9. Complete normed spaces of functions

We are now in a position to define standard spaces of functions.

Let $p \in [1, \infty)$ and define $L^p(X, \mathcal{M}, \mu)$ to be the space of complex-valued measurable functions f on X such that $|f|^p$ is integrable, modulo the equivalence relation of being equal a.e. w.r.t. μ. This is a vector space which we endow with the p-norm

$$\|f\|_p = \Big(\int_X |f(x)|^p \, d\mu \Big)^{1/p}.$$

We have to check that this is a norm. It is straightforward to see that it satisfies strict positivity and homogeneity, but the triangle inequality is far from obvious!

Lemma 9.1 (Hölder's inequality)**.** *If* $p^{-1} + q^{-1} = 1$, *then*

$$\Big| \int_X f(x)g(x)\,d\mu \Big| \le \|f\|_p \|g\|_q.$$

Proof: To show Hölder, it suffices by homogeneity to do this when $\|f\|_p = \|g\|_q = 1$. Then we use Young's equality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \ge 0,$$

on the LHS, and integrate to find that the LHS is bounded by 1. $\square$

By the way this is also valid (in fact, almost trivial) when $p = 1$ and $q = \infty$, where the L^∞ norm is defined by

$$\|f\|_\infty = \inf \big\{ M \mid \mu\{x \mid |f(x)| > M\} \text{ has measure zero} \big\}.$$

Thus $\|f\|_\infty$ is finite if f can be modified on a set of measure zero so that it is bounded by M, and $\|f\|_\infty$ is the smallest M with this property.

To show that $\|\cdot\|_p$ satisfies the triangle inequality, we write

$$S = \Big(\int_X |f + g|^p d\mu \Big)^{1/p}.$$

Then,

$$S^p \leq \int_X |f||f+g|^{p-1}\,d\mu + \int_X |g||f+g|^{p-1}\,d\mu.$$

Applying Hölder's inequality to the expressions on the right hand side:

$$S^p \leq \|f\|_p\|f+g\|_p^{p-1}+\|g\|_p\|f+g\|_p^{p-1} = (\|f\|_p+\|g\|_p)S^{p-1}$$

and rearranging this gives the triangle inequality.

Theorem 9.2. *For $1 \leq p \leq \infty$, the space $L^p(X, \mu)$ is complete under the norm $\| \cdot \|_p$.*

Lemma 9.3. For any Cauchy sequence $\{f_n\}$ in a metric space, there is a subsequence satisfying

$$d(f_{n_{k+1}}, f_{n_k}) < \epsilon(k),$$

for any positive function $\epsilon(k)$.

Proof: For each k, there is some N_k so $d(f_a, f_b) < \epsilon(k)$ for all $a, b \geq N_k$. Take n_k to be the maximum of $\{N_0, \ldots, N_k\}$. $\square$

Proof of Theorem 9.2: We first do $p = 1$. Given a Cauchy sequence f_n in $L^1(X, \mu)$, we show that it has a limit in $L^1(X, \mu)$.

By passing to a subsequence, we may assume that

$$\|f_n - f_{n-1}\|_1 \leq 2^{-n}.$$

Then the function $g = f_1 + \sum_{j=1}^{\infty} |f_{j+1} - f_j|$ is an integrable function. It follows that the series $f_1(x) + \sum_j (f_{j+1}(x) - f_j(x))$ converges a.e., since it converges absolutely a.e. Let $f(x)$ be the limiting function (defined a.e.). Then, $f_n \to f$ pointwise a.e. and f_n is dominated by g, so $\int |f_n - f| \to 0$ by DCT.

For $1 < p < \infty$, we can use essentially the same argument. Passing to a subsequence, we assume

$$\|f_n - f_{n-1}\|_p \leq 2^{-n}.$$

Now let

$$g(x) = |f_1(x)| + \sum_{n=2}^{\infty} |f_n(x) - f_{n-1}(x)|.$$

Then $\|g\|_p \leq \|f_1\|_p + \sum \big\| |f_n - f_{n-1}| \big\|_p < \infty$. Therefore, g is finite a.e. We can see that

$$f_n(x) = f_1(x) + \sum_{j=2}^{n} (f_j(x) - f_{j-1}(x)),$$

and since the sum

$$f_1(x) + \sum_{j=2}^{\infty} (f_j(x) - f_{j-1}(x))$$

converges for a.e. x, the sequence $\{f_n(x)\}$ converges for a.e. x. Let $f(x)$ be the limit of the sequence $\{f_n(x)\}$. Then we have $f_n \to f$ a.e. and $|f_n|^p \leq g^p$, where g^p is an integrable function. Therefore, $|f|^p \leq g^p$, i.e. $f \in L^p$. Also, $|f_n - f| \to 0$ a.e. and

$$|f_n(x) - f(x)|^p \leq \big(|f_n(x) + |f(x)|\big)^p \leq 2^p g^p,$$

which is an integrable function. We conclude that $f_n \to f$ in L^p by the dominated convergence theorem.

The proof for $p = \infty$ is different in character. Assume that the sequence f_n is Cauchy in L^∞. Then given k there exists $N(k)$ such that

$$|f_n(x) - f_m(x)| \leq 2^{-k} \text{ a.e. for } n, m \geq N(k).$$

Let C_{kmn}, for $n, m \geq N(k)$, be the set where this estimate fails. Then C_{kmn} has measure zero and thus $C = \cup_{k,m,n} C_{kmn}$ has measure zero. For $x \in C^c$, we have

$$|f_n(x) - f_m(x)| \leq 2^{-k} \text{ for } n, m \geq N(k). \tag{9.1}$$

Thus, for $x \in C^c$, the sequence $f_n(x)$ is Cauchy. Let $f(x)$ denote the limit of this sequence, and define f arbitrarily (say, equal to 0) for $x \in C$. Then taking $m \to \infty$ in (9.1),

$$|f_n(x) - f(x)| \leq 2^{-k} \text{ for } x \in C^c,\ n \geq N(k).$$

Therefore $\|f\|_\infty \leq \|f_{N(k)}\|_\infty + 2^{-k}$, so $f \in L^\infty$, and $f_n \to f$ in the L^∞ norm. □