**Math 3325, 2013 — Assignment 4**

**Hand in by 5pm on Nov 1**

This assignment is worth 100 marks: 80 for the the best 4 out of the 5 questions below, and 20 for writing quality.

To assist with grading, could you please staple questions 1 & 2 separately from questions 3, 4, & 5, and ensure your name is on both.

1. (a) (10 marks) Let $A$ be a bounded self-adjoint operator. Show that $U = (A - iI)(A + iI)^{-1}$ is unitary.

(b) (5 marks) Let $X$ be a bounded operator. Show that $\sigma(X^*) = \{\bar{\lambda} | \lambda \in \sigma(X)\}$ and that if $X$ is invertible then $\sigma(X^{-1}) = \{\lambda^{-1} | \lambda \in \sigma(X)\}$.

(c) (5 marks) Show that the only positive unitary operator is $I$.

2. Let $A$ be a bounded self-adjoint operator.

(a) (10 marks) Show that $A \geq kI$ for $k \in \mathbb{R}$ if and only if $\sigma(A) \subset [k, \infty)$.

(b) (10 marks) Show that if $A \geq I$, $A^n \geq I$ for every positive integer $n$.

3. Let $A$ be a bounded self adjoint operator on $\mathcal{H}$. Fix some vector $f \in \mathcal{H}$, and let $V$ be the closure of the span of the elements $\{A^n f\}_{n=0}^{\infty}$.

(a) (5 marks) Show $V$ is invariant under $A$, and that if $P$ is the orthogonal projection on $V$, that $AP = PA$.

(b) (5 marks) In our functional calculus for $A$ on upper semicontinuous functions, show that for any operator $B$ such that $AB = BA$, also $f(A)B = Bf(A)$. (Bonus: also do this for the Borel functional calculus.)

(c) (5 marks) If $A_0$ denotes the restriction of $A$ to $V$, and if $E(\lambda)$ is the spectral resolution of $A$, show that $PE(\lambda)$ is the spectral resolution of $A_0$.

(d) (5 marks) Show that $\sigma(A_0) \subset \sigma(A)$.

4. Let $S$ be a linear subspace of $C([0,1])$. Since $C([0,1])$ is a subset of $L^2([0,1])$ we can also regard it as a subspace of $L^2([0,1])$. We assume that $S$ is closed as a subspace of $L^2([0,1])$, i.e., in the $L^2$ topology.

(a) (5 marks) Show that $S$ is a closed subspace of $C([0,1])$ (under the sup norm).

(b) (5 marks) Show that there exists $M > 0$ such that for all $f \in S$,

$$\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2.$$

(Use the closed graph theorem.)

(c) (5 marks) Fix $y \in [0,1]$. Show that there exists a function $k_y \in L^2([0,1])$, with $\|k_y\|_{L^2([0,1])} \leq M$, such that

$$f(y) = \int_0^1 k_y(x) f(x)\, dx$$

for all $f \in S$. (Use the Hilbert space Riesz representation theorem.)

(d) (5 marks) Show that the $L^2$ unit ball of $S$ is compact, and hence that $S$ finite dimensional. (Show that a sequence in the unit ball which converges weakly converges in norm.)

5. (a) (6 marks) Let $(A_n)$ be a decreasing sequence of nonempty closed balls in a Banach space. Show that the intersection of the $A_n$ is nonempty. (Do not assume that the radii converge to zero.)

(b) (6 marks) Let $(B_n)$ be a decreasing sequence of closed, bounded, nonempty convex sets in a reflexive Banach space $Y$. Show that the intersection of the $B_n$ is nonempty. (Hint: first show that the sets $B_n$ are weakly closed, using the Separating Hyperplane theorem. If you can't, assume it and complete the rest of the problem, using Banach-Alaoglu.)

(c) (8 marks) Let $X = L^1(\mathbb{R})$. Find a decreasing sequence of closed, bounded, nonempty convex sets $C_n \subset X$ whose intersection is empty.