

3. REVIEW OF ‘CALCULUS’

- Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous. The support of ϕ is the closure of the set where $\phi(x) \neq 0$. If the support of ϕ is compact then we say that ϕ is compactly supported.

- There exist functions $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$, ϕ is C^∞ , and ϕ is compactly supported. The set of compactly supported, smooth functions on \mathbb{R}^n is denoted $C_c^\infty(\mathbb{R}^n)$.

- L^p norms. The L^p norm, $p \geq 1$, of a measurable function f on a measurable set E is defined to be

$$\|f\|_{L^p(E)} := \left(\int_E |f(x)|^p dx \right)^{1/p}.$$

It is a norm (homogeneous, nonnegative, obeys triangle inequality) provided we identify functions which differ on a set of measure zero. The normed space of (equivalence classes of) functions with finite L^p norm is denoted $L^p(E)$. A very important property is that $L^p(E)$ is complete; we will prove this later in the course. We also define $L^\infty(E)$ to be the set of essentially bounded (equivalence classes of) functions, i.e. those for which

$$\|f\|_{L^\infty(E)} := \sup \{M \mid \text{the set } \{x \mid |f(x)| > M\} \text{ has positive measure.}\}$$

is finite. This is also a complete normed space.

- If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous and compactly supported, then it is in L^p for every $1 \leq p \leq \infty$.

- Hölder’s inequality: if $p^{-1} + q^{-1} = 1$,

$$\left| \int_E f(x)g(x) dx \right| \leq \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

To prove Hölder’s inequality, we begin with Jensen’s inequality (stating that secants of convex functions stay above the

function) for the function $x \mapsto b^x$, obtaining

$$b \leq \frac{1}{p} + \frac{b^q}{q}.$$

Next, we take advantage of the fact that this inequality holds for all b , but the different terms scale differently in b . (You should read Terry Tao’s blog post ‘Amplification, arbitrage, and the tensor product trick!’) In particular, replacing b with $a^{1-p}b$ and rearranging we obtain Young’s inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

From this, Hölder’s inequality follows easily — first prove it for functions with $\|f\|_p = 1$ and $\|g\|_q = 1$.

- Dominated convergence theorem. (SS Chapter 2 Theorem 1.13).

Let f_n be a sequence of functions in $L^1(E)$ converging pointwise a.e. to f . Suppose that $|f_n(x)| \leq g(x)$ for a fixed L^1 function g . Then

$$\int_E f_n \rightarrow \int_E f.$$

Sketch: Consider the sets E_N on which $|x| \leq N$ and $|g(x)| \leq N$. Eventually, every point is in some E_N , and so by the monotone convergence theorem $\int_{E_N} g$ becomes arbitrarily small. Estimate $\int_E |f_n - f|$ as the sum of the integral on one of these sets and the integral on the complement; use the bounded convergence theorem on the first integral and $|f_n - f| \leq 2g$ on the second. \square

The bounded convergence theorem is now a special case of the dominated convergence theorem, but of course one needs to prove it first!

The bounded convergence theorem follows easily from Egorov’s theorem (SS Chapter 1 Theorem 4.4) which says that any pointwise limit of functions actually converges uniformly, off some arbitrarily small open set.

Sketch: [Egorov] Define

$$E_k^n = \{x \in E \mid |f_j(x) - f(x)| < 1/n \text{ for all } j > k\}.$$

Choose k_n large enough that $m(E - E_{k_n}^n) < 2^{-n}$. Let \tilde{A} be the intersection of some tail of the sets $\{E_{k_n}^n\}$, choosing the tail so that \tilde{A} has almost full measure. Finally let A be a closed subset of \tilde{A} , omitting only a small set. \square

- Fubini-Tonelli theorem (in \mathbb{R}^n):

Theorem 3.1.

(i) Suppose that $f : \mathbb{R}^{n+m} \rightarrow \mathbb{C}$ is nonnegative and measurable. Then

$$(3.1) \quad \int_{\mathbb{R}^{n+m}} f = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y) dy \right) dx = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x, y) dx \right) dy.$$

Note: this is an equality in extended real numbers: the left hand side might be $+\infty$, but this happens if and only if the right hand side is also $+\infty$.

(ii) Suppose that $f \in L^1(\mathbb{R}^{n+m})$. Then (??) holds.

The first part is Tonelli's, the second part Fubini's.

Often we use these in conjunction. Suppose we are asked to integrate some function f on \mathbb{R}^d , but don't even know it is integrable. We first apply Tonelli's theorem to $|f|$, justifying the use of multiple integrals. Maybe we can calculate them, or if not, at least estimate them. Thus we can establish that f is integrable. Finally we apply Fubini's theorem to justify using multiple integrals in the actual calculation.

- Polar coordinates:

Let f be a real-valued integrable function on \mathbb{R}^n . Define the $(n - 1)$ -sphere by

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

Let $\tilde{f}(r, \omega) = f(r\omega)$, so $\tilde{f} : \mathbb{R}_+ \times S^{n-1} \rightarrow \mathbb{R}$. Also, for a measurable subset E of \mathbb{R}^n , and $r > 0$, define $E_r \subset S^{n-1}$ by

$$E_r = \{\omega \in S^{n-1} \mid r\omega \in E\}.$$

Then

$$\int_E f(x) dx = \int_0^\infty \left(\int_{E_r} \tilde{f}(r, \omega) d\omega \right) r^{n-1} dr.$$

- Using polar coordinates we see the following: Let B be the unit ball in \mathbb{R}^n . The function $|x|^{-\alpha}$ is in $L^1(B)$ iff $\alpha < n$ and it is in $L^1(\mathbb{R}^n \setminus B)$ iff $\alpha > n$.

- Absolutely continuous functions and the fundamental theorem of calculus.

Definition 3.2. A function $f : [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous* if for any $\epsilon > 0$ there exists a $\delta > 0$ so that

$$\sum_{k=1}^N |f(b_k) - f(a_k)| < \epsilon \quad \text{whenever} \quad \sum_{k=1}^N (b_k - a_k) < \delta$$

and the intervals (a_k, b_k) are disjoint.

Theorem 3.3 (SS Chapter 3, Theorem 3.8). *An absolutely continuous function is differentiable almost everywhere. Moreover, if its derivative is zero almost everywhere, the function is constant.*

Theorem 3.4 (SS Chapter 3, Theorem 3.11). *The derivative of an absolutely continuous function F is integrable, and*

$$F(x) - F(a) = \int_a^x F'(y) dy.$$

Conversely, if f is integrable on $[a, b]$, then $F(x) = \int_a^x f(y) dy$ is absolutely continuous and $F'(x) = f(x)$ almost everywhere.

- Differentiating under the integral sign:

Proposition 3.5. *Suppose that U is an open set in \mathbb{R}^n , E is a measurable set in \mathbb{R}^k , $f : U \times E \rightarrow \mathbb{R}$ is a function so that*

- (i) $f(x, \cdot) : E \rightarrow \mathbb{R}$ is measurable for each $x \in U$,
- (ii) $\partial_{x_i} f(x, y)$ exists and is continuous for all (x, y)

and

- (iii) (the crucial condition)

$$|\partial_{x_i} f(x, y)| \leq g(y) \text{ for some } g \in L^1(E).$$

Then

$$\frac{\partial}{\partial x_i} \int_E f(x, y) dy = \int_E \frac{\partial f}{\partial x_i}(x, y) dy.$$

Proof: (sketch) The LHS is, for a fixed x ,

$$\lim_{h \rightarrow 0} \int_E \frac{f(x + he_i, y) - f(x, y)}{h} dy.$$

Use (ii) and the mean value theorem to write the integrand as $\partial_{x_i} f(x + \theta(h)e_i, y)$ for some $0 \leq \theta(h) \leq h$ and conclude that it is pointwise bounded by $g(y)$. Then by the dominated convergence theorem, we can take the pointwise limit inside the integral. This is just $\partial_{x_i} f(x, y)$ using (ii) again, which gives us the RHS. \square

- Surface measure. Let S be a hypersurface given by the graph of a C^1 function:

$$S = \{(x_1, \dots, x_n) \mid x_n = u(x_1, x_2, \dots, x_{n-1})\}, \\ u \in C^1(\mathbb{R}^{n-1}).$$

Then, in terms of the coordinates (x_1, \dots, x_{n-1}) on S , surface measure on S is defined to be

$$(3.3) \quad d\sigma = \sqrt{1 + |\nabla u(x')|^2} dx', \quad x' = (x_1, \dots, x_{n-1}).$$

Proposition 3.7. *The measure $d\sigma$ on S is invariant under a Euclidean change of coordinates. That is, suppose that (y_1, \dots, y_n) are another set of Euclidean coordinates. This means that there is an orthonormal basis e'_i such that (y_1, \dots, y_n) represents the point $\sum_i y_i e'_i$. If S can also be written as a graph in the y coordinates,*

$$S = \{(y_1, \dots, y_n) \mid y_n = v(y_1, y_2, \dots, y_{n-1})\}, \quad v \in C^1, \\ \text{then we have}$$

$$d\sigma = \sqrt{1 + |\nabla v(y')|^2} dy', \quad y' = (y_1, \dots, y_{n-1}).$$

- Change of variable formula:

Theorem 3.6. *Let $R \subset \mathbb{R}^n$ be a rectangle, and $F : R \rightarrow \mathbb{R}^n$ a C^1 function. Then for every continuous function f defined on $F(R)$, we have the change of variable formula*

$$(3.2) \quad \int_{F(R)} f(y) dy = \int_R (f \circ F)(x) |\det DF(x)| dx.$$

We sometimes write this differently: we think of F as relating two different sets of coordinates, the y coordinates on $F(R)$ and the x coordinates on R . We sometimes write $y = y(x)$ instead of $y = F(x)$. Also, the Jacobian matrix DF is sometimes written $\partial y / \partial x$.

So we have

$$\int_{F(R)} f(y) dy = \int_R f(y(x)) \left| \det \frac{\partial y}{\partial x} \right| dx.$$

The key to proving this proposition is showing that, if the y' coordinates on S are given in terms of x' by $y' = F(x')$, then

$$(3.4) \quad \det DF(x_0) = \frac{\sqrt{1 + |\nabla u(x'_0)|^2}}{\sqrt{1 + |\nabla v(y'_0)|^2}}, \quad y'_0 = F(x'_0).$$

We then use Theorem ??.

The identity (??) can be proved by considering two Euclidean sets of coordinates $y = (y_1, \dots, y_n)$ and $x = (x_1, \dots, x_n)$. Change to $\tilde{y} = (y_1, \dots, y_{n-1}, Y_n)$ and $\tilde{x} = (x_1, \dots, x_{n-1}, X_n)$ where $Y_n = y_n - v(y')$, $X_n = x_n - u(x')$. Then show that, on the surface,

$$\det \frac{\partial y'}{\partial x'} = \left(\frac{\partial Y_n}{\partial X_n} \right)^{-1}.$$

This can be computed explicitly, to be equal to

$$\frac{\sqrt{1 + |\nabla u(x')|^2}}{\sqrt{1 + |\nabla v(y')|^2}}.$$

- The result above allows us to define surface measure for any C^1 hypersurface, not just a graph.

• Integration by parts: the following result will be adequate for now; it is possible to weaken the assumptions.

Proposition 3.8.

(i) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Then if $f, g \in C^1(\overline{\Omega})$, we have

$$\int_{\Omega} \left(f \frac{\partial g}{\partial x_i} + g \frac{\partial f}{\partial x_i} \right) dx = \int_{\partial\Omega} f g n_i d\sigma$$

where $n_i = n \cdot e_i$ is the i th component of the outward pointing normal vector n and σ is surface measure on $\partial\Omega$.

(ii) Assume that f, g are C^1 functions on \mathbb{R}^n , such that $f, \partial_{x_i} f \in L^p(\mathbb{R}^n)$, while $g, \partial_{x_i} g \in L^q(\mathbb{R}^n)$, with $p^{-1} + q^{-1} = 1$. Then

$$\int_{\mathbb{R}^n} f \frac{\partial g}{\partial x_i} dx = - \int_{\mathbb{R}^n} g \frac{\partial f}{\partial x_i} dx.$$

Notice that $dx' = (n \cdot e_n) d\sigma$ in the notation of (??), where n is the upward pointing unit normal to S .