphism $f \in \operatorname{Hom}_{\mathcal{C}}(X \rightarrow Y)$ is $\bar{f} \in \operatorname{Hom}_{\mathcal{C}}(Y \rightarrow X)$ which is an anti-linear map.
We define:


Then we have the lemma.

Lemma 5.12. The following graphic calculus relations hold:
(Bigon 2)


In each relation, it is a summations of morphisms at each vertex and vertices with different colour have different summations. For example, the relation ( $I=$ $H)$ interpreted by summations is:

### 5.4 Projection $H_{F}$

If $X, Y, Z$ are simple objects $X, Y, Z \in \operatorname{IrrC}$, let $\gamma_{X, Y, Z}$ be an orthonormal basis for $\operatorname{Hom}_{\mathcal{C}}(1 \rightarrow X \otimes Y \otimes Z)$. For the Hilbert space $\mathcal{H}_{F}$, we first define a projection $H_{F, X}$ with $X \in \operatorname{Irr} C$ satisfies:

$$
H_{F, X}: \mathcal{H}_{F} \rightarrow \mathcal{H}_{F}
$$

We then introduce a graphical notation for states, writing:

$$
\left.\bigotimes_{i=1}^{n} f_{i} \otimes \bigotimes_{i=1}^{n} X_{i}\right)
$$

as:


Now, heuristically we define:

by:

where

$$
f_{i} \in \mathcal{H}_{n\left(V_{i}\right)}=\bigoplus_{\begin{array}{c}
\text { labelling the } \\
\text { edges } \\
\text { to } V_{i} \text { adjacent } \\
Y_{j} \in \operatorname{Irrc}
\end{array}} \operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{i=1}^{n} Y_{j}\right)
$$

The Hamiltonian $H_{F, X}$ defined returns 0 if such $Y_{j}^{\prime} s$ don't match with $X_{i}^{\prime} s$. Under these definition, we don't need to worry about such state will be counted in kernel of $H$, since such state is not in the kernel of some $H_{E}$ which include these vertex and edge.
This picture doesn't literally make sense since it is not in the Hilbert space we defined. But it intends that we interpret the pieces in the dashed boxes around the face via Lemma (Fusion):
by which we just get

$$
\begin{aligned}
& H_{F, X}\left(\bigotimes_{i=1}^{n} f_{i} \otimes \bigotimes_{i=1}^{n} X_{i}\right) \\
& =\sum_{\left.P_{i} \in \operatorname{IrrC}\right)} \prod_{i=1}^{n} \sqrt{\frac{d_{P_{i}}}{d_{X_{i}} d_{X}}} P_{P_{n}}^{X_{n}} \underbrace{X_{n}}_{f_{1}} \underbrace{X_{n-1}}_{X_{1}}
\end{aligned}
$$

We are going to show that we can build the projection $H_{F}$ as a certain linear combination of the $H_{F, X}$. The $H_{F, X}$ commute with each other and with $H_{E}$ and this commutativity property is inherited by $H_{F}$.

Lemma 5.13. Suppose for a face $F$ in the cellulation, we have two Hamiltonians $H_{F, X}$ and $H_{F, Y}$. then we have:

$$
\begin{equation*}
H_{F, Y} H_{F, X}=\sum_{Z} N_{X, Y}^{Z} H_{F, Z} \tag{5.6}
\end{equation*}
$$

where $N_{X, Y}^{Z}$ is then the multiplicity of $Z$ in $X \otimes Y$.

Proof.

$$
H_{F, Y} H_{F, X}\left(\bigotimes_{i=1}^{n} f_{i} \otimes \bigotimes_{i=1}^{n} X_{i}\right)
$$

Using Lemma $(\mathrm{I}=\mathrm{H})$ in graphs with same structure as dashed area around the face:

Notice that only the $P_{1}=P_{2}=\cdots=P_{n}$ terms survive by using Lemma ( $\mathrm{I}=\mathrm{H}$ ). Denote them all by $P_{1}$. Then consider Lemma (Bigon 2) in the dashed areas above. Remark that there are three dashed boxes here since we need them all to
apply Lemma (Bigon 2):

$$
=\sum_{\left.P_{1} \in \operatorname{IrrC}\right)} \sum_{\left.Q_{i} \in \operatorname{IrrC}\right)} \prod_{i=1}^{n} \sqrt{\frac{d_{Q_{i}}}{d_{X_{i}} d_{X} d_{Y}} \sqrt{\frac{d_{X} d_{Y}}{d_{P_{1}}}}} Q_{Q_{n}}^{X_{n}} \overbrace{P_{1}}^{P_{1}} \underbrace{X_{1}}_{P_{1}}
$$

Then we can apply the same lemma around the face, actually $n-1$ times in all:


At the end we apply Lemma (Bigon 1) to dashed area:

Substitute $P_{1}$ by $Z$ :

$$
\begin{aligned}
& =\sum_{Z \in \operatorname{IrrC})} \sum_{\left.Q_{i} \in \operatorname{IrrC}\right)} \prod_{i=1}^{n} \sqrt{\frac{d_{Q_{i}}}{d_{X_{i}} d_{Z}}} N_{X, Y}^{Z} \\
& =\sum_{Z \in \operatorname{IrrC})} N_{X, Y}^{Z} H_{F, Z}\left(\bigotimes_{i=1}^{n} f_{i} \otimes \bigotimes_{i=1}^{n} X_{i}\right)
\end{aligned}
$$

which proves the proposition. In the original literature by Levin and Wen [LW05], the proof isn't done right. Their proof was not in the Hilbert space they defined.

Proposition 5.14. Any Hamiltonians $H_{F, X}$ and $H_{F^{\prime}, Y}$ commute and $H_{F, X}$ also commutes with $H_{E}$ for any edge $E \in \Gamma$.

Proof. For Hamiltonians $H_{F, X}$ and $H_{E}$, they either share one edge and two vertices or they are not adjacent. When they are not adjacent, $H_{E}$ and $H_{F, X}$ they just acts on different Hilbert spaces and then commute. If $E$ is an edge around face by definition of $H_{E}$, suppose $f_{1} \otimes X_{1} \otimes f_{2}$ is share by both projections. Then we get:

$$
H_{E} H_{F, X}\left(\bigotimes_{i=1}^{n} f_{i} \otimes \bigotimes_{i=1}^{n} X_{i}\right)
$$

$=\sum_{\left.P_{i} \in \operatorname{IrrC}\right)} \prod_{i=1}^{n} \sqrt{\frac{d_{P_{i}}}{d_{X_{i}} d_{X}}} H_{E} \quad P_{n}$

$=\sum_{\left.P_{i} \in \operatorname{IrC} \mathcal{C}\right)} \prod_{i=1}^{n} \sqrt{\frac{d_{P_{i}}}{d_{X_{i}} d_{X}}}$



$$
=H_{F, X} H_{E}\left(\bigotimes_{i=1}^{n} f_{i} \otimes \bigotimes_{i=1}^{n} X_{i}\right)
$$

and they commute.
For Hamiltonians $H_{F, X}$ and $H_{F^{\prime}, Y}, X, Y \in \operatorname{Irr\mathcal {C}}$ again if two faces do not adjacent, then the Hamiltonians act on different spaces so they commute. If two faces
adjacent, they must adjacent at one edge. Then consider:

$$
H_{F^{\prime}, Y} H_{F, X}\left(\bigotimes_{i=1}^{2 n-1} f_{i} \otimes \bigotimes_{i=1}^{2 n-1} X_{i}\right)
$$

We just need to consider the Hamiltonians acting on the sharing edge. Suppose the sharing edge is $X_{n}$, then after applying $H_{F, X}$ first and then $H_{F^{\prime}, Y}$, we get:


Use Lemma $(\mathrm{I}=\mathrm{H})$ (from right hand side to left hand side).

which is equivalent to apply $H_{F^{\prime}, Y}$ and $H_{F, X}$ in order. Therefore these two Hamiltonians commute.

Recall the definition of Grothendieck group, since $\mathcal{C}$ is a fusion category, it is semisimple. Then let $S=\left\{\left[X_{i}\right]\right\}$ be the collection of all simple objects in $\mathcal{C}$, we have the grothendieck group $K_{0}(\mathcal{C})$. Using the inner product $\langle\cdot, \cdot\rangle$ defined in the proof of Theorem 5.10, we have:

$$
\left\langle\left[X_{i}\right],\left[X_{j}\right]\right\rangle=\left[X_{i}\right]\left[X_{j}\right]=\sum_{X_{k} \in S} N_{X_{i}, X_{j}}^{X_{k}}\left[X_{k}\right]
$$

where $N_{X_{i}, X_{j}}^{X_{k}}=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}}\left(X_{i} \otimes X_{j} \rightarrow X_{k}\right)$ which now is also the multiplicity of $\left[X_{k}\right]$ in $\left[X_{i}\right]\left[x_{j}\right]$.
Combine with Lemma 5.13, we define an algebra map, for any $\left[X_{i}\right] \in S$ :

$$
\begin{aligned}
\phi: K_{0}(\mathcal{C}) & \rightarrow \mathbb{C}\left\langle H_{F, X}\right\rangle \\
\phi\left(\left[X_{i}\right]\right) & =H_{F, X_{i}}
\end{aligned}
$$

Then by Theorem 5.10, this algebra $\operatorname{map} \phi_{F}$ is well-defined and is an isomorphism.

Theorem 5.15. In $K_{0}(\mathcal{C})$ we can define an idempotent $R$ by:

$$
R=\frac{1}{\mathcal{D}} \sum_{[X] \in S} d_{[X]}[X]
$$

where $d_{[X]}=d_{X}=\epsilon(X) \circ \epsilon^{*}(X)$ is the dimension of $X$ in $\mathcal{C}, \mathcal{D}=\sum_{Y \in S}\left(d_{Y}\right)^{2}$ is the global dimension and $S$ is the base of $K_{0}(\mathcal{C})$.

We need the dollowing Lemma to show such idempotent $R$ exists.
Lemma 5.16. For $[X],[Y] \in K_{0}(\mathcal{C})$,

$$
d_{[X]} d_{[Y]}=\sum_{Z} N_{X, Z}^{Y} d_{[Z]}
$$

Proof. In unitary fusion category $\mathcal{C}$, for object $X$, we have $d_{X}^{*}=d_{X}$. For categorical dimension $d_{X Y}=\operatorname{co\epsilon }(X Y) \circ \cot ^{*}(X Y)$ since $\tau: A \rightarrow A^{* *}$ is a natural isomorphism in fusion category. For $\operatorname{co\epsilon }(X Y)$ :


Similarly for $\cot ^{*}(X Y)$ :


Therefore $d_{X Y}=\operatorname{co\epsilon }(X Y) \circ \operatorname{co\epsilon }^{*}(X Y)=\operatorname{co\epsilon }(X) \operatorname{co\epsilon }(Y) \circ \operatorname{co\epsilon }^{*}(Y) \circ \operatorname{co\epsilon }^{*}(X)=d_{X} d_{Y}$.
Which then give us:

$$
d_{X} d_{Y}=d_{X, Y}
$$

$$
\begin{aligned}
& =d_{[X][Y]} \\
& =\operatorname{dim}\left(\sum_{Z \in S} N_{X, Y}^{Z} Z\right) \\
& =\sum_{Z \in S} N_{X, Y}^{Z} d_{Z} \\
& =\sum_{Z \in S} \operatorname{dim} \operatorname{Hom}(X Y \rightarrow Z) d_{Z} \\
& =\sum_{Z \in S} \operatorname{dim} \operatorname{Hom}\left(X Z^{*} \rightarrow Y^{*}\right) d_{Z} \\
& =\sum_{Z \in S} N_{X, Z^{*}}^{Y_{Z}^{*}} d_{Z} \\
& =\sum_{Z \in S} N_{X, Z}^{Y} d_{[Z]}
\end{aligned}
$$

where the objects $X, Y, Z$ are self-adjoint in unitary fusion category [Gal12]. Then we can prove the theorem.

Proof.

$$
\begin{aligned}
R^{2} & =\frac{1}{\mathcal{D}^{2}} \sum_{[X],[Y] \in S} d_{[X]} d_{[Y]}[X][Y] \\
& =\frac{1}{\mathcal{D}^{2}} \sum_{[X],[Y],[Z] \in S} d_{[X]} d_{[Y]} N_{X, Y}^{Z}[Z] \\
& =\frac{1}{\mathcal{D}^{2}} \sum_{[Y],[Z] \in S} d_{[Y]} d_{[Y]} d_{[Z]} N_{X, Y}^{Z}[Z] \\
& =\frac{1}{\mathcal{D}} \sum_{[Z] \in S} N_{X, Y}^{Z} d_{[Z]}[Z] \\
& =R
\end{aligned}
$$

So $R$ is an idempotent in $K_{0}(\mathcal{C})$.
Recall we have algebra map $\phi_{F}$ then $\phi_{F}(R)=\frac{1}{\mathcal{D}} \sum_{X \in \operatorname{IrrC}} d_{X} H_{F, X}$ gives an idempotent on $\mathcal{H}_{F}$. Then we define the Hamiltonian $H_{F}$ as:

$$
\begin{equation*}
H_{F}=I-\frac{1}{\mathcal{D}} \sum_{X \in \operatorname{IrrC}} d_{X} H_{F, X} \tag{5.7}
\end{equation*}
$$

where $I$ is the identity map on $\mathcal{H}_{F}$.

Proposition 5.17. The Hamiltonian $H_{F}$ is a projection and it commutes with $H_{F^{\prime}}$ and $H_{V}$ for any face $F^{\prime}$ and vertex $V$

Since $\phi_{F}(R)=\frac{1}{\mathcal{D}} \sum_{X \in \operatorname{IrrC}} d_{X} H_{F, X}$ is an projection on $\mathcal{H}_{F}$, so is $I-\frac{1}{\mathcal{D}} \sum_{X \in \operatorname{IrrC}} d_{X} H_{F, X}=H_{F}$. The commutation just inherits from $H_{F, X}$.

### 5.5 Hamiltonian over Hilbert space $\mathcal{H}$

Recall we define the Hamiltonian $H$ as the sum of projections run over all edges and faces over the cellulation $\Gamma$, which is:

$$
\begin{equation*}
H=\sum_{E} H_{E}+\sum_{F} H_{F} \tag{5.8}
\end{equation*}
$$

It is a sum of commuting projections, so it is diagonalisable with eigenvalues lying in $\mathbb{N}$. The ground state for $H$ therefore is:

$$
\begin{equation*}
\operatorname{Ker}(H)=\left(\bigcap_{E} \operatorname{Ker}\left(H_{E}\right)\right) \cap\left(\bigcap_{F} \operatorname{Ker}\left(H_{F}\right)\right) \tag{5.9}
\end{equation*}
$$

## Chapter 6

## The toric code and its ground states

Toric code introduced by Kitaev is the simplest example which contains the general features of Levin-Wen models. It is an exactly solvable spin $1 / 2$ model on the lattice. In this chapter we will show that the ground states of the toric code model are isomorphic to the skein module of the quotient category $T L(\delta=1)$.

### 6.1 The toric code model

The toric code is defined on a specific cellulation of surface $\Gamma \subset \Sigma$ called square lattice with periodic boundary conditions and each edge is assigned a spin- $1 / 2$ state. A spin- $1 / 2$ model is just a model with two eigenstates. As each $1 / 2$-spin


Figure 6.1: The toric code is defined in terms of square lattice of surface $\Sigma$. A vertex $V$ and face $F$ are highlighted. Each circle is a $1 / 2$-spin state.
model only has two eigenstates, the toric code can be seen as a Levin-Wen model based on category $\mathcal{C}=\mathcal{R} e p \mathbb{Z} / 2 \mathbb{Z}$. Then we have two simple objects 1 and $g$ where 1 is the trivial representation and $g$ is the sign representation. It gives the Hilbert space of each edge:

$$
\mathcal{H}_{E}=\mathbb{C}^{2}
$$

In toric code, each vertex is 4 -valent. The Hilbert space of each vertex is then:

$$
\mathcal{H}_{V}=\bigoplus_{\substack{\text { labelling the } \\ \text { edges } E_{i} \text { adjacent } \\ \text { ob } \\ X_{i} \in \operatorname{br} \text { IrrC }}} \operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{i=1}^{4} X_{i}\right)
$$

Since $X_{i} \in\{1, g\}$, we have:

$$
\operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow X_{1} \otimes X_{2} \otimes X_{3} \otimes X_{4}\right)= \begin{cases}\mathbb{C}, & \text { if even number of } X_{i} \text { are } g \\ 0, & \text { otherwise }\end{cases}
$$

Then we get :

$$
\mathcal{H}_{V} \cong \bigoplus_{i=1}^{8} \mathbb{C} \cong \mathbb{C}^{8}
$$

The Hilbert space of the whole cellulation therefore is:

$$
\mathcal{H}_{\Gamma}=\bigotimes_{V \in \text { vertices of } \Gamma} \otimes \bigotimes_{E \in \operatorname{edges} \text { of } \Gamma} \mathcal{H}_{E}
$$

For each edge $E$ and surface $F$, consider Hamiltonian $H_{\Gamma}$ of the form:

$$
\begin{equation*}
H_{\Gamma}=\sum_{\text {vertices } \mathrm{E} \text { of } \Gamma} H_{V} \quad \text { and } \quad+\sum_{\text {faces } \mathrm{F} \text { of } \Gamma} H_{F} \tag{6.1}
\end{equation*}
$$

where the Hamiltonians of edges $E$ are defined by 5.1 and the Hamiltonians of faces $F$ are defined by 5.4. From Proposition 5.6 and 5.14, all of these Hamiltonians commute. Therefore Hamiltonians $H_{E}$ and $H_{F}$ are diagonalizable.
As Hamiltonians $H_{F}$ and $H_{E}$ commute, to get the kernel of $H$, we have the following relation:

$$
\begin{align*}
\operatorname{Ker}(H) & =\left(\bigcap_{E} \operatorname{Ker}\left(H_{E}\right)\right) \cap\left(\bigcap_{F} \operatorname{Ker}\left(H_{F}\right)\right)  \tag{6.2}\\
& =\operatorname{Ker}\left(\sum_{E} H_{E}\right) \cap \operatorname{Ker}\left(\sum_{F} H_{F}\right)
\end{align*}
$$

Recall when we define the Hamiltonians $H_{F, X}$, if one state is not in the kernel of $H_{F, X}$, it will also not be in the kernel of some $H_{E}$. therefore $\operatorname{Ker}\left(\sum_{F} H_{F}\right) \subset$ $\operatorname{Ker}\left(\sum_{E} H_{E}\right)$. So for determining $\operatorname{Ker}(H)$ it is enough to determine $\operatorname{Ker}\left(\sum_{F} H_{F}\right)$. But for complement, the kernel $\operatorname{Ker}\left(\sum_{E} H_{E}\right)$ is still be determined.

### 6.2 Kernel of $\sum_{E} H_{E}$

By the general formula of $\operatorname{Ker}\left(\sum_{E} H_{E}\right)$ in Proposition 5.7 , in toric code we have:




Recall in toric code:

$$
\operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{\substack{\text { labeledededeces } \\ X_{i} \text { adtacent } \\ \text { to } V}} X_{i}\right)= \begin{cases}\mathbb{C}, & \text { if even number of } X_{i} \text { are } g \\ 0, & \text { otherwise }\end{cases}
$$

Which gives $\operatorname{Ker}\left(\Sigma_{E \in \Gamma} H_{E}\right)$ has basis: all ways to label edges such that each vertex has even number adjacent edges labelled by $g$.

### 6.3 Kernel of $\sum_{F} H_{F}$

From equation 5.4, in toric code $H_{F}$ for any face $F$ is defined as:

$$
\begin{aligned}
H_{F} & =I-\frac{1}{\mathcal{D}} \sum_{X \in \operatorname{IrrC}}\left(d_{X}\right) H_{F, X} \\
& =I-\frac{1}{2} H_{F, 1}-\frac{1}{2} H_{F, g}
\end{aligned}
$$

where $I$ is the identity morphism and $d_{1}=\left(d_{g}\right)^{2}=1$.
Lemma 6.1. The Hamiltonian $H_{F, g}$ in toric code model reverse labels on all edges around $F$, while $H_{F, g}=I$ preserve the states.

Proof. Let us apply the Hamiltonian $H_{F, g}$



Consider the dashed area, since both $X_{4}$ and $P_{4}$ in $\{1, g\}$, there are only 4 different states. Observe that when $P_{4}=X_{4}$, the morphism in dashed area is zero, but when $P_{4}=g X_{4}$, the morphism collection $\operatorname{Hom}_{\mathcal{C}}\left(X_{4} \otimes g \rightarrow g X_{4}\right) \cong \mathbb{C}$. Repeat such analysis around the disc, we then have:



For $f_{i}$,

$$
f_{i} \in \bigoplus_{X_{j} \in \operatorname{Irr\mathcal {C}}} \operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{j=1}^{4} X_{j}\right)
$$

When $\sum_{j=1}^{4} X_{j}=1, \operatorname{Hom}_{\mathcal{C}}\left(1 \rightarrow \bigotimes_{j=1}^{4} X_{j}\right) \cong \mathbb{C}$ is 1-dimension, otherwise $\operatorname{Hom}_{\mathcal{C}}(1 \rightarrow$ $\bigotimes_{j=1}^{4} X_{j}$ ) is zero. Therefore $f_{i}$ is just a morphism over 1-dimensional vector space. Therefore $f_{i}$ just a scalar $\alpha$.
In $\mathcal{R e p}(\mathbb{Z} / 2 \mathbb{Z})$, every morphism space $\operatorname{Hom}_{\mathcal{C}}\left(\bigotimes_{i} X_{i} \rightarrow \bigotimes_{j} Y_{j}\right)$ can only be zero or 1-dimensional space $\mathbb{C}$ with identity as basis. Then for any two morphisms $\beta \in \operatorname{Hom}_{\mathcal{C}}\left(\bigotimes_{i} X_{i} \rightarrow \bigotimes_{j} Y_{j}\right)$ with $\beta(i d)=m \cdot i d$ and $\gamma \in \operatorname{Hom}_{\mathcal{C}}\left(\bigotimes_{k} W_{k} \rightarrow \bigotimes_{i} X_{i}\right)$ with $\gamma(i d)=n \cdot i d, \beta \circ \gamma(i d)=m n \cdot i d$. For any two morphisms $\beta \in \operatorname{Hom}_{\mathcal{C}}\left(\otimes_{i} X_{i} \rightarrow\right.$ $\left.\bigotimes_{j} Y_{j}\right)$ with $\beta(i d)=n \cdot i d$ and $\gamma \in \operatorname{Hom}_{\mathcal{C}}\left(\bigotimes_{k} W_{k} \rightarrow \bigotimes_{l} Z_{l}\right)$ with $\gamma(i d)=n \cdot i d$, $\beta \otimes \gamma(i d)=n \cdot i d$.
Also we have $\mathcal{R} e p(\mathbb{Z} / 2 \mathbb{Z})$ is rigid. Then we have $\epsilon(X): X^{*} \otimes X \rightarrow 1$ and $\operatorname{co\epsilon }(X): 1 \rightarrow X \otimes X^{*}$. Since $1^{*}=1$ and $g^{*}=g$, then $\epsilon(1)=1$ so $\epsilon$ and $\operatorname{co\epsilon }$ are all identities.
Now consider the dashed area. The green point and blue point represent two identity morphisms and we can multiply each morphism to the scalar morphism $f_{i}$ at the red vertex. Therefore we then have:



Then apply such relation around the disc.


Use the same method we can get $H_{F, 1}\left(\bigotimes_{i=1}^{4} f_{i} \otimes \bigotimes_{i=1}^{4} X_{i}\right)=\bigotimes_{i=1}^{4} f_{i} \otimes \bigotimes_{i=1}^{4} X_{i}$ so $H_{F, 1}=I$.

The Hamiltonian of faces is then:

$$
\begin{equation*}
H_{F}=I-\frac{1}{2} I-\frac{1}{2} V=\frac{1}{2} I-\frac{1}{2} R e_{F} \tag{6.3}
\end{equation*}
$$

where we use $R e_{F}$ to denote the reverse function which reverse the states of all edges around face $F$.
To find the kernel $\operatorname{Ker}\left(\sum_{F} H_{F}\right)$ we can use the projection $I-H_{F}=\frac{1}{2} I+\frac{1}{2} R e_{F}$. When such projection apply to the ground states, it just preserves the ground states. Thus, let us define:

$$
\prod_{F}\left(I-H_{F}\right)=\prod_{F}\left(\frac{1}{2} I+\frac{1}{2} R e_{F}\right)
$$

which is the projection onto $\operatorname{Ker}\left(\sum_{F} H_{F}\right)$. Therefore the ground state $x \in \mathcal{H}$ of $H$ is the same as the ground state of $\sum_{F} H_{F}$, which is the state satisfies

$$
R e_{F}(x)=I(x)
$$

for every face $F$.
Theorem 6.2. The ground states of $H$ in toric code model satisfies:

$$
\begin{equation*}
\operatorname{Ker}(H)=\mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right] \tag{6.4}
\end{equation*}
$$

So the ground states of toric code do not depends on the square lattice $\Gamma \in \Sigma$, but only depends on $\Sigma$.

Proof. To show the isomorphism holds, it is enough to define two injective maps $\phi: \operatorname{Ker}(H) \rightarrow \mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]$ and $\left.\psi: \mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]\right) \rightarrow \operatorname{Ker}(H)$.
For any element in $\left.\mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]\right)$, it is then a linear combination of 1-chains in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$, since $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ modulo $\operatorname{Im}\left(\partial C_{2}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right)$, we define $\psi^{\prime}$ : $\left.\mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]\right) \rightarrow \mathcal{H}$ which maps every 1 -chain in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ to loop on square lattice quotient by the boundaries formed by edges in lattice and the 1chain, and all homological equivalent 1-chains maps to the same loop on square lattice. Then label each corresponding edge by $g$.
After applying the map $\psi^{\prime}$ we then apply the projection $\prod_{F}\left(I-H_{F}\right)$ to the image


Figure 6.2: The way to map 1-chains in $H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})$ to loops on square lattice
of $\psi^{\prime}$. then by definition of $\prod_{F}\left(I-H_{F}\right)$, it gives a linear composition of elements in $\operatorname{Ker}(H)$ which is itself also in $\operatorname{Ker}(H)$. Thus we define $\psi=\prod_{F}\left(I-H_{F}\right) \circ \psi^{\prime}$ which is injective since both $\psi^{\prime}$ and $\prod_{F}\left(I-H_{F}\right)$ are injective.
Inversely, recall $\operatorname{Ker}(H)=\operatorname{Ker}\left(\sum_{F} H_{F}\right) \subset \operatorname{Ker}\left(\sum_{E} H_{E}\right)$, then there is an identity map $i: \operatorname{Ker}(H) \rightarrow \operatorname{Ker}\left(\sum_{E} H_{E}\right)$ canonically. The basis for $\operatorname{Ker}\left(\sum_{E} H_{E}\right)$ is the collection of states where each states satisfies the condition that every vertex has an even number of edges labelled by $g$. Then let us define a map $\phi^{\prime}: \operatorname{Ker}\left(\sum_{E} H_{E}\right) \rightarrow \mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]$ such that each edge labelled by $g$ transfers to an edge assigned by 1 and each edge labelled by 1 transfers to edge assigned by 0 in $\mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]$. Since each vertex has even number of adjacent edges labelled by $g$, then after apply the map $\phi^{\prime}$, the transferred vertices still have even number of edges assigned by 1 . Therefore, applying boundary map $\partial_{1}$ will give even number of 0 -chains at each vertices. For any $x \in \operatorname{Ker}(H)$, we then define $\phi(x)=\phi^{\prime} \circ i(x) \in \mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]$. For any element $x \in \operatorname{Ker}(H)$, we have $\operatorname{Re}_{F}(x)=x$ for any faces $F$. By the definition of $R e_{F}$, it reverses the labelling of edges around $F$ then $\phi \circ R e_{F}$ is the same as add an extra 2-boundary in $\mathbb{C}\left[H_{1}(\Sigma ; \mathbb{Z} / 2 \mathbb{Z})\right]$. So $\phi \circ \operatorname{Re}_{F}(x)=\phi(x)$ for any faces gives $\phi(x)$ is the homological
equivalent class. So $\phi$ is injective over $\operatorname{Ker}(H)$.

Then combine with Theorem 4.14, we have:

$$
\mathcal{S}_{\Sigma}(T L(\delta=1) ; \mathbb{Z} / 2 \mathbb{Z})=\operatorname{Ker}(H)
$$

which shows the Temperley-Lieb category $T L(\delta=1)$ generates toric code model.

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