Math 3325, 2013 — Assignment 2

Discuss in tutorial on Sept 2, and hand in by noon on Friday Sept 6

(Please note I'll be out of town Thursday and Friday September 5-6, so if you have questions about the assignment please ask them before Thursday!)

This assigment is out of 100: 4 questions worth a total of 80 marks, and 20 marks for your writing quality. Acknowledge any help that you receive, either from a book, another student, or an internet source. Discussion of the problems with other students is allowed, but you must write your solutions yourself. Do not look at anyone else's solutions, and do not show your solutions to another student.

1. (30 marks) Note: attempt part (iii) even if you cannot do parts (i) and (ii).

(i) (10 marks) Suppose that f_n and f are functions in $L^1(\mathbb{R}^n)$, such that $f_n \to f$ in L^1. Show that $\hat{f}_n \to \hat{f}$ uniformly. Here the Fourier transform is defined by the convergent integral

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

(ii) (10 marks) Suppose that a sequence of bounded continuous functions g_n converge uniformly to the function h, and in L^2 to the function k. Show that $h = k$ a.e.

(iii) (10 marks) Now suppose that $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then we have two, potentially different, definitions of the Fourier transform of f: firstly by a convergent integral,

$$\mathcal{F}_1 f(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$$

and secondly by taking a sequence of Schwartz functions f_n converging to f in L^2, and taking the L^2 limit of the sequence $\mathcal{F}_1 f_n$ (call this limit $\mathcal{F}_2 f$). Use parts (i) and (ii) to show that $\mathcal{F}_1 f = \mathcal{F}_2 f$ a.e. Suggestion: use the fact, proved in the text, that there is a sequence of Schwartz functions f_n converging to f in both the L^1 and the L^2 sense.

2. (20 marks) Suppose that $g \in L^1(\mathbb{R})$ is compactly supported.

(i) (10 marks) Show that $\hat{g}(\xi)$ is C^∞.

(ii) (10 marks) Show that $\hat{g}(\xi)$ is analytic and entire. That is, show that the Taylor series of $\hat{g}$ at $\xi = 0$ has infinite radius of convergence, and converges to $\hat{g}$.

3. (20 marks)

(i) (10 marks) Show that the only $f \in L^1(\mathbb{R})$ satisfying $f = f * f$ is the zero function.

(ii) (10 marks) Find a nonzero $g \in L^2(\mathbb{R})$ such that $g = g * g$.

4. (10 marks) Show that bounded continuous functions are not dense in $L^\infty(\mathbb{R})$.