0 Towards functional calculus

The most powerful tool for describing linear transformations in finite dimensions is the theory of eigenvalues and (generalized) eigenvectors.

What can we say about linear transformations in infinite dimensions?

First, why do we want to say anything at all?

1. Several of the essential equations of fundamental physics (e.g. Schrödinger's equation from quantum mechanics, the heat equation, but not Einstein's equations from general relativity or the Navier-Stokes equations from fluid mechanics) are *linear* differential equations, and an abstract understanding of the corresponding linear operators allows us to understand the solutions to these equations.

2. It leads naturally to the study of Banach algebras, C^*-algebras, and von Neumann algebras, essential tools in the study of quantum mechanics and quantum field theory.

3. It's a beautiful mathematical theory, and a great starting place to learn functional analysis and operator algebra.

Until we've set up a lot of definitions, we won't be able to precisely state the infinite dimensional analogues of the theory of eigenvalues and eigenvectors. However, we can ask already a very closely related question! (Why it's so closely related, we'll see along the way.)

As soon as we know how to diagonalize a matrix, $M = UDU^{-1}$, we can also *take functions* of that matrix, by making the definition $f(M) = Uf(D)U^{-1}$, where for a diagonal matrix D, $f(D)$ means taking f of each diagonal entry.

In fact, suppose we knew how to take arbitrary functions of some matrix M (but hadn't yet tried to diagonalize it). From these functions we could actually recover all the information coming from diagonalization. For example, if we wanted to know if M had the number λ as a eigenvalue, we would take the function

$$f(x) = \begin{cases} 1 & \text{if } x = \lambda \\ 0 & \text{otherwise,} \end{cases}$$

and look at $f(M)$. We see that $f(M)$ is the zero matrix exactly if λ is not an eigenvalue! (And otherwise, it's the projection onto the λ eigenspace.)

This somewhat abstract idea — that we understand a linear transformation exactly if we can apply functions to it — motivates the main question for this course.

Some quick notation. Let's write $\mathcal{T}(\mathcal{H})$ for all the linear transformations of some infinite dimensional vector space $\mathcal{H}$. Recall an *algebra* is a vector space which additionally has a multiplicative structure. The linear transformations of $\mathcal{H}$ form an algebra under composition. Complex-valued functions $f : \mathbb{C} \to \mathbb{C}$ form an algebra under point-wise multiplication, and by 'an algebra of functions' we mean some subalgebra of this one. The first, and most important, example of an algebra of functions is the polynomials.

> **Given a linear operator T on an infinite dimensional vector space $\mathcal{H}$, and an algebra of functions $\mathcal{U}$, when can we produce a homomorphism of algebras $\Psi : \mathcal{U} \to \mathcal{T}(\mathcal{H})$, sending the identity function to T?**

Such a homomorphism of algebras is called a 'functional calculus'. (Recall that an algebra homomorphism is a linear map respecting multiplication, so $\Psi(fg) = \Psi(f)\Psi(g)$, where on the left hand side we mean pointwise multiplication of functions, and on the right hand side composition of linear transformations.)

Along the way we'll prove increasingly powerful theorems giving partial answers to this question. A complete answer is still very much an open problem! These theorems will, at the same time, provide analogues of the theory of eigenvalues in finite dimensions.

Infinite dimensional vector spaces, in full generality, are hopelessly intractable objects; we're going to begin by assuming a lot of additional structure. We'll first study *Hilbert spaces*, and their linear transformations. Later we'll move to *Banach spaces*: there's less to work with, so we can't prove as much, but for some problems they are essential.

Before we turn to the details of Hilbert spaces, I want to point out a few relatively easy facts, which in fact provide the core tools for creating functional calculuses.

Theorem 0.1. *For any linear operator T on any vector space, there is a functional calculus for the algebra of polynomial functions.*

Proof: We send a polynomial $p(x) = \sum_{i=0}^{k} a_i x^i$ to $p(T) = \sum_{i=0}^{k} a_i T^k$, where T^k means the k-fold composition of T with itself. It's easy to verify this is an algebra homomorphism. □

Theorem 0.2 (Weierstrauss approximation theorem). *Any continuous function on $[a, b]$ can be uniformly approximated by polynomials.*

We've always got a functional calculus for polynomials, and polynomials are good enough to approximate arbitrary continuous functions (at least on bounded intervals). This naturally suggests an approach to defining a functional calculus for all continuous functions. We just need to develop of theory of limits of linear transformations, powerful enough that if we define $f(T)$ as the limit of $p_n(T)$, where $\{p_n\}$ is a sequence of polynomials approximating f, that this actually results in an algebra homomorphism! (This will only work for sufficiently nice T, because we need to work on bounded intervals of $\mathbb{R}$.)

Hilbert spaces provide the most excellent example of such a theory, and so we will begin our study there.