

## 5. FUNDAMENTAL SOLUTIONS

The Fourier transform is the perfect tool for finding fundamental solutions of constant coefficient differential operators in  $\mathbb{R}^n$ .

Consider the problem of solving

$$P(D)u = f$$

in  $\mathbb{R}^n$ , where  $D$  stands for  $(D_1, \dots, D_n)$ ,  $D_i = -i\partial_{x_i}$  is the partial derivative in the  $i$ th direction, and  $P$  is a polynomial. We suppose that  $f \in \mathcal{S}(\mathbb{R}^n)$  is given and want to find a solution  $u$ . We might also want to know, for example, if  $f \in L^2$  implies that  $u \in L^2$ .

If there is a solution, then Fourier transforming, we have

$$P(\xi)\hat{u} = \hat{f}.$$

Therefore, if  $P(\xi)$  never vanishes, there is a solution  $\hat{u}(\xi) = P(\xi)^{-1}\hat{f}(\xi)$  to this equation. Taking the inverse Fourier transform we get our solution  $u$ .

This works well, provided that  $|P(\xi)| \geq c > 0$  everywhere for some fixed  $c$ . This implies that  $P(\xi)^{-1} \in BC^\infty(\mathbb{R}^n)$  and hence that  $\hat{u}(\xi) = P(\xi)^{-1}\hat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n)$  which in turn gives  $u \in \mathcal{S}(\mathbb{R}^n)$ .

Moreover, using our results on convolutions, if  $\hat{u}(\xi) = P(\xi)^{-1}\hat{f}(\xi)$ , then  $u = \mathcal{G}(P(\xi)^{-1}) * f$ , so if we can compute  $\mathcal{G}(P(\xi)^{-1})$  then we get a solution without explicit mention of the Fourier transform.

### The Laplacian

Let's consider the most important PDE of all — Laplace's equation  $-\Delta u = f$ . Here  $\Delta$  is the Laplacian, given by

$$\Delta f(x) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(x).$$

If we try to solve  $-\Delta u = f$  in this way, we get  $\hat{u}(\xi) = |\xi|^{-2}\hat{f}(\xi)$ , and the singularity at  $\xi = 0$  causes some difficulties. To avoid these let me look instead at  $(-\Delta + \lambda^2)u = f$  where  $\lambda > 0$ . Now  $P(\xi) = \lambda^2 + |\xi|^2$  has no zeroes. Thus  $P(\xi)^{-1} \in BC^\infty(\mathbb{R}^n)$ . Can we compute the inverse Fourier transform of  $P(\xi)^{-1}$ ?

Let's do this just in dimension 3, which is interesting both because it describes our physical world and because we can compute the inverse Fourier transform exactly. Note that in dimension 3,  $(|\xi|^2 + \lambda^2)^{-1}$  is in  $L^2$ , so the inverse Fourier transform is well defined, but it is not in  $L^1$ , so it is not defined as a convergent integral. Rather, it is defined as a limit of the inverse Fourier transform of Schwartz functions.

To compute it, choose a function  $\phi \in C_c^\infty(\mathbb{R}_{\geq 0})$  which is equal to 1 on the interval  $[0, 1]$  and is zero outside  $[0, 2]$ . Then the function  $\phi(|\xi|/R)(|\xi|^2 + \lambda^2)^{-1}$  is in  $\mathcal{S}(\mathbb{R}^3)$  for all  $R > 0$  and converges in  $L^2$  to  $(|\xi|^2 + \lambda^2)^{-1}$  as  $R \rightarrow \infty$ .

Now consider the integral

$$\mathcal{G}(\phi(|\xi|/R)(|\xi|^2 + \lambda^2)^{-1})(\xi) = (2\pi)^{-3} \int e^{ix \cdot \xi} \frac{\phi(|\xi|/R)}{|\xi|^2 + \lambda^2} d\xi.$$

Changing to polar coordinates, this is

$$(2\pi)^{-3} \int_0^\infty \frac{\phi(r/R)}{r^2 + \lambda^2} r^2 dr \int_0^\pi e^{i|x|r \cos \theta} \sin \theta d\theta \int_0^{2\pi} d\varphi.$$

The  $\varphi$  integral is trivial and gives a factor of  $2\pi$ . To do the theta integral, let  $u = -\cos \theta$ . Then  $\sin \theta d\theta = du$  and we obtain

$$-i(2\pi)^{-2}|x|^{-1} \int_0^\infty (e^{i|x|r} - e^{-i|x|r}) \frac{\phi(r/R)}{r^2 + \lambda^2} r dr.$$

We can write this as an integral from  $-\infty$  to  $\infty$ :

$$-i(2\pi)^{-2}|x|^{-1} \int_{-\infty}^\infty e^{i|x|r} \frac{\phi(|r|/R)}{r^2 + \lambda^2} r dr.$$

We want the limit of this integral as  $R \rightarrow \infty$ . We can check that

$$v_R(x) \stackrel{\text{def}}{=} -i(2\pi)^{-2}|x|^{-1} \int_{|r| \geq R} e^{i|x|r} \frac{\phi(r/R)}{r^2 + \lambda^2} r dr$$

tends to zero in  $L^2(\mathbb{R}^3)$  (as a function of  $x$ ) as  $R \rightarrow \infty$ . In fact the integral can be bounded by  $C|x|^{-1}$ , or, by integrating by parts (integrate the exponential and differentiate the rest), by  $C|x|^{-2}R^{-1}$ . Therefore the function  $v_R(x)$  can be bounded by

$$(5.1) \quad \begin{cases} \frac{1}{|x|}, & \text{if } |x| \leq R^{-1}; \\ \frac{1}{|x|^2 R}, & \text{if } |x| \geq R^{-1} \end{cases}$$

and one can check that the  $L^2$  norm here is  $O(R^{-1/2})$ , which certainly tends to zero as  $R \rightarrow \infty$ .

Thus we can throw out the contribution to the integral for  $|r| > R$ . This expression no longer depends on our cutoff function  $\phi$  at all, since  $\phi$  is identically 1 for  $|r| \leq R$ . We obtain

$$\mathcal{G}((|\xi|^2 + \lambda^2)^{-1}) = \lim_{R \rightarrow \infty} -i(2\pi)^{-2}|x|^{-1} \int_{-R}^R e^{i|x|r} \frac{1}{r^2 + \lambda^2} r \, dr.$$

So, the solution of the equation  $(-\Delta + \lambda^2)u = f$  on  $\mathbb{R}^3$  is

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\lambda|x-y|}}{|x-y|} f(y) \, dy.$$

If we formally take the pointwise limit  $\lambda \rightarrow 0$  in this integral, we obtain the putative formula

$$(5.2) \quad u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) \, dy$$

for ‘the’ solution to  $-\Delta u = f$ . This can be justified when, for example,  $f$  is compactly supported and  $L^2$ . Then  $u$  given by (??) is the unique solution to this equation that tends to zero at infinity. However, generally  $u$  will not be in  $L^2$ . In fact, we will have  $u(x) = c/|x| + O(|x|^{-2})$  as  $x \rightarrow \infty$  with  $c$  usually  $\neq 0$ .

Next I claim that the function

$$w_R(x) \stackrel{\text{def}}{=} -i(2\pi)^{-2}|x|^{-1} \int_{\gamma} e^{i|x|r} \frac{r}{r^2 + \lambda^2} \, dr$$

goes to zero in  $L^2(\mathbb{R}^3)$  when  $\gamma$  is the contour from  $r = R$  to  $r = -R$  anticlockwise along the circle  $|r| = R$  in the complex  $r$ -plane. In fact, we can also bound  $w_R(x)$  by (??), using a similar argument as for  $v_R$ , from which the claim follows.

We now have a closed contour around which we integrate the analytic function  $e^{i|x|r} r (r^2 + \lambda^2)^{-1}$ . By Cauchy’s residue theorem, the integral is given by the  $2\pi i$  times the residue of the function at its unique pole in this region, which is at  $r = i\lambda$ . Hence the value of the integral is given by

$$-i \cdot 2\pi i \cdot |x|^{-1} (2\pi)^{-2} \cdot \frac{i\lambda e^{-\lambda|x|}}{2i\lambda} = \frac{1}{4\pi} \frac{e^{-\lambda|x|}}{|x|}.$$

We have thus shown that

$$\mathcal{G}((|\xi|^2 + \lambda^2)^{-1}) = \frac{1}{4\pi} \frac{e^{-\lambda|x|}}{|x|}.$$

Let us check directly that if  $f \in C_c^2(\mathbb{R}^3)$ , then  $u$  given by (??) satisfies  $\Delta u = f$ : We compute

$$\begin{aligned} -\Delta_x \int \frac{1}{|x-y|} f(y) \, dy &= -\Delta_x \int \frac{1}{|y|} f(x-y) \, dy \\ &= \int \frac{1}{|y|} (-\Delta_x f(x-y)) \, dy \\ &= \int \frac{1}{|y|} (-\Delta_y f(x-y)) \, dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3 \setminus B(0, \epsilon)} \frac{1}{|y|} (-\Delta_y f(x-y)) \, dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3 \setminus B(0, \epsilon)} \sum_i \partial_{y_i} \frac{1}{|y|} (\partial_{y_i} f(x-y)) \, dy + O(\epsilon). \end{aligned}$$

The  $O(\epsilon)$  term is from the boundary integral, and we discard it since we are taking the limit  $\epsilon \rightarrow 0$ . Now the integrand is equal to

$$(\Delta_y \frac{1}{|y|}) f(x-y) - \sum_i \partial_{y_i} \left( (\partial_{y_i} \frac{1}{|y|}) f(x-y) \right)$$

and the first term vanishes since  $\Delta_y \frac{1}{|y|} = 0$  away from the singularity at 0, while the second term becomes a

boundary term using Green's Theorem. So we get

$$\begin{aligned}
 &= \lim_{\epsilon \rightarrow 0} \int_{\partial B(0, \epsilon)} \sum_i \nu_i \left( \partial_{y_i} \frac{1}{|y|} \right) f(x-y) dy \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\partial B(0, \epsilon)} \frac{1}{|y|^2} f(x-y) dy \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\partial B(0, 1)} f(x-\epsilon y) dy \\
 &= |\partial B(0, 1)| f(x) = 4\pi f(x).
 \end{aligned}$$

• We cannot apply the  $\Delta_x$  operator directly to the  $1/|x-y|$  term in the first line, since the second partial derivatives of  $1/|x-y|$  are not integrable as is required by the DUTIS theorem. If you illegally did this, you would end up proving that  $\Delta u = 0$ , which is false!

• The second line of this derivation shows that if  $f \in C^2$ , with compact support, then  $u \in C^2$ . This can be improved to  $f \in C^1 \implies u \in C^2$ , but it is not true that  $f \in C^0 \implies u \in C^2$  as you might expect. However, there are two analogous statements that *are* true: if  $f \in L^2$ , then  $u$  has all its second derivatives in  $L^2$ ; and if  $f \in C^\alpha$ , then  $u \in C^{2,\alpha}$ . (This expresses that  $u$  has continuous 2nd partial derivatives, which are Hölder continuous with exponent  $\alpha$ , i.e. satisfy  $|f(x) - f(y)| \leq C|x-y|^\alpha$ .) These statements express the 'ellipticity' of  $\Delta$ .

### Heat equation

The heat equation on  $\mathbb{R}^n \times \mathbb{R}_+$  is the equation

$$u_t(x, t) = \Delta u(x, t), \quad x \in \mathbb{R}^n, t > 0$$

supplemented with the initial condition

$$u(x, 0) = f(x).$$

Assume that  $f$  is a Schwartz function. Then we can find a solution that is a continuous function of  $t$  with values in Schwartz functions of  $x$ . Fourier transforming in the  $x$  variable but not the  $t$  variable (which would not make sense since the solution is only defined for  $t \geq 0$ ) we get

$$\hat{u}_t = -|\xi|^2 \hat{u}, \quad \hat{u}(\xi, 0) = \hat{f}(\xi).$$

This is an ODE in  $t$  for each fixed  $\xi$ , and the solution is

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t} \hat{u}(\xi, 0) = e^{-|\xi|^2 t} \hat{f}(\xi).$$

Hence, the function  $u$  is given by a convolution:

$$u(x, t) = \mathcal{G}(e^{-|\xi|^2 t}) * f.$$

So we need to know the inverse Fourier transform of  $e^{-|\xi|^2 t}$ . But we have already worked this out, and the answer is

$$\mathcal{G}(e^{-|\xi|^2 t}) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

To summarize, the solution of the PDE is

$$u(x, t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy.$$

The function  $(4\pi t)^{-n/2} e^{-|x-y|^2/4t}$  is called the 'heat kernel' on  $\mathbb{R}^n$ . It may be regarded as the solution to the heat equation with initial condition  $f = \delta_y(x)$ .