

## 10. RADON-NIKODYM DERIVATIVES

### Signed measures

Let  $(X, \mathcal{M})$  be a measurable space. A signed measure is a map  $\nu$  from  $\mathcal{M}$  to  $(-\infty, \infty]$  with the property that if  $E_1, E_2, \dots$  are disjoint elements of  $\mathcal{M}$ , then

$$\nu(\cup_j E_j) = \sum_{j=1}^{\infty} \nu(E_j).$$

Notice that this implies that if  $\nu(\cup_j E_j) < \infty$ , then the sum on the RHS is absolutely convergent, for otherwise it would not be independent of the ordering of the  $E_j$ . Sometimes we refer to (unsigned) measures as positive measures to make the distinction clear.

An example of a signed measure is

$$\nu(E) = \int_E f d\mu$$

where  $(X, \mathcal{M}, \mu)$  is a measure space and  $f$  is a fixed real-valued function such that  $f_-$ , the negative part of  $f$ , is integrable. (This ensures that  $\nu$  can never take the value  $-\infty$ , which is not allowed.) In fact, we shall soon prove that this is the only possibility.

Given a signed measure  $\nu$ , we define the total variation  $|\nu| : \mathcal{M} \rightarrow \mathbb{R}$  as follows:

$$|\nu|(E) = \sup \sum_{j=1}^{\infty} |\nu(E_j)|,$$

where we sup over all ways of decomposing  $E$  into a countable disjoint union of measurable sets  $E_j$ .

**Proposition 10.1.** *The total variation  $|\nu|$  is a positive measure satisfying*

$$\nu \leq |\nu|.$$

Proof: We need to show that

$$|\nu|(E) \leq \sum_{j=1}^{\infty} |\nu|(E_j) \text{ and } |\nu|(E) \geq \sum_{j=1}^{\infty} |\nu|(E_j)$$

whenever  $E$  is written as a countable disjoint union of measurable sets  $E_j$ .

To prove  $\geq$ , we choose numbers  $\alpha_j < |\nu|(E_j)$ . Then, we can find a partition  $E_j = \cup_i F_{i,j}$  into measurable sets such that

$$\alpha_j \leq \sum_j |\nu(F_{i,j})|.$$

Then  $\cup_{i,j} F_{i,j}$  is a partition of  $E$ , so we get

$$\sum_j \alpha_j \leq \sum_{i,j} |\nu(F_{i,j})| \leq |\nu|(E).$$

Taking the sup over all possible  $\alpha_j$  proves  $\geq$ .

To prove  $\leq$ , we take a partition of  $E$  into measurable sets  $F_k$ . Then we have

$$\begin{aligned} \sum_k |\nu(F_k)| &= \sum_k \left| \sum_j \nu(F_k \cap E_j) \right| \\ &\leq \sum_{k,j} |\nu(F_k \cap E_j)| \leq \sum_j |\nu|(E_j). \end{aligned}$$

Since this is true for each way of partitioning  $E$ , we find that

$$|\nu|(E) \leq \sum_j |\nu|(E_j)$$

as required.

The statement  $\nu \leq |\nu|$  is obvious. □

We can then write any signed measure as the difference of two positive measures, by writing

$$\nu = \frac{\nu + |\nu|}{2} + \frac{\nu - |\nu|}{2} = \nu_+ + \nu_-.$$

We say that  $\nu$  is  $\sigma$ -finite if  $|\nu|$  is, and then  $\nu_+$  and  $\nu_-$  automatically are as well.

Notice that the finite signed measures on a measurable space  $(X, \mathcal{M})$  form a vector space, denoted  $M(X)$ .

**Theorem 10.2.**  *$M(X)$  is a complete normed space under the norm*

$$\|\nu\|_{M(X)} = |\nu|(X).$$

Proof: It is straightforward to show that  $\|\cdot\|_{M(X)}$  is a norm. Suppose that  $\nu_j$  is a Cauchy sequence in  $M(X)$ . Then for each  $E \in \mathcal{M}$ ,  $|\nu_n(E) - \nu_m(E)| \rightarrow 0$  as  $m, n \rightarrow \infty$ , so  $\lim_n \nu_n(E)$  exists for each  $E$ . Define  $\nu(E)$  to be  $\lim_n \nu_n(E)$ . We need to show that  $\nu$  is a finite signed measure.

To show countable additivity, suppose that  $E = \cup_i E_i$  is a disjoint union of measurable sets.

**Lemma 10.3.** *For  $\epsilon > 0$  there exists an  $N(\epsilon)$  so*

$$\sum_{i=1}^{\infty} |\nu_n(E_i) - \nu_m(E_i)| < \epsilon \text{ for } n, m \geq N(\epsilon).$$

Proof: To begin, choose  $M(\epsilon)$  so that  $\| |\nu_n| - |\nu_m| \| < \epsilon/5$  for all  $n, m \geq M(\epsilon)$ . Then choose  $I(\epsilon)$  so that  $|\nu_{M(\epsilon)}| \left( \cup_{i \geq I(\epsilon)} E_i \right) < \epsilon/5$ . Then  $|\nu_m| \left( \cup_{i \geq I(\epsilon)} E_i \right) < 2\epsilon/5$  for all  $m \geq M(\epsilon)$ . Finally, choose  $N(\epsilon) \geq M(\epsilon)$  large enough so that  $\sum_{i < I(\epsilon)} |\nu_n(E_i) - \nu_m(E_i)| < \epsilon/5$  for all  $n, m \leq N(\epsilon)$ . (This is possible precisely because it is a finite sum.) Finally then we have for all  $n, m \geq N(\epsilon)$

$$\begin{aligned} & \sum_i |\nu_n(E_i) - \nu_m(E_i)| \\ &= \sum_{i < I(\epsilon)} |\nu_n(E_i) - \nu_m(E_i)| + \sum_{i \geq I(\epsilon)} |\nu_n(E_i) - \nu_m(E_i)| \\ &< \epsilon/5 + \sum_{i \geq I(\epsilon)} |\nu_n(E_i)| + |\nu_m(E_i)| \\ &< 5\epsilon/5 = \epsilon. \end{aligned} \quad \square$$

Certainly then, for every integer  $M$ ,

$$\sum_{i=1}^M |\nu_n(E_i) - \nu_m(E_i)| < \epsilon \text{ for } n, m \geq N(\epsilon).$$

Taking  $m$  to infinity, we find that

$$\sum_{i=1}^M |\nu_n(E_i) - \nu(E_i)| \leq \epsilon \text{ for } n \geq N(\epsilon).$$

Since this is true for all  $M$ , we get

$$(10.1) \quad \sum_{i=1}^{\infty} |\nu_n(E_i) - \nu(E_i)| \leq \epsilon \text{ for } n \geq N(\epsilon).$$

Now we can compute

$$\begin{aligned} |\nu(E) - \sum_i \nu(E_i)| &= \lim_n |\nu_n(E) - \sum_i \nu(E_i)| \\ &= \lim_n \left| \sum_i (\nu_n(E_i) - \nu(E_i)) \right| \\ &\leq \limsup_n \sum_i |\nu_n(E_i) - \nu(E_i)| \\ &\leq \epsilon \end{aligned}$$

by (??). Since this is true for all  $\epsilon$ , we see that  $\nu(E) = \sum_i \nu(E_i)$ , so  $\nu$  is countably additive, and hence a signed measure. Now (??) with  $E_i$  a partition of  $X$  shows that  $\|\nu_n - \nu\|_{M(X)} \rightarrow 0$ . This shows that  $\nu$  is a finite measure and that  $\nu_n \rightarrow \nu$  under the total variation norm, completing the proof.  $\square$

## Absolute continuity

### Definition 10.4.

- (1) We say that a signed measure  $\mu$  is supported on a set  $A$  if  $\mu(E) = \mu(E \cap A)$  for all  $E \in \mathcal{M}$ .
- (2) Two signed measures  $\mu$  and  $\nu$  are mutually singular if they are supported on disjoint subsets. This is denoted  $\mu \perp \nu$ .
- (3) If  $\nu$  is a signed measure and  $\mu$  a positive measure, we say that  $\nu$  is absolutely continuous w.r.t.  $\mu$  if

$$\mu(E) = 0 \implies \nu(E) = 0.$$

If  $|\nu|$  is a finite measure then this last condition is equivalent to the assertion that for each  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\mu(E) < \delta \implies |\nu|(E) < \epsilon,$$

while in general this is a strictly stronger assertion.

**Example.** Lebesgue measure, delta measures, and  $E \mapsto \int_E f$  on  $\mathbb{R}^n$ .

**Exercise.** Give an example where  $|\nu|$  is not finite, and the first assertion does not imply the second.

**Theorem 10.5** (Radon-Nikodym).

Let  $\mu$  be a  $\sigma$ -finite positive measure on the measurable space  $(X, \mathcal{M})$  and  $\nu$  a  $\sigma$ -finite signed measure. Then we can write  $\nu = \nu_a + \nu_s$  where  $\nu_a$  is absolutely continuous w.r.t.  $\mu$ , and  $\nu_s$  and  $\mu$  are mutually singular. Moreover, there exists an extended  $\mu$ -integrable function  $f$  such that

$$\nu_a(E) = \int_E f d\mu.$$

- A function is extended  $\mu$ -integrable if its negative part is integrable.

Proof: We first prove when  $\mu$  and  $\nu$  are both positive and finite measures. Once we have done that, the general case is then not difficult.

We use Hilbert space ideas. Consider the Hilbert space  $L^2(X, \rho)$  where  $\rho = \mu + \nu$ . Consider the map

$$L^2(X, \rho) \ni \psi \mapsto l(\psi) = \int \psi d\nu.$$

This is a bounded linear functional, since

$$|l(\psi)| \leq \int |\psi| d\nu \leq \int |\psi| d\rho \leq \rho(X)^{1/2} \|\psi\|_{L^2}$$



using Cauchy-Schwarz. Therefore  $l$  is inner product with some element  $g$  of  $L^2(X, \rho)$ :

$$(10.2) \quad \int \psi d\nu = \int \psi g d\rho \text{ for all } \psi \in L^2(X, \rho).$$

For any measurable set  $E$ , with  $\rho(E) > 0$ , set  $\psi = 1_E$ . Then we find that

$$\nu(E) = \int 1_E d\nu = \int 1_E g d\rho,$$

so

$$0 \leq \int 1_E g d\rho \leq \rho(E),$$

which implies that  $g \leq 1$  a.e. w.r.t.  $\rho$ . By changing  $g$  on a set of  $\rho$ -measure zero, we can assume that  $g \leq 1$  everywhere.

Now we define  $A$  to be the set where  $g < 1$  and  $B$  to be the set where  $g = 1$ . Putting  $\psi = 1_B$ , we find that

$$\nu(B) = \int 1_B d\nu = \int 1_B g d\rho = \int 1_B d\rho = \nu(B) + \mu(B).$$

Therefore,  $\mu(B) = 0$ . Since  $\mu$  is a positive measure, this means that  $\mu$  is supported in  $B^c = A$ . So define

$$\nu_a(E) = \nu(E \cap A), \quad \nu_s(E) = \nu(E \cap B).$$

We have just shown that  $\nu_s$  and  $\mu$  are mutually singular. Now we show that  $\nu_a$  is absolutely continuous w.r.t.  $\mu$ .

First we reformulate Equation (??) as

$$\int \psi(1 - g) d\nu = \int \psi g d\mu.$$

It is tempting to try  $\psi = (1 - g)^{-1}$ , which would then give

$$\nu(E) = \int_E d\nu = \int_E (1 - g)^{-1}(1 - g) d\nu = \int_E (1 - g)^{-1} g d\mu$$

and the desired conclusion.

However, this is not allowed since  $(1 - g)^{-1} \notin L^2(X, \rho)$  necessarily. Instead, we approximate, setting

$$\psi = (1 + g + g^2 + \dots + g^n)1_{E \cap A}$$

which is bounded and therefore in  $L^2$ . We obtain

$$\int_{E \cap A} (1 - g^{n+1}) d\nu = \int_{E \cap A} g \frac{1 - g^{n+1}}{1 - g} d\mu.$$

Since  $g < 1$  on  $A$ ,  $1 - g^{n+1} \uparrow 1$  pointwise, so by MCT we get

$$\nu_a(E) = \nu(E \cap A) = \int_{E \cap A} d\nu = \int_{E \cap A} \frac{g}{1 - g} d\mu.$$

This shows that  $\nu_a$  is absolutely continuous and we may take  $f = g(1 - g)^{-1}$ , which (by putting  $E = X$ ) the above equation shows is integrable w.r.t.  $\mu$ .

To prove for  $\sigma$ -finite, positive measures  $\mu, \nu$ , we write  $X$  as the disjoint union of a countable family  $E_j$  of sets of finite measure. Let  $\mu_j, \nu_j$  be the restrictions of  $\mu, \nu$  to  $E_j$ . Then we can decompose  $\nu_j$  as  $\nu_{j,a} + \nu_{j,s}$  as above. Setting  $\nu_a = \sum_j \nu_{j,a}$  and  $\nu_s = \sum_j \nu_{j,s}$  we satisfy the conditions of the theorem. To treat the case of a signed measure, we treat the positive and negative parts of  $\nu$  separately.  $\square$