7. Measure Theory

You have already studied Lebesgue measure in $\mathbb{R}^n$. Here we consider a more abstract setup. This will allow us to develop a unified framework to deal with many different situations, such as measures supported on submanifolds or fractals, or infinite dimensional spaces, such as Wiener measure used in the definition of Brownian motion.

Definition 7.1. A measure space is a set X together with a σ-algebra $\mathcal{M}$, a collection of subsets of X, and a measure $\mu : \mathcal{M} \to [0, \infty]$, such that

• $\mathcal{M}$ is required to contain the empty set and to be closed under countable unions and complementation (hence also closed under countable intersections).

• μ is required to be countably additive: if $E_1, E_2, \dots$ is a countable family of disjoint measurable sets, then

$$\mu\big(\cup_n E_n \big) = \sum_n \mu(E_n).$$

Example.

• Lebesgue measure on $\mathbb{R}^n$, with the σ-algebra of Lebesgue measurable sets.

• Counting measure on any set, where every set is measurable.

• Surface measures.

• Hausdorff measures, which are 'fractional dimensional' measures of subsets of $\mathbb{R}^n$. We will introduce these in due course.

How to construct measure spaces

Measures can be constructed from 'exterior measures'. Let X be any set. An **exterior measure** μ_* is a function from $\mathcal{P}(X)$ to $[0, \infty]$ such that

(i) $\mu_*(\emptyset) = 0$;

(ii) If $E_1 \subset E_2$, then $\mu_*(E_1) \le \mu_*(E_2)$;

(iii) if $E_1, E_2, \dots$ is a countable family of sets, then

$$\mu_*\big(\cup_n E_n \big) \le \sum_n \mu_*(E_n).$$

There is a general construction, due to Carathéodory, of obtaining a measure from an exterior measure. We say that $E \subset X$ is measurable if, for every subset $A \subset X$, we have

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c).$$

That is, E separates every set efficiently: we have equality in the equation above rather than '$\le$' which holds automatically. Heuristically, we can think of a non-measurable set as being one that looks big 'from the outside' but small 'from the inside', and the exterior measure as measuring the size of the set as viewed from the outside. But $\mu_*(A) - \mu_*(A \cap E^c)$ in some sense measures the size of $A \cap E$ from the inside, so if we get equality, then this indicates that E should be measurable.

Theorem 7.2. *Let μ_* be an exterior measure on X. Then the subset $\mathcal{M}$ of $\mathcal{P}(X)$ consisting of measurable sets forms a σ-algebra, and μ_* restricted to $\mathcal{M}$ is a measure.*

Proof: 1. It is clear that $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.

2. To show that the union of two sets $E_1, E_2 \in \mathcal{M}$ is in $\mathcal{M}$, we use

$$\mu_*(A) = \mu_*(A \cap E_1) + \mu_*(A \cap E_1^c)$$

and then write each of these two terms as a sum as follows

$$\mu_*(A \cap E_1) = \mu_*(A \cap E_1 \cap E_2) + \mu_*(A \cap E_1 \cap E_2^c)$$
$$\mu_*(A \cap E_1^c) = \mu_*(A \cap E_1^c \cap E_2) + \mu_*(A \cap E_1^c \cap E_2^c).$$

Now using subadditivity of μ_*, we have

$$\mu_*(A \cap (E_1 \cup E_2)) \le \mu_*((A \cap E_1 \cap E_2)) + \mu_*((A \cap E_1 \cap E_2^c)) + \mu_*((A \cap E_1^c \cap E_2))$$

(Draw a diagram!) Combining the two we get

$$\mu_*(A \cap (E_1 \cup E_2)) + \mu_*(A \cap E_1^c \cap E_2^c) \le \mu_*(A)$$

or equivalently

$$\mu_*(A) \ge \mu_*(A \cap (E_1 \cup E_2)) + \mu_*(A \cap (E_1 \cup E_2)^c),$$

since $A \cap E_1^c \cap E_2^c = A \cap (E_1 \cup E_2)^c$. Equality in this equation follows from subadditivity, which gives the '$\le$' direction.

2.' Additivity for two sets follows directly from the definition of measurability: put $A = E_1 \cup E_2$ and $E =$

E_1, and obtain

$$\mu_*(E_1 \cup E_2) = \mu_*(E_1) + \mu_*(E_2).$$

One then uses induction to obtain finite additivity.

3. To show closure under countable unions, let $E_i \in \mathcal{M}$. Without loss of generality, the E_i are disjoint. Let $G_n = E_1 \cup \cdots \cup E_n$, and let $G = \cup E_n$. Then each $G_n \in \mathcal{M}$, and we have

$$\begin{aligned}\mu_*(A) &= \mu_*(A \cap G_n) + \mu_*(A \cap G_n^c) \\ &\geq \mu_*(A \cap G_n) + \mu_*(A \cap G^c)\end{aligned}$$

and since $A \cap G_n = \cup_{j=1}^n A \cap E_j$ so $\mu_*(A \cap G_n) = \sum \mu_*(A \cap E_j)$, using finite additivity as proved earlier, we now have

$$\sum_{j=1}^{n} \mu_*(A \cap E_j) \leq \mu_*(A) - \mu_*(A \cap G^c).$$

Hence, taking the limit as $n \to \infty$,

$$\sum_{j=1}^{\infty} \mu_*(A \cap E_j) \leq \mu_*(A) - \mu_*(A \cap G^c).$$

By countable subadditivity, $\mu_*(A \cap G) \leq \sum_{j=1}^\infty \mu_*(A \cap E_j)$, and so

$$\mu_*(A \cap G) \leq \mu_*(A) - \mu_*(A \cap G^c)$$

The other inequality is automatic, so we have

$$\mu_*(A \cap G) = \mu_*(A) - \mu_*(A \cap G^c).$$

Thus $G \in \mathcal{M}$.

3' Finally we need to show countable additivity. Since $\sum_{j=1}^\infty \mu_*(A \cap E_j)$ is pinched between two quantities we now know are equal, we also have

$$\sum_{j=1}^{\infty} \mu_*(A \cap E_j) = \mu_*(A \cap G).$$

Finally putting $A = G$ we get countable additivity of μ_* on $\mathcal{M}$. □

This measure μ has the property of being complete: if Z is a measurable set with measure zero, then every subset of Z is measurable (with measure zero, of course). Any measure that is not complete can be completed in a natural way: see exercise 2 in the text.

Measures on metric spaces

The theorem above is very nice, but how do you find out which sets are measurable? Suppose that X is a metric space. Then we would like at least all open sets to be measurable. This is guaranteed by the following condition that relates the metric and measure properties of X. Before stating it we make the following

Definition 7.3. The Borel σ-algebra $\mathcal{B}_X$ is the intersection of all σ-algebras containing all the open sets of X. (Note that the intersection of any collection of σ-algebras is itself a σ-algebra.) A measure defined on $\mathcal{B}_X$ is called a Borel measure on X.

Said another way, the Borel σ-algebra is the smallest σ-algebra containing all the open sets of X.

Proposition 7.4. *Let μ_* be an outer measure on the metric space X. Suppose that*

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B) \tag{7.1}$$

whenever $\operatorname{dist}(A, B) > 0$. *Then all open sets in X are measurable, and hence the induced measure μ is a Borel measure.*

• Such an exterior measure is called a **metric exterior measure**.

• Recall that $\operatorname{dist}(A, B)$ is defined to be the inf of distances $d(a, b)$ where $a \in A$ and $b \in B$. For example the distance in $\mathbb{R}$ between $(0, 1)$ and $(1, 2)$ is zero, even though the sets are disjoint.

Proof: Let O be an open set. Define O_n to be the subset of O of points distance $> 1/n$ from the boundary of O. We first claim that $\mu_*(O) = \lim_n \mu_*(O_n)$. (Note that $\geq$ is trivial here). Consider the 'shells'

$$S_n = O_n \setminus O_{n-1} = \left\{ x \,\middle|\, \frac{1}{n-1} \geq d(x, \partial O) > \frac{1}{n} \right\},$$

for $n \geq 2$ with $S_1 = O_1$. Note that since O is open, we have a formula

$$O = O_N \cup \bigcup_{n>N} S_n$$

for each N.

For even n, these sets are a positive distance apart, by an easy triangle inequality argument. If $x \in S_n$ and $y \in S_{n+2}$ we have $d(x, \partial O) \leq d(x,y) + d(y, \partial O)$, and so $d(x,y) > \frac{1}{n(n+1)}$. (Similarly for the odd n.)

We now split into two cases, depending on whether $\mu_*(O)$ is finite or infinite. First suppose that $\mu_*(O) < \infty$. Then, by condition (7.1), we have

$$\sum_n \mu_*(S_{2n}) = \mu_*(\cup_n S_{2n}) \leq \mu_*(O) < \infty.$$

Similarly for the odd shells:

$$\sum_n \mu_*(S_{2n+1}) \leq \mu_*(O) < \infty.$$

Hence the sum of the tails of these infinite series can be made arbitrarily small. In particular, for any $\epsilon > 0$ there exists N such that

$$\sum_{n>N} \mu_*(S_n) < \epsilon.$$

Using subadditivity for $O = O_N \cup \bigcup_{n>N} S_n$

$$\mu_*(O) \leq \mu_*(O_N) + \sum_{n>N} \mu_*(S_n)$$

and thus

$$\mu_*(O_N) \geq \mu_*(O) - \epsilon,$$

proving the claim. If on the other hand, $\mu_*(O) = \infty$, then we have by subadditivity

$$\mu_*(O) \leq \sum_{n\geq 1} \mu_*(S_n)$$

and so

$$\sum_n \mu_*(S_n) = \infty,$$

and using the positive distance between the even and odd slices as before, this implies that $\mu(O_n) \to \infty$.

We can apply the same argument to $\mu(A \cap O)$, for any set A, obtaining $\mu_*(A \cap O) = \lim_n \mu_*(A \cap O_n)$. (Note here that even though $A \cap O$ is not open, the identity $O = O_N \cup \bigcup_{n>N} S_n$ implies

$$A \cap O = A \cap O_N \cup \bigcup_{n>N} (A \cap S_n).$$

Now, take any set A. Using condition (7.1) we see that for any n

$$\mu_*(A) \geq \mu_*(A \cap O_n) + \mu_*(A \setminus O).$$

Taking the limit as $n \to \infty$, we obtain

$$\mu_*(A) \geq \mu_*(A \cap O) + \mu_*(A \setminus O).$$

The inequality $\leq$ is immediate from subadditivity, so we see that O is measurable. □

We also give a result about approximation of measurable sets by open and closed sets.

Proposition 7.5. *Let X be a metric space and μ a Borel measure which is finite on all balls in X of finite radius. Then for each Borel set E and each $\epsilon > 0$ there is a bigger open set $O \supset E$ with $\mu(O \setminus E) < \epsilon$, and a smaller closed set $F \subset E$ such that $\mu(E \setminus F) < \epsilon$.*

Let us refer to the two properties as outer regularity and inner regularity.

Proof: We show that closed sets have these properties, and the collection $\mathcal{C}$ of Borel sets having these properties is a σ-algebra. The Borel σ-algebra is, by definition, a sub-σ-algebra of $\mathcal{C}$, proving our claim.

For a closed set F, inner regularity is trivial. For outer regularity, we write $F_k = F \cap \overline{B(x_0, k)}$ for each $k \in \mathbb{N}$, where x_0 is an arbitrarily chosen point. Each F_k is closed and has finite measure, since it is a subset of $\overline{B(x_0, k)}$. which by hypothesis has finite measure. For each F_k we define a family of open neighbourhoods

$$O_{k,l} = \{x \in X \mid d(x, F_k) < 2^{-l}\},$$

which 'shrink' down to F_k.

Then, since F_k is closed, we have $\cap_l O_{k,l} = F_k$. Hence there exists $l(k)$ such that $\mu(O_{k,l(k)}) < \mu(F_k) + \epsilon 2^{-k}$. To see this, observe that for any l we have the disjoint decomposition

$$O_{k,l} = F_k \cup \bigcup_{i \geq l} O_{k,i} \setminus O_{k,i+1}$$

(where every set here is Borel). Looking at this for $l = 1$, we see that there is some $N(k)$ so $\mu(\bigcup_{i \geq N} O_{k,i} \setminus O_{k,i+1}) < \epsilon 2^{-k}$. Then we have

$$\mu(O_{k,N(k)}) < \mu(F_k) + \epsilon 2^{-k}.$$

Then define $O = \cup_k O_{k,N(k)}$, which is an open set, and observe

$$\mu(O \backslash F) \leq \sum_k \mu(O_{k,N(k)} \backslash F) \leq \sum_k \mu(O_{k,N(k)} \backslash F_k) = \epsilon.$$

It is easy to check that $\mathcal{C}$ is closed under complementation, so it remains to check closedness under countable unions. Outer regularity is straightforward using an $\epsilon 2^{-k}$ argument: Given $E = \cup E_i$, choose O_i with $\mu(O_i \setminus E_i) < \epsilon 2^{-k}$. Then define $O = \cup O_i$, which is automatically an open set. Then

$$\begin{aligned} \mu(O \setminus E) &= \mu(\cup_i O_i \setminus E) \\ &\leq \mu(\cup O_i \setminus E_i) \\ &\leq \sum \mu(O_i \setminus E_i) \\ &< \epsilon. \end{aligned}$$

The same argument doesn't work for inner regularity, because an infinite union of closed sets need not be closed. It is enough to establish inner regularity for finite unions, however.

This observation means that given a countable family E_k in $\mathcal{C}$, we may assume that it is increasing. Let $E = \cup E_k$. Now define $E'_{k,n} = E_k \cap (\overline{B(x_0, n)} \backslash B(x_0, n-1))$ and analogously E'_n. Since $\mu(E'_{k,n})$ is increasing (in k) and $\cup_k E'_{k,n} = E'_n$ has finite measure, there exists a $k(n)$ such that $\mu(E'_{k(n),n}) > \mu(E'_n) - \epsilon 2^{-n-1}$. We approximate $E'_{k(n),n}$ with a closed set $F_n \subset E'_{k(n),n}$ to within $\epsilon 2^{-n-1}$. Then the union of the F_n is closed (since at most finitely many intersect in any given ball $B(x_0, n)$, and F is closed iff $F \cap \overline{B(x_0, n)}$ is closed for all n) and approximates E within ϵ. □

Premeasures

We have seen how to construct a measure from an exterior measure. An exterior measure may in turn be constructed from a more basic object called a premeasure. This is defined on an algebra, rather than a σ-algebra, of subsets of X, that is, a collection of subsets of X containing the empty set and closed under *finite* unions and complementation. Let $\mathcal{A}$ be an algebra of subsets of X. Then a premeasure $\mu_0 : \mathcal{A} \to [0, \infty]$ is a function satisfying

(i) $\mu_0(\emptyset) = 0$;

(ii) If $E_1, E_2, \dots$ is a countable collection of disjoint subsets of $\mathcal{A}$, **and** $\cup_k E_k \in \mathcal{A}$, then

$$\mu_0\big(\cup_n E_n\big) = \sum_n \mu_0(E_n).$$

Given a premeasure μ_0, we define for any set $E \subset X$

$$\mu_*(E) = \inf \Big\{ \sum_{n=1}^{\infty} \mu_0(E_j) \mid E \subset \cup_j E_j, \text{ and each } E_j \in \mathcal{A} \Big\}.$$

(Observe the infimum is over a nonempty set, because $X \in \mathcal{A}$ covers any set E.)

Theorem 7.6. *The function μ_* is an exterior measure, such that*

(i) $\mu_*(E) = \mu_0(E)$ for all $E \subset \mathcal{A}$;

(ii) Every $A \subset \mathcal{A}$ is measurable.

Consequently, μ_0 extends to a measure on the σ-algebra generated by $\mathcal{A}$.

This, if you recall, was precisely how Lebesgue measure was defined.

Proof: It is straightforward to show that μ_* is an exterior measure. To show that $\mu_*(E) = \mu_0(E)$ for all $E \subset \mathcal{A}$, we take a cover of E by elements $E_j \in \mathcal{A}$. Then define $E'_k = E \cap (E_k \setminus \cup_{j=1}^{k-1} E_j)$. Now $E = \cup E'_k$m and the sets E'_k are disjoint and all in $\mathcal{A}$. We then have $\mu_0(E) = \sum_j \mu_0(E'_j)$, and therefore, $\mu_0(E) \leq \sum_j \mu_0(E_j)$, since $E'_j \subset E_j$. Taking the inf of the right hand side shows that $\mu_0(E) \leq \mu_*(E)$. The $\geq$ inequality is obvious.

To show that $E \in \mathcal{A}$ is measurable, take any set $A \subset X$, and any covering E_j of A. Using finite additivity of μ_0 on $\mathcal{A}$, we have

$$\mu_0(E_j) = \mu_0(E_j \cap E) + \mu_0(E_j \cap E^c).$$

Summing in j, we find that

$$\sum_j \mu_0(E_j) = \sum \mu_0(E_j \cap E) + \sum \mu_0(E_j \cap E^c)$$
$$\geq \mu_*(A \cap E) + \mu_*(A \cap E^c).$$

Since this is true for every cover of A, we find that

$$\mu_*(A) \geq \mu_*(A \cap E) + \mu_*(A \cap E^c),$$

which shows that E is measurable since the $\leq$ inequality is obvious. □

For later use we note the following. Let $\mathcal{A}_\sigma$ be the collection of countable unions of elements of $\mathcal{A}$ and let $\mathcal{A}_{\sigma\delta}$ denote the collection of countable intersections of elements of $\mathcal{A}_\sigma$. (Of course, if $\mathcal{A}$ were actually a σ-algebra, then we would have $\mathcal{A} = \mathcal{A}_\sigma = \mathcal{A}_{\sigma\delta}$.) Then

Proposition 7.7. *For any set A, and any $\epsilon > 0$, there exists an $E_1 \in \mathcal{A}_\sigma$ such that $A \subset E_1$ and $\mu_*(E_1) < \mu_*(A) + \epsilon$, and there exists $E_2 \in \mathcal{A}_{\sigma\delta}$ such that $A \subset E_2$ and $\mu_*(A) = \mu_*(E_2)$.*

Exercise. What is the relation between the Borel sets of $\mathbb{R}$ and the Lebesgue measurable sets of $\mathbb{R}$?