Fixed Point Theorems

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1 Introduction

Fixed point problems consist of determining conditions such that a function f from a space E to itself has a fixed point: a point $x \in E$ such that $f(x) = x$. The conditions can involve both the space E and the function f. These problems often provide interesting connections between several areas of mathematics, such as analysis, topology and differential equations, with widespread applications. The aim of this paper is to provide an accessible exposition of the proofs of two significant fixed point theorems: the Markov-Kakutani fixed point theorem and the Kakutani fixed point theorem, and to briefly explore some of their applications to other areas of mathematics.

2 Statement of the theorems

Definitions of terms used in each of the following theorems can be found in section 3.

Theorem 1 (Markov-Kakutani). *Let E be a Hausdorff topological vector space, C a nonempty compact convex set in E. Suppose that Γ is a set of continuous maps of C to itself satisfying the following conditions:*

(a) The set Γ is a family of continuous affine transformations ϕ: if $\phi \in \Gamma$, $x, y \in C$, and α and β are two positive real numbers satisfying $\alpha + \beta = 1$, then

$$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y).$$

(b) The set Γ is abelian, in that for every $\phi, \psi \in \Gamma$, and every $x \in C$, we have

$$\phi\left(\psi(x)\right) = \psi\left(\phi(x)\right).$$

Then there exists a point $x_0 \in C$ such that $\phi(x_0) = x_0$ for all $\phi \in \Gamma$.

This theorem was originally by Markov, with an alternative proof provided in 1938 by Kakutani [6]. In 1941, Kakutani extended the theorem to set-valued functions Φ from a set to its power set:

Theorem 2 (Kakutani 1941). *Let Φ be an upper semi-continuous point-to-set mapping of a nonempty bounded closed convex set S in a Euclidean space to its power set 2^S such that $\Phi(x)$ is non-empty, closed and convex for all $x \in S$. Then there exists some $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

The need for a Euclidean space E in Kakutani's theorem is due to the use of Brouwer's theorem (stated in section 3) in the proof. An extension to a more general topological space is provided in section 5.
There are many variants of fixed point theorems (FPTs) each with different strengths. The Markov-Kakutani theorem is powerful in that it determines a single fixed point for a whole family of functions, while theorems such as the Schauder-Tychonoff fixed point theorem determine conditions on the space such that the restriction on the function is minimal, namely that we only require the function f to be continuous. Spaces such as those defined in the Schauder-Tychonoff theorem are said to possess the fixed point property. For the sake of interest only (the proof will not be covered here), the statement of the Schauder-Tychonoff theorem is as follows:

Theorem 3 (Schauder-Tychonoff)**.** *Let E be a Hausdorff locally convex topological vector space, C a nonempty compact convex subset of E, and f a continuous map of C into itself. Then f admits at least one fixed point.*

3 Brouwer's fixed point theorem and definitions

Theorem 4 (Brouwer [1])**.** *Let f be a continuous mapping of the closed unit ball $\overline{B_1(0)}$ in a Euclidean space to itself. Then there exists a point $x \in \overline{B_1(0)}$ such that $f(x) = x$.*

Indeed, this holds for any convex, compact set of a Euclidean space. The theorem originally arose in a purely topological context, and many of its proofs require a level of algebraic topology beyond the scope of this paper. However, a simple special case of the theorem where we take the interval $[0, 1]$ in the real numbers and f any continuous function from the interval to itself can be proved by applying the intermediate value theorem to the continuous function ϕ defined by $\phi(x) = f(x) - x$ on $[0, 1]$. Brouwer's theorem is assumed in the following proofs. The two FPTs considered in this paper are notable in that they extend Brouwer's finite-dimensional theorem to infinite dimensions.

Definition A set S is *convex* if for every $x, y \in S$ $\alpha x + \beta y \in S$ for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

This definition corresponds exactly with our intuitive idea of convexity, where any line segment joining two points in a set is contained within the set. The definition clearly extends by induction to a set of n points x_i in S, and positive reals α_i such that $\sum_{i=1}^{n} \alpha_i = 1$ so that $\sum_{i=1}^{n} \alpha_i x_i \in S$.

Definition A *point-to-set mapping*, or *set-valued function*, is a function that to every point in its domain associates a set of points.

Definition A set-valued function $\Phi : S \to \Re(S)$ where $\Re(S)$ denotes the set of all closed convex subsets of S, is *upper semi-continuous* if given sequences (x_n) and (y_n) converging to x_0 and y_0 respectively and $y_n \in \Phi(x_n)$ for each n implies $y_0 \in \Phi(x_0)$.

4 Proof of the Markov-Kakutani theorem

The following proof was originally outlined by Kakutani in [6].

Proof. The main idea of the proof is to construct the following family of funtions on the compact and convex set C:

$$U = \{u_n : C \to E \text{ defined by } u_n = \frac{1}{n}(1 + u + \ldots + u^{n-1}) : n \in \mathbb{N}^*, u \in \Gamma\}.$$

We then define $\Gamma^* = \{v = \prod_{j=1}^{m} u_{n_j}^{(j)} : u^{(j)} \in U, n_j, m \in \mathbb{N}^*\}$. Here, the product signifies successive compositions of elements in U.
Then we have the following facts:
(i) The functions u_n map C into itself. This is due to the convexity of C and the weighting of the u^i, which by assumption map C into itself, in the sum.
(ii) The elements of Γ^* are affine functions. This is an immediate consequence of the fact that the elements of Γ are affine.
(iii) Γ^* is abelian. This is due to Γ being abelian.
(iv) The elements of Γ^* map C into itself. This follows from (i) and the fact that each element of Γ^* is a composition of elements from U.
(v) If v, v' and v'' are elements of Γ^* such that $v = v' \circ v'' = v'' \circ v'$, then $v(C) \subset v'(C)$ and $v(C) \subset v''(C)$, so in particular, $v(C) = (v' \circ v'')(C) \subset v'(C) \cap v''(C)$.
(vi) For every element v of Γ^*, $v(C)$ is compact. This follows from the continuity of elements of Γ and the compactness of C.
We now consider the sets $v(C)$ for each element v of Γ^*. The following lemmas enable us to determine the fixed points of the family Γ.

Lemma 1. *Finite intersections of the $v(C)$ are nonempty.*

Proof. This is shown by finite induction on the number of elements $v(C)$ in the intersection. Each $v(C)$ is non-empty for $v \in \Gamma^*$.
The case $n = 2$ was considered in (v), since for any elements $v_1, v_2 \in \Gamma^*$, $v_1(C) \cap v_2(C) \supset (v_1 \circ v_2)(C)$, and $v_1 \circ v_2$ is an element of Γ^*, so $(v_1 \circ v_2)(C)$ is nonempty.
Now suppose that for $n \in \mathbb{N}$, each intersection of n elements $v_1, \ldots, v_n$ of Γ^* is nonempty. Consider any $n+1$ elements $v_1, \ldots, v_n$ of Γ^*. The intersection $\bigcap_{i=1}^{n} v_i(C)$ is non-empty, so $\bigcap_{i=1}^{n+1} v_i(C) = (\bigcap_{i=1}^{n} v_i(C)) \bigcap v_{n+1} \supset (v_1 v_2 \ldots v_{n+1})(C)$, since $v_1 \ldots v_{n+1}$ is a finite product of elements from Γ^*, so is itself an element of Γ^*.
Thus $\bigcap_{i=1}^{n+1} v_i(C) \supset (v_1 v_2 \ldots v_{n+1})(C)$ is nonempty, and by induction every finite intersection of the $v(C)$ for $v \in \Gamma^*$ is nonempty. □

We say that the family $\Sigma = \{v(C) : v \in \Gamma^*\}$ has the *finite intersection property* (FIP), namely that every finite subset of Σ has nonempty intersection. We use the following lemma to show that the intersection of all $v(C)$ over all elements of Γ^* is nonempty.

Lemma 2. *All sets Σ of closed subsets of a topological space X with the finite intersection property are such that the intersection of all elements of Σ is nonempty if and only if X is compact.*

Proof. Let X be a compact topological space and Σ some collection of closed subsets of X with the FIP.
Suppose $I := \bigcap_{\sigma\in\Sigma}\sigma$ is empty. Then $I^c = \left(\bigcap_{\sigma\in\Sigma}\sigma\right)^c = \bigcup_{\sigma\in\Sigma}\sigma^c = X$. Since each $\sigma \in \Sigma$ is closed, σ^c is open, so I^c forms an open cover of X. The space X is compact, so I^c admits a finite subcover $J = \bigcup_{j=1}^{r}\sigma_j$ of X. But then $J^c = \left(\bigcup_{j=1}^{r}\sigma_j\right)^c = \bigcap_{j=1}^{r}\left(\sigma_j^c\right)^c = \bigcap_{j=1}^{r}\sigma_j = \emptyset$. This contradicts the assumption that Σ has the FIP.
Conversely, let X be a topological space such that every collection of closed subsets Σ with the FIP has nonempty intersection, and suppose that X is not compact, so there exists some collection of open sets $O = \{X_\lambda : \lambda \in \Lambda\}$ for some index set Λ that forms an open cover of X: $\bigcup_{\lambda\in\Lambda}X_\lambda = X$ such that for all finite subsets J of Λ, $\bigcup_{j\in J}X_j \subsetneq X$. For each $\lambda \in \Lambda$, we have that X_λ^c is closed, so that $L := \{X_\lambda^c : X_\lambda \in O\}$ is a collection of closed subsets of X. The set L has the FIP, since for any finite subset J of Λ, $\bigcap_{j\in J}X_j^c = \left(\bigcup_{j\in J}X_j\right)^c$, which is nonempty given $\bigcup_{j\in J}X_j$ is a proper subset of X for all finite subsets $J \subset \Lambda$. Thus, the intersection of L is nonempty: $\bigcap_{\lambda\in\Lambda}X_\lambda^c \neq \emptyset$, so $\left(\bigcap_{\lambda\in\Lambda}X_\lambda^c\right)^c = \bigcup_{\lambda\in\Lambda}X_\lambda \neq X$. This contradicts the assumption that O is an open cover of X. Therefore X is compact.

□

Here, E is Hausdorff, and each $v(C)$ is compact (by (vi)), so in particular the $v(C)$ are closed, and Σ is a set of closed sets in the compact set C with the FIP. Therefore by lemma 2, the intersection of all the $v(C)$ is nonempty, and contains at least one point x_0. The claim is that this element x_0 is a common fixed point for all elements u of Γ. Since each u_n is an element of Γ^* for every $u \in \Gamma$ and every natural number n, there exists an element $y \in C$ such that

$$x_0 = \frac{1}{n}(y + u(y) + \ldots + u^{n-1}(y))$$

for each u and n. Then

$$u(x_0) = \frac{1}{n}(u(y) + u^2(y) + \ldots + u^n(y))$$

since u is affine, and we have

$$u(x_0) - x_0 = \frac{1}{n}(u^n(y) - y) \in \frac{1}{n}(C - C).$$

Here, for any sets U and V, $U + V$ (respectively $U - V$) denotes the set $\{u + v : u \in U, v \in V\}$ (respectively $\{u - v : u \in U, v \in V\}$). Similarly $\beta S = \{\beta s : s \in S\}$ for any constant β. We digress briefly to make the following definition:

Definition A subset S of a vector space E over the field F of real or complex numbers is *balanced* if for all $s \in S$ and all $\alpha \in F$ such that $|\alpha| \leq 1$, $\alpha s \in S$.

Intuitively, this means that the set S has no holes: all balanced sets must contain 0, and every point on the line joining any element s of S and 0 must also be contained in S.

Now we let V be any balanced neighbourhood of 0, and W some balanced neighbourhood of 0 such that $W+W \subset V$. The set $C-C$ is compact by compactness of C, so the cover $\{x+W\}_{x\in C-C}$ admits a finite subcover $\{x_k+W\}$. The convex set $C-C$ contains 0, so it is balanced, the map defined by multiplication by a real scalar is continuous, and by definition of a neighbourhood, W contains some open set containing 0, so in particular W contains some ball of radius $\epsilon > 0$ about 0. Therefore there exists for each k some $\alpha_k \in (0,1]$ such that $\alpha x_k \in W$. Taking $\alpha = min_k\{\alpha_k\}$, $\alpha x_k \in W$ for all k.
Thus $\alpha(C-C) \subset W+\alpha W \subset W+W \subset V$. Setting $\frac{1}{n} < \alpha$, we have that $u(x_0)-x_0 \in V$, for any V a neighbourhood of 0.
Therefore $u(x_0) - x_0 = 0$, as required.

□

The Markov-Kakutani theorem can in fact be extended to cases where Γ is no longer abelian, as seen in the following corollary.

Corollary 1. *Let E and C be defined as in the Markov-Kakutani theorem, and Γ a family of continuous affine functions as in condition (a) of the theorem. Suppose further that there exists a natural number m and subsets Γ_i with $0 \le i \le m$ of Γ such that*

$$\{I\} = \Gamma_m \subset \Gamma_{m-1} \subset \ldots \subset \Gamma_0 = \Gamma,$$

where I is the identity map on C and for each pair of elements u, v in Γ_{i-1} there exists an element w of Γ_i such that $u \circ v = v \circ u \circ w$.
Then there exists a point x_0 of C such that $u(x_0) = x_0$ for all u in Γ.

Proof. If we let $m = 1$, then we have the chain $\{I\} = \Gamma_1 \subset \Gamma$, such that every pair of elements u, v in Γ there exists an element w of $\{I\}$ such that $u \circ v = v \circ u \circ w$, but since I is the identity on C, $w = 1$, and Γ is abelian, so we are in the case of theorem 1, which has already been shown. We proceed by induction on m.
Suppose now that the corollary holds for some natural number m.
Consider some sequence $\{I\} = \Gamma_{m+1} \subset \Gamma_m \subset \ldots \subset \Gamma_1 \subset \Gamma_0 = \Gamma$ satisfying the properties in the statement of the corollary. Then by assumption, the corollary holds for Γ_1, so that the set F of fixed points of the functions in Γ_1 is nonempty. Since every $u \in \Gamma$ is continuous and E is Hausdorff, the set of fixed points F is closed. Furthermore, F is convex since every $u \in \Gamma$ is affine. Therefore F is compact, by compactness of C.
Now let u and v be elements of Γ. By the assumption on the chain of Γ_i, there exists some $w \in \Gamma_1$ such that $u \circ v = v \circ u \circ w$. Thus for any $x \in F$, $u \circ v(x) = v \circ u(x)$ since F is by definition the set of points where $w(x) = x$. Therefore, when functions in Γ are restricted to the set F, they commute in pairs.
In fact, given that $\Gamma_1 \subset \Gamma$, the same equality holds taking any $v \in \Gamma_1$, so that $v \circ u(x) = u \circ v(x) = u(x)$ for any $x \in F$. Thus, for any $x \in F$, $u(x)$ is a fixed point for $v \in \Gamma_1$. This holds for any such v, so $u(F) \subset F$. We are now back in safe territory, within the case $m = 1$. Therefore there exists some $x_0 \in F \subset C$ such that $u(x_0) = x_0$ for all $u \in \Gamma$.

□

There are two significant cases when the conditions on the family of functions Γ in the corollary are satisfied:
(1) If Γ is an abelian semigroup of continuous functions from C into itself containing the identity, where a semigroup satisfies all the axioms of a group except for the existence of inverses. Indeed, we are again in the case $m = 1$ (where m is the number of elements in the chain of subsets of Γ between I and Γ), thereby fulfilling the conditions of the theorem.
(2) If Γ is a solvable group of bijective maps on C, where a solvable group is such that there exists a chain of subgroups Γ_i as specified in the corollary, where Γ_{i+1} is a normal subgroup of Γ_i and Γ_i/Γ_{i+1} is abelian.

5 Proof of Kakutani's theorem

Some details of the proof have been omitted, though all the general steps used in Kakutani's proof [7] are provided here. For further explanation, see [5]. We first suppose that the set S is an n-dimensional simplex (defined below), then extend S to the general case in the statement of the theorem.

Definition An *n-dimensional simplex* in $\mathbb{R}^m$ defined by the $n+1$ points $u_0, \ldots, u_n$ and denoted $S = \langle u_0, \ldots, u_n \rangle$ is the smallest convex set containing the $n+1$ vertices. Thus $S = \{x = \theta_0 u_0 + \ldots + \theta_n u_n : \theta_i \geq 0 \text{ for } 0 \leq i \leq k, \sum_{i=0}^{n} \theta_i = 1\}$. The θ_j are called the *barycentric coordinates* of the point x in S.

Concretely, this should just be thought of as an n-dimensional generalisation of a triangle, which is a two-dimensional simplex.

Definition The *barycentre* of an n-dimensional simplex $\langle u_0, \ldots, u_n \rangle$ is the point $x = \sum_{i=0}^{n} \frac{1}{n+1} u_i$.

Definition We define the *barycentric subdivision* of an n-dimensional simplex into $(n+1)!$ simplices of the same dimension by induction on n.
For $n = 0$, the simplex consists of a single point, and is its own barycentre, so it is already subdivided.
For $n = 1$, $\langle x_0, x_1 \rangle$ is simply a line segment, and we subdivide it into the two one-dimensional simplices $\langle x_0, x \rangle$ and $\langle x, x_1 \rangle$ where x is the barycentre of the simplex $\langle x_0, x_1 \rangle$.
Then for $n \in \mathbb{N}^*$, given the subdivision on each $(n-1)$-dimensional face of a simplex S, we form the $(n+1)!$ n-dimensional simplices $\langle x_0, \ldots, x_{n-1}, x \rangle$, where the x_i are the vertices of one of the cells in the subdivision of the faces and x is the barycentre of S.
Iterating the construction of the barycentric subdivision m times, gives the *mth barycentric subdivision.*

Proof. Step 1 Suppose S is an n-dimensional nondegenerate simplex $\langle v_0, \ldots, v_n \rangle$ and consider the mth barycentric subdivision of S. We define the continuous function $f^m : S \to S$ by $f^m(x) = y$ where y is some element in $\Phi(x)$ if x is a vertex of one of the simplices in the subdivision. Otherwise, x is in at least one simplex of the subdivision, say with vertices $\langle x_0, \ldots, x_n \rangle$, so that $x = \sum_{j=0}^{n} \theta_j x_j$, where the θ_j are as in the definition of a simplex. In this case we define $y = \sum_{j=0}^{n} \theta_j f^m(x_j)$. From this construction, we can now apply Brouwer's theorem to f^m , remembering that E is a Euclidean space, so there exists some element of S, say z^m, that is a fixed point of f^m. There are two possible cases:

1) z^m is a vertex of the subdivision, in which case $f^m(z^m) = z^m \in \Phi(z^m)$ by construction of f^m, so we are done.

2) If z^m is not a vertex, then $z^m = \sum_{j=0}^{n} \theta_j^m y_j^m$, where we have the usual restrictions on the θ_j^m and $\langle y_0^m, \ldots, y_n^m \rangle$ is a simplex containing z^m. Furthermore, since z^m is a fixed point of the function f^m, we have that $z^m = f^m(z^m) = \sum_{j=1}^{n} \theta_j^m z_j^m$, where $z^m = f^m(y_j^m) \in \Phi(y_j^m)$. Since S is compact in the Euclidean space E, we also have sequential compactness, so letting $m \to \infty$, there is some subsequence (m_k) of (m) such that $z^{m_k} \to z$, $\theta_j^{m_k} \to \theta_j$ and $z_j^{m_k} \to z_j$. As m tends to infinity we are shrinking down elements of the subdivision to points, so that $z_j^{m_k} \to z$ and we have $z = \sum_{j=0}^{n} \theta_j z_j$, where the θ_j still give a convex weighting of the vertices z_j. Finally, since Φ is upper semicontinuous, the z_j are elements of $\Phi(z)$. Using convexity of $\Phi(z)$, we therefore have that z, a convex linear combination of the z_j, is an element of $\Phi(z)$, and z is a fixed point for Φ.

Step 2 Given an arbitrary nonempty bounded closed convex set S in E, we simply consider a simplex S' containing S in E, and define the retraction mapping $\psi : S' \to S'$ such that ψ restricted to S is the identity, and ψ is continuous. Then defining the set-valued function $\Psi \to 2^{S'}$ by $\Psi(x) = \Phi(\psi(x))$, we can now apply step one to Ψ, knowing that any fixed point of Ψ must be an element of S since $\psi(x) \in S$. Therefore we have a fixed point of Φ as required. □

Furthermore, Kakutani's theorem can be generalised to the infinite dimensional case, so that it applies to more general topological spaces, as in the Markov-Kakutani theorem. The extension was given by Fan [3] and Glicksberg [4]:

Theorem 5 (Kakutani-Fan-Glicksberg). *Let S be a nonempty compact convex subset of a locally convex Hausdorff topological vector space. Let $\Phi : S \to 2^S$ be an upper semi-continuous point-to-set function such that for every x in S, $\Phi(x)$ is a nonempty closed convex subset of S. Then Φ has a fixed point in S.*

The definition of upper-semi-continuous functions is somewhat different in this case, as we are no longer in a Euclidean space so first-countability is no longer assured, and we might not be able to consider sequences converging in S. For further details, refer to [3] or [4].

6 An application of fixed point theorems

Fixed point theorems have proven to be very useful in many areas of mathematics. Theorems such as the Schauder-Tychonoff theorem have been used extensively in proving the existence and uniqueness of solutions to differential equations. The outreach of Kakutani's theorem is particularly wide, from proofs of the existence of the Haar measure to proofs in game theory and economics, which will be considered briefly here.
In 1950, John Nash proved the existence of the Nash equilibrium in game theory, using Kakutani's fixed point theorem [8]. He considered the case of an n-person game, wherein each of the players has a finite set of pure strategies from which to choose with each n-tuple of strategies corresponding to a definite pay-off. Each such n-tuple can be considered as a point in the product space consisting of the n strategy spaces of each of the players. One point P counters another point Q if the strategy of each player in P is the best choice of strategy against the $n-1$ other strategies in Q. A self-countering strategy n-tuple, wherein each player has the best possible strategy against all other players, is called a Nash equilibrium point. Kakutani's fixed point theorem proves the existence of such an equilibrium, since the correspondence of the strategy to the convex set of all possible countering strategies is a point-to-set mapping of the product space to itself, and the pay-off funtions are continuous, so the mapping is upper semi-continuous.

References

[1] Brouwer, L.E.J., *Uber Abbildung von Mannigfaltigkeiten*, Math. Ann. 7 (1912) 97 -115 .

[2] Edwards, R.E., "Functional Analysis: Theory and Applications," pp. 149-173, Holt, Rinehart and Winston, New York, 1965.

[3] Fan, K.,(1952): *Fixed-Point and Minimax Theorems in Locally Convex Linear Spaces*, Proceedings of the National Academy of Sciences, USA, 38, 121–126 (1952).

[4] Glicksberg, L.,"A Further Generalization of the Kakutani Fixed-Point Theorem with applications to Nash Equilibrium Points", *Proceedings of the American Mathematics Society*, 3, (1952) 170-174.

[5] Hillas, J., "Fixed Point Theorems", `http://homes.eco.auckland.ac.nz/jhil044/public_html/econ781/fpb_01.pdf`, accessed 23 October 2013.

[6] Kakutani, S., *Two Fixed-point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Volume 14, Number 7 (1938), 242-245.

[7] Kakutani, S., *A generalisation of Brouwer's fixed point theorem*, Duke Math. J. Volume 8, Number 3 (1941), 457-459.

[8] Nash, J.F., Jr. (1950). *Equilibrium Points in N-Person Games*, Proc. Nat. Acad. Sci. U.S.A. 36 (1): 48–49.