Math 3325, 2013 — Assignment 3

Discuss in tutorial on Sept 23 and 30, and hand in by 5pm on October 4

This assignment is worth 100 marks: 20 for each question below, and 20 for writing quality.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \in L^p(X, \mu)$ be a nonnegative real-valued function. Let $\lambda_f : [0, \infty) \to [0, \infty)$ be the distribution function of f, defined by

$$\lambda_f(\alpha) = \mu\{x \mid f(x) > \alpha\}.$$

(a) (12 marks) First assume that $p = 1$. Show that the integral of f is equal to

$$\int_0^\infty \lambda_f(\alpha)\, d\alpha.$$

Hint: consider the product measure space $(X \times \mathbb{R}, \mathcal{M} \times \mathcal{L}, \mu \times \lambda)$ where $(\mathbb{R}, \mathcal{L}, \lambda)$ is the real line with the σ-algebra of Lebesgue measurable sets and the Lebesgue measure. Relate the integral of f to the set

$$\{(x, \alpha) \in X \times \mathbb{R} \mid 0 \leq \alpha \leq f(x)\}$$

and use Fubini's theorem.

(b) (8 marks) In a similar fashion, show that for $1 \leq p < \infty$

$$\|f\|_p^p = \int_0^\infty p\alpha^{p-1}\lambda_f(\alpha)\, d\alpha.$$

2. Use polar coordinates to prove that

(a) (10 marks) the volume of the sphere S^{n-1} (i.e. the boundary of the unit ball in $\mathbb{R}^n$), with respect to the standard measure on S^{n-1}, is

$$\frac{2\pi^{n/2}}{\Gamma(n/2)},$$

and

(b) (10 marks) the Lebesgue measure of the unit ball B^n is

$$\frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

Hint: obtain (b) from (a). To get (a), compute

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx$$

using polar coordinates. Apply integration by parts to the resulting integral and relate the measure of S^{n-1} to that of S^{n-3}. Also use the fact that

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \alpha > 0,$$

which you can take as the definition of $\Gamma(n/2)$ for half-integers $n/2$ if you wish.

3. Let μ be a Borel measure on $\mathbb{R}^n$ that is translation invariant, such that the μ-measure of the unit cube $(0,1]^n$ is equal to 1. The aim of this question is to show that μ agrees with Lebesgue measure on the Borel sets.

(a) (10 marks) First show that it agrees with Lebesgue measure on rectangles (start with rectangles with rational side lengths).

(b) (10 marks) Show that it is absolutely continuous with respect to Lebesgue measure (restricted to the Borel σ-algebra), and hence that it can be written in the form

$$\mu(E) = \int_E f d\lambda,$$

where $d\lambda$ denotes Lebesgue measure, for some λ-measurable function f.

(You can either do exactly what the question says, or achieve the results of parts (b) and (c) together by another means.)

(c) (0 marks) Finally, $f = 1$ a.e. using the Lebesgue Differentiation theorem: given a measurable function f, for a.e. x,

$$f(x) = \lim_{\epsilon \to 0} \frac{1}{\lambda(Q(x,\epsilon))} \int_{Q(x,\epsilon)} f(y)\, d\lambda(y).$$

Here $Q(x,\epsilon)$ is a cube of side length ϵ centred at x.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a sequence f_n of measurable functions *converges in measure* to f if, for every $\epsilon > 0$,

$$\mu\big(\{x \in X \mid |f_n(x) - f(x)| > \epsilon\}\big) \to 0.$$

This is also called convergence in probability.

(a) (5 marks) Show that if $\|f_n - f\|_{L^1(X)} \to 0$, then f_n converges to f in measure.

(b) (5 marks) Give an example to show that the converse is false.

(c) (10 marks) Suppose that $(X, \mathcal{M}, \mu)$ is a measure space, (f_n) is a sequence of real-valued measurable functions on X such that all f_n are dominated by a fixed integrable function g. If f_n converges to f in measure, show that f is integrable and that

$$\int f_n \to \int f.$$

That is, DCT holds with the hypothesis of pointwise a.e. convergence replaced by convergence in measure.