3. Review of 'calculus'

• Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be continuous. The support of ϕ is the closure of the set where $\phi(x) \neq 0$. If the support of ϕ is compact then we say that ϕ is compactly supported.

• There exist functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$, ϕ is C^∞, and ϕ is compactly supported. The set of compactly supported, smooth functions on $\mathbb{R}^n$ is denoted $C_c^\infty(\mathbb{R}^n)$.

• L^p norms. The L^p norm, $p \geq 1$, of a measurable function f on a measurable set E is defined to be

$$\|f\|_{L^p(E)} := \Big(\int_E |f(x)|^p \, dx \Big)^{1/p}.$$

It is a norm (homogeneous, nonnegative, obeys triangle inequality) provided we identify functions which differ on a set of measure zero. The normed space of (equivalence classes of) functions with finite L^p norm is denoted $L^p(E)$. A very important property is that $L^p(E)$ is complete; we will prove this later in the course. We also define $L^\infty(E)$ to be the set of essentially bounded (equivalence classes of) functions, i.e. those for which

$$\|f\|_{L^\infty(E)} := \sup \Big\{ M | \text{ the set } \{x \mid |f(x)| > M\} \text{ has positive measure.} \Big\}$$

is finite. This is also a complete normed space.

• If $\phi : \mathbb{R}^n \to \mathbb{R}$ is continuous and compactly supported, then it is in L^p for every $1 \leq p \leq \infty$.

• Hölder's inequality: if $p^{-1} + q^{-1} = 1$,

$$\Big| \int_E f(x) g(x) \, dx \Big| \leq \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

To prove Hölder's inequality, we begin with Jensen's inequality (stating that secants of convex functions stay above the function) for the function

$x \mapsto b^x$, obtaining

$$b \leq \frac{1}{p} + \frac{b^q}{q}.$$

Next, we take advantage of the fact that this inequality holds for all b, but the different terms scale differently in b. (You should read Terry Tao's blog post 'Amplification, arbitrage, and the tensor product trick'!) In particular, replacing b with $a^{1-p}b$ and rearranging we obtain Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

From this, Hölder's inequality follows easily --- first prove it for functions with $\|f\|_p = 1$ and $\|g\|_q = 1$.

• Dominated convergence theorem. (SS Chapter 2 Theorem 1.13).

Let f_n be a sequence of functions in $L^1(E)$ converging pointwise a.e. to f. Suppose that $|f_n(x)| \leq g(x)$ for a fixed L^1 function g. Then

$$\int_E f_n \to \int_E f.$$

Sketch: Consider the sets E_N on which $|x| \leq N$ and $|g(x)| \leq N$. Eventually, every point is in some E_n, and so by the monotone convergence theorem $\int_{E_N^c} g$ becomes arbitrarily small. Estimate $\int_E |f_n - f|$ as the sum of the integral on one of these sets and the integral on the complement; use the bounded convergence theorem on the first integral and $|f_n - f| \leq 2g$ on the second. □

The bounded convergence theorem is now a special case of the dominated convergence theorem, but of course one needs to prove it first!

The bounded convergence theorem follows easily from Egorov's theorem (SS Chapter 1 Theorem 4.4) which says that any pointwise limit of functions actually converges uniformly, off some arbitrarily small open set.

Sketch: [Egorov] Define

$$E_k^n = \{x \in E \, | \, |f_j(x) - f(x)| < 1/n \text{ for all } j > k\}.$$

Choose k_n large enough that $m(E - E^n_{k_n}) < 2^{-n}$. Let $\tilde{A}$ be the intersection of some tail of the sets $\{E^n_{k_n}\}$, choosing the tail so that $\tilde{A}$ has almost full measure. Finally let A be a closed subset of $\tilde{A}$, omitting only an small set. $\square$

- Fubini-Tonelli theorem (in $\mathbb{R}^n$):

Theorem 3.1.

(i) Suppose that $f : \mathbb{R}^{n+m} \to \mathbb{C}$ is nonnegative and measurable. Then

$$\int_{\mathbb{R}^{n+m}} f = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} f(x,y)\, dy \Big)\, dx = \int_{\mathbb{R}^m} \Big(\int_{\mathbb{R}^n} f(x,y)\, dx \Big)\, dy. \tag{3.1}$$

Note: this is an equality in extended real numbers: the left hand side might be $+\infty$, but this happens if and only if the right hand side is also $+\infty$.

(ii) Suppose that $f \in L^1(\mathbb{R}^{n+m})$. Then (3.1) *holds.*

The first part is Tonelli's, the second part Fubini's.

Often we use these in conjunction. Suppose we are asked to integrate some function f on $\mathbb{R}^d$, but don't even know it is integrable. We first apply Tonelli's theorem to $|f|$, justifying the use of multiple integrals. Maybe we can calculate them, or if not, at least estimate them. Thus we can establish that f is integrable. Finally we apply Fubini's theorem to justify using multiple integrals in the actual calculation.

• Polar coordinates:

Let f be an real-valued integrable function on $\mathbb{R}^n$. Define the $(n-1)$-sphere by

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

Let $\tilde{f}(r, \omega) = f(r\omega)$, so $\tilde{f} : \mathbb{R}_+ \times S^{n-1} \to \mathbb{R}$. Also, for a measurable subset E of $\mathbb{R}^n$, and $r > 0$, define $E_r \subset S^{n-1}$ by

$$E_r = \{\omega \in S^{n-1} \mid r\omega \in E\}.$$

Then

$$\int_E f(x)\, dx = \int_0^\infty \Big(\int_{E_r} \tilde{f}(r, \omega)\, d\omega \Big) r^{n-1}\, dr.$$

• Using polar coordinates we see the following: Let B be the unit ball in $\mathbb{R}^n$. The function $|x|^{-\alpha}$ is in $L^1(B)$ iff $\alpha < n$ and it is in $L^1(\mathbb{R}^n \setminus B)$ iff $\alpha > n$.

• Absolutely continuous functions and the fundamental theorem of calculus.

Definition 3.2. A function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for any $\epsilon > 0$ there exists a $\delta > 0$ so that

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta$$

and the intervals (a_k, b_k) are disjoint.

Theorem 3.3 (SS Chapter 3, Theorem 3.8)**.** *An absolutely continuous function is differentiable almost everywhere. Moreover, if its derivative is zero almost everywhere, the function is constant.*

Theorem 3.4 (SS Chapter 3, Theorem 3.11)**.** *The derivative of an absolutely continuous function F is integrable, and*

$$F(x) - F(a) = \int_a^x F'(y)dy.$$

Conversely, if f is integrable on $[a, b]$, then $F(x) = \int_a^x f(y)dy$ is absolutely continuous and $F'(x) = f(x)$ almost everywhere.

- Differentiating under the integral sign:

Proposition 3.5. *Suppose that U is an open set in $\mathbb{R}^n$, E is a measurable set in $\mathbb{R}^k$, $f : U \times E \to \mathbb{R}$ is a function so that*

(i) $f(x, \cdot) : E \to \mathbb{R}$ is measurable for each $x \in U$,

(ii) $\partial_{x_i} f(x, y)$ exists and is continuous for all (x, y) and

(iii) (the crucial condition)

$$|\partial_{x_i} f(x, y)| \le g(y) \text{ for some } g \in L^1(E).$$

Then

$$\frac{\partial}{\partial x_i} \int_E f(x, y)\, dy = \int_E \frac{\partial f}{\partial x_i}(x, y)\, dy.$$

Proof: (sketch) The LHS is, for a fixed x,

$$\lim_{h \to 0} \int_E \frac{f(x + he_i, y) - f(x, y)}{h}\, dy.$$

Use (ii) and the mean value theorem to write the integrand as $\partial_{x_i} f(x+\theta(h)e_i, y)$ for some $0 \le \theta(h) \le h$ and conclude that it is pointwise bounded by $g(y)$. Then by the dominated convergence theorem, we can take the pointwise limit inside the integral. This is just $\partial_{x_i} f(x, y)$ using (ii) again, which gives us the RHS. $\square$

- Change of variable formula:

Theorem 3.6. *Let $R \subset \mathbb{R}^n$ be a rectangle, and $F : R \to \mathbb{R}^n$ a C^1 function. Then for every continuous function f defined on $F(R)$, we have the change of variable formula*

$$\int_{F(R)} f(y)dy = \int_R (f \circ F)(x)|\det DF(x)|dx. \tag{3.2}$$

We sometimes write this differently: we think of F as relating two different sets of coordinates, the y coordinates on $F(R)$ and the x coordinates on R. We sometimes write $y = y(x)$ instead of $y = F(x)$. Also, the Jacobian matrix DF is sometimes written $\partial y/\partial x$. So we have

$$\int_{F(R)} f(y)\, dy = \int_R f(y(x))\left|\det \frac{\partial y}{\partial x}\right| dx.$$

• Surface measure. Let S be a hypersurface given by the graph of a C^1 function:

$$S = \{(x_1, \dots, x_n) \mid x_n = u(x_1, x_2, \dots, x_{n-1})\},$$
$$u \in C^1(\mathbb{R}^{n-1}).$$

Then, in terms of the coordinates $(x_1, \dots, x_{n-1})$ on S, surface measure on S is defined to be

(3.3)
$$d\sigma = \sqrt{1 + |\nabla u(x')|^2}\, dx', \quad x' = (x_1, \dots, x_{n-1}).$$

Proposition 3.7. *The measure $d\sigma$ on S is invariant under a Euclidean change of coordinates. That is, suppose that $(y_1, \dots, y_n)$ are another set of Euclidean coordinates. This means that there is an orthonormal basis e_i' such that $(y_1, \dots, y_n)$ represents the point $\sum_i y_i e_i'$. If S can also be written as a graph in the y coordinates,*

$$S = \{(y_1, \dots, y_n) \mid y_n = v(y_1, y_2, \dots, y_{n-1})\}, \quad v \in C^1,$$

then we have

$$d\sigma = \sqrt{1 + |\nabla v(y')|^2}\, dy', \quad y' = (y_1, \dots, y_{n-1}).$$

The key to proving this proposition is showing that, if the y' coordinates on S are given in terms of x' by $y' = F(x')$, then

(3.4)

$$\det DF(x_0) = \frac{\sqrt{1 + |\nabla u(x_0')|^2}}{\sqrt{1 + |\nabla v(y_0')|^2}}, \quad y_0' = F(x_0').$$

We then use Theorem 3.6.

The identity (3.4) can be proved by considering two Euclidean sets of coordinates $y = (y_1, \ldots, y_n)$ and $x = (x_1, \ldots, x_n)$. Change to $\tilde{y} = (y_1, \ldots, y_{n-1}, Y_n)$ and $\tilde{x} = (x_1, \ldots, x_{n-1}, X_n)$ where $Y_n = y_n - v(y')$, $X_n = x_n - u(x')$. Then show that, on the surface,

$$\det \frac{\partial y'}{\partial x'} = \Big(\frac{\partial Y_n}{\partial X_n}\Big)^{-1}.$$

This can be computed explicitly, to be equal to

$$\frac{\sqrt{1 + |\nabla u(x')|^2}}{\sqrt{1 + |\nabla v(y')|^2}}.$$

• The result above allows us to define surface measure for any C^1 hypersurface, not just a graph.

• Integration by parts: the following result will be adequate for now; it is possible to weaken the assumptions.

Proposition 3.8.

(i) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Then if $f, g \in C^1(\overline{\Omega})$, we have

$$\int_\Omega \left(f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i} \right) \, dx = \int_{\partial\Omega} fgn_i \, d\sigma$$

where $n_i = n \cdot e_i$ is the ith component of the outward pointing normal vector n and σ is surface measure on $\partial\Omega$.

(ii) Assume that f, g are C^1 functions on $\mathbb{R}^n$, such that $f, \partial_{x_i} f \in L^p(\mathbb{R}^n)$, while $g, \partial_{x_i} g \in L^q(\mathbb{R}^n)$, with $p^{-1} + q^{-1} = 1$. Then

$$\int_{\mathbb{R}^n} f\frac{\partial g}{\partial x_i} \, dx = -\int_{\mathbb{R}^n} g\frac{\partial f}{\partial x_i} \, dx.$$

Notice that $dx' = (n \cdot e_n) d\sigma$ in the notation of (3.3), where n is the upward pointing unit normal to S.