1. Hilbert spaces

A complex Hilbert space H is a complete normed space over $\mathbb{C}$ whose norm is derived from an inner product. That is, we assume that there is a sesquilinear form $(\cdot,\cdot) : H \times H \to \mathbb{C}$, linear in the first variable and conjugate linear in the second, such that

$$(f,g) = \overline{(g,f)},$$

$$(f,f) \geq 0 \; \forall f \in H, \text{ and } (f,f) = 0 \implies f = 0.$$

The norm and inner product are related by

$$(f,f) = \|f\|^2.$$

We will always assume that H is separable (has a countable dense subset).

• As usual for a normed space, the distance on H is given by $d(f,g) = \|f-g\| = \sqrt{(f-g,f-g)}$.

• The Cauchy-Schwarz and triangle inequalities,

$$|(f,g)| \leq \|f\|\|g\|, \quad \|f+g\| \leq \|f\| + \|g\|,$$

can be derived fairly easily from the inner product.

To prove the Cauchy-Schwarz inequality we use the fact that

$$z(t) = (f+tg, f+tg)$$

is a norm and hence positive, and in particular positive at its minimum value (as a function of t). The minimum occurs at $t = -\frac{\Re(f,g)}{(g,g)}$, and if (f,g) happens to be real substituting this into $z(t) \geq 0$ immediately gives the desired inequality. For the general case, we can choose g' to be some unit complex number multiple of g so that (f,g') is real, and see that the Cauchy-Schwarz inequality for f and g is equivalent to the corresponding statement for f and g'.

The triangle inequality then comes from

$$\begin{aligned}\|f+g\|^2 &= (f+g, f+g) \\ &= (f,f) + 2\Re(f,g) + (g,g) \\ &\leq (f,f) + 2|(f,g)| + (g,g) \\ &\leq \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2.\end{aligned}$$

• You should have seen some examples last semester. The simplest (finite-dimensional) example is $\mathbb{C}^n$ with its standard inner product. It's worth recalling from linear algebra that if V is an n-dimensional (complex) vector space, then from any set of n linearly independent vectors we can manufacture an orthonormal basis $e_1, e_2, \ldots, e_n$ using the Gram-Schmidt process. In terms of this basis we can write any $v \in V$ in the form

$$v = \sum a_i e_i, \quad a_i = (v, e_i)$$

which can be derived by taking the inner product of the equation $v = \sum a_i e_i$ with e_i. We also have

$$\|v\|^2 = \sum_{i=1}^{n} |a_i|^2.$$

• Standard infinite-dimensional examples are $l^2(\mathbb{N})$ or $l^2(\mathbb{Z})$, the space of square-summable sequences, and $L^2(\Omega)$ where Ω is a measurable subset of $\mathbb{R}^n$.

Orthogonality

We say that $f, g \in H$ are orthogonal (perpendicular) if $(f,g) = 0$. An orthonormal set E is one such that for all $e_1 \neq e_2 \in E$, we have $\|e_1\| = \|e_2\| = 1$ and $(e_1, e_2) = 0$. Such a set is automatically linearly independent. We say that E is an orthonormal basis if it is an orthonormal set, and in addition, that the set of all finite linear combinations of elements of E is dense in H. Note that if H is separable, then E is countable.

Exercise. Verify that a Hilbert space orthonormal basis in a finite dimensional Hilbert space is exactly the same thing as a orthonormal basis in the sense of linear algebra.

How do we see that an orthonormal set in a separable Hilbert space has at most countably many elements? Fix a countable dense set $\mathcal{D}$. For each basis element e_i pick some element $x_i \in \mathcal{D}$ with $d(e_i, x_i) < 1/2$. We claim that if $i \neq j$, $x_i \neq x_j$; this follows from $d(e_i, e_j) = \sqrt{2}$, and the triangle inequality. Thus we've found an injective function from our basis to a countable set. (You may want to read about the Schroeder-Bernstein theorem. Which categories does it hold in?)

Note that this argument used the Axiom of Choice — was that necessary?

Theorem 1.1 (Theorem 2.3 of Stein-Shakarchi)**.** *The following properties of an orthonormal set $E = \{e_1, e_2, \dots\}$ are equivalent:*

(i) E is an orthonormal basis.

(ii) If $(f, e_j) = 0$ for all j, then $f = 0$.

(iii) If $f \in H$ and we define

$$S_N(f) = \sum_{j=1}^{N} (f, e_j) e_j,$$

then $S_N(f) \to f$ as $N \to \infty$.

(iv) If $a_j = (f, e_j)$ then $\|f\|^2 = \sum_j |a_j|^2$.

Proof:
(i) $\implies$ (ii). We'll try to show that $f = 0$ by showing that it has arbitrarily small norm. Do the only thing you can do, and begin by picking a finite linear combination $g = \sum_{i=1}^N a_i e_i$ within ϵ of f. The trick is that we can replace (f, f) with $(f, f - g)$, and then by Cauchy-Schwarz we have

$$\|f\|^2 = (f, f - g) \leq \|f\| \, \|f - g\| < \epsilon \|f\| .$$

(ii) $\implies$ (iii). Here we need

Lemma 1.2 (Bessel's inequality)**.** *If $\{x_i\}$ is an orthonormal set (whether a basis or not),*

$$\sum_i |(f, x_i)|^2 \leq \|f\|^2 .$$

Proof: Note that $f - S_N(f)$ is perpendicular to $S_N(f)$, so we have

$$\|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2 = \|f - S_N(f)\|^2 + \sum_{i=1}^{N} |(f, x_i)|^2$$

giving the result. □

Then we have

$$\sum_{j=1}^{\infty} |(f, e_j)|^2 \leq \|f\|^2 < \infty$$

and so in particular $\sum_{j=M}^{\infty} |(f, e_j)|^2 \to 0$ as $M \to \infty$. Thus easily $S_N(f)$ is a Cauchy sequence and converges to something, say g. But then (ii) applied to $f - g$ shows $g = f$.
(iii) $\implies$ (iv). is trivial.
(iv) $\implies$ (i). We have $\|f - S_N(f)\|^2 = \|f\| - \sum_{j=1}^N |(f, e_j)|^2 \to 0$ as $N \to \infty$, so finite linear combinations of the e_i are dense, as required. □

We see that when an orthonormal set is also a basis, Bessel's inequality becomes an equality, usually referred to as Parseval's identity.

Theorem 1.3. *Every (separable) Hilbert space has an orthonormal basis.*

Proof: To prove this, we start with a countable dense subset $\{f_1, f_2, \dots\}$. In this list, let us eliminate every f_j that is not linearly independent from f_i, $i < j$. Then we apply the Gram-Schmidt process to the resulting list to generate an orthonormal set $e_1, e_2, \dots$. Suppose that this is not a basis. Then by part (ii) above, there would be a nonzero f such that $(f, e_j) = 0$ for all j. Equivalently, $(f, f_j) = 0$ for all j since f_j is a linear combination of $e_1, \dots, e_j$. But then, $\|f - f_j\|^2 = \|f\|^2 + \|f_j\|^2 \geq \|f\|^2$ so f is not in the closure of the f_j, contradicting the density of the set of f_j. Hence we have constructed an orthonormal basis. □

Closed subspaces

Subspaces of a Hilbert space need not be closed. For example, the subspace of C^∞ functions inside $L^2(\mathbb{R})$ is not closed. Nor is the subspace of simple functions (those taking on only finitely many values, except on a set of measure zero). Note that closed subspaces are Hilbert spaces in their own right.

Closed subspaces are important due to the following two results. (These are proved in the opposite order in the text, with a completely different proof. Either one follows quite easily from the other.)

Given any subset $E \subset H$, the orthogonal complement $E^\perp$ is the set

$$\{f \in H \mid (f,e) = 0 \text{ for all } e \in E\}.$$

A fundamental property of $E^\perp$ is that it is always closed. This shows that H has lots of proper closed subspaces, in contrast to many complete normed spaces.

Theorem 1.4. *Let S be a closed subspace of a Hilbert space H, and let $S^\perp$ be its orthogonal complement. Then $H = S \oplus S^\perp$ as inner product spaces.*

Recall that $H = S \oplus T$ means that every $f \in H$ can be expressed uniquely as $f = s + t$ with $s \in S$ and $t \in T$, with s and t depending continuously on f. The statement 'as inner product spaces' means that if $f_1 = s_1 + t_1$, $f_2 = s_2 + t_2$ are decomposed as above, then $(f_1, f_2) = (s_1, s_2) + (t_1, t_2)$. Continuity of the decomposition is a direct consequence of this.

Proof: To prove this we can observe first that S and $S^\perp$ are Hilbert spaces in their own right. So we can choose orthonormal bases $e_1, e_2, \ldots$ for S and $e'_1, e'_2, \ldots$ for $S^\perp$. Then using property (ii) of Theorem 2.3 above, we show that the union of these two orthonormal bases is an orthonormal basis for H. Expressing any $f \in H$ in terms of this basis automatically gives a decomposition of f as $s + t, s \in S, t \in S^\perp$. Uniqueness follows easily since if $f = s + t = s' + t'$, then $s - s' = t' - t$ then $s - s'$ is both in S and orthogonal to it, hence $s - s' = 0$. □

The second reason that closed subspaces are important is that we have the following result:

Theorem 1.5. *Let S be a closed subspace of a Hilbert space H, and let $f \in H$. Then there is a unique point s in S closest to f, characterized by the property that $s - f$ is orthogonal to S.*

Proof: Using the decomposition $H = S \oplus S^\perp$, we write $f = s + t$ with $s \in S$ and $t \in S^\perp$. Notice that the distance from f to s is $\|t\|$. For any other $s' \in S$, say $s' = s + x$, we have $\|f - s'\|^2 = \|f - s - x\|^2 = \|f - s\|^2 + \|x\|^2 \geq \|t\|^2$, with equality exactly when $x = 0$. □

Linear functionals

A continuous linear functional on H is a continuous linear map l from H to $\mathbb{C}$. Continuity gives that

$$\exists \delta > 0 \text{ such that } \|f\| \leq \delta \implies |l(f)| \leq 1.$$

Then by linearity, we have for all f

$$|l(f)| \leq \delta^{-1}\|f\|.$$

So there exists M such that $|l(f)| \leq M\|f\|$, namely $M = \delta^{-1}$. Such an M is called a bound on l and l is called bounded. The least upper bound for l is written $\|l\|$. We thus see that for linear functionals, continuity and boundedness are equivalent.

There are 'obvious' such functionals, given by the inner product with a fixed vector g, i.e. the linear functional

$$l : f \mapsto (f, g).$$

Notice that by Cauchy-Schwarz, we have

$$|l(f)| \leq \|f\|\|g\| \implies \|l\| \leq \|g\|.$$

Putting $f = g$ we see that $\|l\| = \|g\|$.

Theorem 1.6 (Riesz representation theorem)**.** *The space of linear functionals is isomorphic, as a normed* *space, to H itself, i.e. every continuous linear functional is given by the inner product with a fixed vector.*

Proof: Given a continuous linear functional $l \neq 0$, we look at the orthogonal complement of its null space N, which is a closed subspace of H. This is one dimensional: Otherwise, suppose $x, y \in N^\perp$ are linearly independent. We must have $l(x) \neq 0$ and $l(y) \neq 0$, but then $\frac{x}{l(x)} - \frac{y}{l(y)} \in N$.

Next, choose $g \in N^\perp$ so that $l(g) = \|g\|^2$. Using the decomposition $H = N \oplus N^\perp$, we see that any $f \in H$ can be written as $f = h + zg$, for some $h \in N$ and $z \in \mathbb{C}$. In fact, $z = \frac{(f,g)}{(g,g)}$. Then $l(f) = l(h + \frac{(f,g)}{(g,g)} g) = (f, g)$, as desired. $\square$

Linear Transformations between Hilbert spaces

Let H_1 and H_2 be two Hilbert spaces. A continuous linear transformation is a linear map $T : H_1 \to H_2$, continuous in the sense of metric spaces. As for linear functionals, continuity is equivalent to boundedness (exercise). We say that T is bounded with bound M if

$$\|Tf\|_{H_2} \leq M \|f\|_{H_1} \text{ for all } f \in H_1.$$

The least upper bound is called the norm of T and denoted $\|T\|$. Notice that we can also express the norm of T by

(1.1)
$$\|T\| = \sup_{\|f\| \leq 1, \|g\| \leq 1} |(Tf, g)| \text{ where } f \in H_1, g \in H_2.$$

Let's write $\|T\|'$ for the supremum above. We're going to prove $\|T\| = \|T\|'$ by showing $\|T\|' \leq \|T\|$ and $\|T\| \leq \|T\|'$.

First, if M is a bound for T, then by Cauchy-Schwarz $|(Tf, g)| \leq M \|f\| \|g\|$, and so $\|T\|' \leq \|T\|$.

Second, write $\|Tf\|^2 = (Tf, Tf)$ as $\|f\| \|Tf\| (T\frac{f}{\|f\|}, \frac{Tf}{\|Tf\|}) \leq \|f\| \|Tf\| \|T\|'$. Cancelling a factor of $\|Tf\|$, we then have $\|Tf\| \leq \|f\| \|T\|'$ for all f, and so $\|T\|'$ is a bound for T and hence at least $\|T\|$.

Exercise. Find a linear transformation $T : H \to H$ so $\|T\| > \sup_{\|f\| \leq 1} |(Tf, f)|$. (Hint: $H = \mathbb{C}^2$ will suffice.)

Examples:

- A convolution operator. For example, if $k \in L^1(\mathbb{R})$, then
$$f(x) \mapsto \int_{-\infty}^{\infty} k(x-y) f(y)\, dy : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$
- Integration on $L^2([0,1])$:
$$f(x) \mapsto \int_0^x f(s)\, ds.$$
- Let S be a closed subspace in H. We write $H = S \oplus S^\perp$ and send $f \mapsto Pf = s$ where $f = s + t$, with $s \in S, t \in S^\perp$. This is an example of a projection operator.
- Let $e_1, e_2, \dots$ be an orthonormal basis of H. Write $f = \sum a_j e_j$. Then the map $f \mapsto (a_1, a_2, \dots)$ is a linear transformation from H to $l^2(\mathbb{N})$.

Exercise. Prove that the first two examples really do give bounded linear transformations.

If $T : H_1 \to H_2$ is a BLT (bounded linear transformation), then the **adjoint** BLT is the operator $T^* : H_2 \to H_1$ defined by

$$(f, T^* g)_1 = (Tf, g)_2 \text{ for all } f \in H_1, g \in H_2.$$

Check that this really does define a unique operator. Notice that (**??**) shows that $\|T^*\| = \|T\|$. Show that the kernel of T^* is the orthogonal complement of the range of T, and that the adjoint of T^* is T.

Projections and unitaries

An BLT (bounded linear transformation) $P : H \to H$ is called a **projection** if $P^2 = P$ and an **orthogonal projection** if, in addition, $P = P^*$. Notice that any projection satisfies $Pf = f$ for $f \in \text{ran}(P)$. It follows that the range of P is closed, since if $Pf_n \to g$, then $Pf_n \to Pg$. If it is an orthogonal projection, then $f - Pf$ is orthogonal to Pf, since

$$\begin{aligned}(f - Pf, Pf) &= (f, Pf) - (f, P^* Pf) \\ &= (f, Pf) - (f, P^2 f) = 0.\end{aligned}$$

Therefore, if $R = \text{ran}(P)$, then Pf is the first component of f in the decomposition $H = R \oplus R^\perp$. There is thus a one-to-one correspondence between closed subspaces of H and orthogonal projections P.

Exercise. Suppose that S is a closed subspace of H with orthogonal complement $S^{\perp}$, and let $P, P^{\perp}$ be the corresponding orthogonal projections. Show that

$$P + P^{\perp} = \mathrm{Id}\,.$$

Exercise. Suppose that P is an orthogonal projection with one dimensional range S. Write down an expression for P in terms of (i) a nonzero vector $s \in S$ and (ii) the inner product on H.

Theorem 1.7. *Let S be a closed subspace of a Hilbert space H, and let $f \in H$. Then there is a unique point s in S closest to f, and it is given by $s = Pf$ where P is the orthogonal projection onto S. (Compare with Theorem* **??***.)*

The following variation is often useful in the case that $S^{\perp}$ is finite dimensional.

Corollary 1.8. *Let S, H, f and s be above. Then $s = f - P^{\perp} f$ where $P^{\perp}$ is the orthogonal projection onto $S^{\perp}$.*

A BLT $U : H_1 \to H_2$ is called **unitary** if it is invertible, with $U^{-1} = U^*$. It follows that

$$(Uf, Ug)_2 = (U^* U f, g)_1 = (U^{-1} U f, g)_1 = (f, g)_1,$$

so U preserves the inner product and therefore the norm: $\|Uf\|_2 = \|f\|_1$. Two Hilbert spaces are said to be unitarily equivalent if there is a unitary transformation mapping between them.

Theorem 1.9. *Any two infinite dimensional, separable Hilbert spaces are unitarily equivalent.*

The idea of the proof is to choose an orthonormal basis $e_1, e_2, \ldots$ for the first Hilbert space and an orthonormal basis $f_1, f_2, \ldots$ for the second, and then map $e_i \to f_i$. It is not hard to see that there is a unique BLT with this property, and that it is unitary.