

Math 3325, 2014 — Assignment 4

Hand in by 5pm on October 31

This assignment is worth 100 marks: 80 for the the best 4 out of the 5 questions below, and 20 for writing quality.

1. (a) (10 marks) Let A be a bounded self-adjoint operator. Show that $U = (A - iI)(A + iI)^{-1}$ is unitary.
- (b) (5 marks) Let X be a bounded operator. Show that $\sigma(X^*) = \{\bar{\lambda} | \lambda \in \sigma(X)\}$ and that if X is invertible then $\sigma(X^{-1}) = \{\lambda^{-1} | \lambda \in \sigma(X)\}$.
- (c) (5 marks) Show that the only positive unitary operator is I .

2. Let A be a bounded self-adjoint operator.

- (a) (10 marks) Show that $A \geq kI$ for $k \in \mathbb{R}$ if and only if $\sigma(A) \subset [k, \infty)$.
- (b) (10 marks) Show that if $A \geq I$, $A^n \geq I$ for every positive integer n .

3. Let A be a bounded self adjoint operator on \mathcal{H} . Fix some vector $f \in \mathcal{H}$, and let V be the closure of the span of the elements $\{A^n f\}_{n=0}^\infty$.

- (a) (5 marks) Show V is invariant under A , and that if P is the orthogonal projection on V , that $AP = PA$.
- (b) (5 marks) In our functional calculus for A on upper semicontinuous functions, show that for any bounded operator B such that $AB = BA$, also $f(A)B = Bf(A)$. (Bonus: also do this for the Borel functional calculus.)
- (c) (5 marks) If A_0 denotes the restriction of A to V , and if $E(\lambda)$ is the spectral resolution of A , show that $PE(\lambda)$ is the spectral resolution of A_0 .
- (d) (5 marks) Show that $\sigma(A_0) \subset \sigma(A)$.

4. Let S be a linear subspace of $C([0, 1])$. Since $C([0, 1])$ is a subset of $L^2([0, 1])$ we can also regard it as a subspace of $L^2([0, 1])$. We assume that S is closed as a subspace of $L^2([0, 1])$, i.e., in the L^2 topology.

- (a) (5 marks) Show that S is a closed subspace of $C([0, 1])$ (under the sup norm).
- (b) (5 marks) Show that there exists $M > 0$ such that for all $f \in S$,

$$\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2.$$

(Use the closed graph theorem.)

- (c) (5 marks) Fix $y \in [0, 1]$. Show that there exists a function $k_y \in L^2([0, 1])$, with $\|k_y\|_{L^2([0, 1])} \leq M$, such that

$$f(y) = \int_0^1 k_y(x) f(x) dx$$

for all $f \in S$. (Use the Hilbert space Riesz representation theorem.)

- (d) (5 marks) Show that the L^2 unit ball of S is compact, and hence that S finite dimensional. (Show that a sequence in the unit ball which converges weakly converges in norm.)

5. (a) (6 marks) Let (A_n) be a decreasing sequence of nonempty closed balls in a Banach space. Show that the intersection of the A_n is nonempty. (Do not assume that the radii converge to zero.)

(b) (6 marks) Let (B_n) be a decreasing sequence of closed, bounded, nonempty convex sets in a reflexive Banach space Y . Show that the intersection of the B_n is nonempty. (Hint: first show that the sets B_n are weakly closed, using the Separating Hyperplane theorem. If you can't, assume it and complete the rest of the problem, using Banach-Alaoglu.)

(c) (8 marks) Let $X = L^1(\mathbb{R})$. Find a decreasing sequence of closed, bounded, nonempty convex sets $C_n \subset X$ whose intersection is empty.