5. Fundamental solutions

The Fourier transform is the perfect tool for finding fundamental solutions of constant coefficient differential operators in $\mathbb{R}^n$.

Consider the problem of solving

$$P(D)u = f$$

in $\mathbb{R}^n$, where D stands for $(D_1, \ldots, D_n)$, $D_i = -i\partial_{x_i}$ is the partial derivative in the ith direction, and P is a polynomial. We suppose that $f \in \mathcal{S}(\mathbb{R}^n)$ is given and want to find a solution u. We might also want to know, for example, if $f \in L^2$ implies that $u \in L^2$.

If there is a solution, then Fourier transforming, we have

$$P(\xi)\hat{u} = \hat{f}.$$

Therefore, if $P(\xi)$ never vanishes, there is a solution $\hat{u}(\xi) = P(\xi)^{-1}\hat{f}(\xi)$ to this equation. Taking the inverse Fourier transform we get our solution u.

This works well, provided that $|P(\xi)| \geq c > 0$ everywhere for some fixed c. This implies that $P(\xi)^{-1} \in BC^\infty(\mathbb{R}^n)$ and hence that $\hat{u}(\xi) = P(\xi)^{-1}\hat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n)$ which in turn gives $u \in \mathcal{S}(\mathbb{R}^n)$.

Moreover, using our results on convolutions, if $\hat{u}(\xi) = P(\xi)^{-1}\hat{f}(\xi)$, then $u = \mathcal{G}(P(\xi)^{-1}) * f$, so if we can compute $\mathcal{G}(P(\xi)^{-1})$ then we get a solution without explicit mention of the Fourier transform.

The Laplacian

Let's consider the most important PDE of all — Laplace's equation $-\Delta u = f$. Here Δ is the Laplacian, given by

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x).$$

If we try to solve $-\Delta u = f$ in this way, we get $\hat{u}(\xi) = |\xi|^{-2}\hat{f}(\xi)$, and the singularity at $\xi = 0$ causes some difficulties. To avoid these let me look instead at $(-\Delta + \lambda^2)u = f$ where $\lambda > 0$. Now $P(\xi) = \lambda^2 + |\xi|^2$ has no zeroes. Thus $P(\xi)^{-1} \in BC^\infty(\mathbb{R}^n)$. Can we compute the inverse Fourier transform of $P(\xi)^{-1}$?

Let's do this just in dimension 3, which is interesting both because it describes our physical world and because we can compute the inverse Fourier transform exactly. Note that in dimension 3, $(|\xi|^2 + \lambda^2)^{-1}$ is in L^2, so the inverse Fourier transform is well defined, but it is not in L^1, so it is not defined as a convergent integral. Rather, it is defined as a limit of the inverse Fourier transform of Schwartz functions.

To compute it, choose a function $\phi \in C_c^\infty(\mathbb{R}_{\geq 0})$ which is equal to 1 on the interval $[0,1]$ and is zero outside $[0,2)$. Then the function $\phi(|\xi|/R)(|\xi|^2+\lambda^2)^{-1}$ is in $\mathcal{S}(\mathbb{R}^3)$ for all $R>0$ and converges in L^2 to $(|\xi|^2+\lambda^2)^{-1}$ as $R \to \infty$.

Now consider the integral

$$\mathcal{G}\left(\phi(|\xi|/R)(|\xi|^2+\lambda^2)^{-1}\right)(\xi) = (2\pi)^{-3}\int e^{ix\cdot\xi}\frac{\phi(|\xi|/R)}{|\xi|^2+\lambda^2}\,d\xi.$$

Changing to polar coordinates, this is

$$(2\pi)^{-3}\int_0^\infty \frac{\phi(r/R)}{r^2+\lambda^2}\,r^2dr\int_0^\pi e^{i|x|r\cos\theta}\sin\theta\,d\theta\int_0^{2\pi}d\varphi.$$

The φ integral is trivial and gives a factor of 2π. To do the theta integral, let $u=-\cos\theta$. Then $\sin\theta d\theta = du$ and we obtain

$$-i(2\pi)^{-2}|x|^{-1}\int_0^\infty (e^{i|x|r}-e^{-i|x|r})\frac{\phi(r/R)}{r^2+\lambda^2}r\,dr.$$

We can write this as an integral from $-\infty$ to ∞:

$$-i(2\pi)^{-2}|x|^{-1}\int_{-\infty}^\infty e^{i|x|r}\frac{\phi(|r|/R)}{r^2+\lambda^2}\,r\,dr.$$

We want the limit of this integral as $R\to\infty$. We can check that

$$v_R(x) \stackrel{\text{def}}{=} -i(2\pi)^{-2}|x|^{-1}\int_{|r|\geq R} e^{i|x|r}\frac{\phi(r/R)}{r^2+\lambda^2}\,r\,dr$$

tends to zero in $L^2(\mathbb{R}^3)$ (as a function of x) as $R \to \infty$. In fact the integral can be bounded by $C|x|^{-1}$, or, by integrating by parts (integrate the exponential and differentiate the rest), by $C|x|^{-2}R^{-1}$. Therefore the function $v_R(x)$ can be bounded by

$$\begin{cases} \frac{1}{|x|}, \text{ if } |x| \leq R^{-1}; \\ \frac{1}{|x|^2 R}, \text{ if } |x| \geq R^{-1} \end{cases} \tag{5.1}$$

and one can check that the L^2 norm here is $O(R^{-1/2})$, which certainly tends to zero as $R \to \infty$.

Thus we can throw out the contribution to the integral for $|r| > R$. This expression no longer depends on our cutoff function ϕ at all, since ϕ is identically 1 for $|r| \leq R$. We obtain

$$\mathcal{G}\left((|\xi|^2 + \lambda^2)^{-1}\right) = \lim_{R\to\infty} -i(2\pi)^{-2}|x|^{-1} \int_{-R}^{R} e^{i|x|r} \frac{1}{r^2+\lambda^2}\, r\, dr.$$

Next I claim that the function

$$w_R(x) \stackrel{\text{def}}{=} -i(2\pi)^{-2}|x|^{-1} \int_\gamma e^{i|x|r} \frac{r}{r^2+\lambda^2}\, dr$$

goes to zero in $L^2(\mathbb{R}^3)$ when γ is the contour from $r = R$ to $r = -R$ anticlockwise along the circle $|r| = R$ in the complex r-plane. In fact, we can also bound $w_R(x)$ by (**??**), using a similar argument as for v_R, from which the claim follows.

We now have a closed contour around which we integrate the analytic function $e^{i|x|r}r(r^2+\lambda^2)^{-1}$. By Cauchy's residue theorem, the integral is given by the $2\pi i$ times the residue of the function at its unique pole in this region, which is at $r = i\lambda$. Hence the value of the integral is given by

$$-i \cdot 2\pi i \cdot |x|^{-1}(2\pi)^{-2} \cdot \frac{i\lambda e^{-\lambda|x|}}{2i\lambda} = \frac{1}{4\pi}\frac{e^{-\lambda|x|}}{|x|}.$$

We have thus shown that

$$\mathcal{G}\left((|\xi|^2+\lambda^2)^{-1}\right) = \frac{1}{4\pi}\frac{e^{-\lambda|x|}}{|x|}.$$

So, the solution of the equation $(-\Delta + \lambda^2)u = f$ on $\mathbb{R}^3$ is

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\lambda|x-y|}}{|x-y|} f(y)\, dy.$$

If we formally take the pointwise limit $\lambda \to 0$ in this integral, we obtain the putative formula

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\, dy \tag{5.2}$$

for 'the' solution to $-\Delta u = f$. This can be justified when, for example, f is compactly supported and L^2. Then u given by (**??**) is the unique solution to this equation that tends to zero at infinity. However, generally u will not be in L^2. In fact, we will have $u(x) = c/|x| + O(|x|^{-2})$ as $x \to \infty$ with c usually $\neq 0$.

Let us check directly that if $f \in C_c^2(\mathbb{R}^3)$, then u given by (**??**) satisfies $\Delta u = f$: We compute

$$
\begin{aligned}
-\Delta_x \int \frac{1}{|x-y|} f(y)\,dy &= -\Delta_x \int \frac{1}{|y|} f(x-y)\,dy \\
&= \int \frac{1}{|y|}(-\Delta_x f(x-y))\,dy \\
&= \int \frac{1}{|y|}(-\Delta_y f(x-y))\,dy \\
&= \lim_{\epsilon\to 0} \int_{\mathbb{R}^3\setminus B(0,\epsilon)} \frac{1}{|y|}(-\Delta_y f(x-y))\,dy \\
&= \lim_{\epsilon\to 0} \int_{\mathbb{R}^3\setminus B(0,\epsilon)} \sum_i \partial_{y_i} \frac{1}{|y|}(\partial_{y_i} f(x-y))\,dy + O(\epsilon).
\end{aligned}
$$

The $O(\epsilon)$ term is from the boundary integral, and we discard it since we are taking the limit $\epsilon \to 0$. Now the integrand is equal to

$$
(\Delta_y \frac{1}{|y|}) f(x-y) - \sum_i \partial_{y_i}\Big((\partial_{y_i}\frac{1}{|y|}) f(x-y)\Big)
$$

and the first term vanishes since $\Delta_y \frac{1}{|y|} = 0$ away from the singularity at 0, while the second term becomes a boundary term using Green's Theorem. So we get

$$= \lim_{\epsilon\to 0}\int_{\partial B(0,\epsilon)} \sum_i \nu_i(\partial_{y_i}\frac{1}{|y|}) f(x-y)\,dy$$

$$= \lim_{\epsilon\to 0}\int_{\partial B(0,\epsilon)} \frac{1}{|y|^2} f(x-y)\,dy$$

$$= \lim_{\epsilon\to 0}\int_{\partial B(0,1)} f(x-\epsilon y)\,dy$$

$$= |\partial B(0,1)| f(x) = 4\pi f(x).$$

- We cannot apply the Δ_x operator directly to the $1/|x-y|$ term in the first line, since the second partial derivatives of $1/|x-y|$ are not integrable as is required by the DUTIS theorem. If you illegally did this, you would end up proving that $\Delta u = 0$, which is false!

- The second line of this derivation shows that if $f \in C^2$, with compact support, then $u \in C^2$. This can be improved to $f \in C^1 \implies u \in C^2$, but it is not true that $f \in C^0 \implies u \in C^2$ as you might expect. However, there are two analogous statements that *are* true: if $f \in L^2$, then u has all its second derivatives in L^2; and if $f \in C^\alpha$, then $u \in C^{2,\alpha}$. (This expresses that u has continuous 2nd partial derivatives, which are Hölder continuous with exponent α, i.e. satisfy $|f(x) - f(y)| \leq C|x - y|^\alpha$.) These statements express the 'ellipticity' of Δ.

Heat equation

The heat equation on $\mathbb{R}^n \times \mathbb{R}_+$ is the equation

$$u_t(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^n,\ t > 0$$

supplemented with the initial condition

$$u(x,0) = f(x).$$

Assume that f is a Schwartz function. Then we can find a solution that is a continuous function of t with values in Schwartz functions of x. Fourier transforming in the x variable but not the t variable (which would not make sense since the solution is only defined for $t \geq 0$) we get

$$\hat{u}_t = -|\xi|^2 \hat{u}, \quad \hat{u}(\xi,0) = \hat{f}(\xi).$$

This is an ODE in t for each fixed ξ, and the solution is

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}(\xi,0) = e^{-|\xi|^2 t}\hat{f}(\xi).$$

Hence, the function u is given by a convolution:

$$u(x,t) = \mathcal{G}(e^{-|\xi|^2 t}) * f.$$

So we need to know the inverse Fourier transform of $e^{-|\xi|^2 t}$. But we have already worked this out, and the answer is

$$\mathcal{G}(e^{-|\xi|^2 t}) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

To summarize, the solution of the PDE is

$$u(x,t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\, dy.$$

The function $(4\pi t)^{-n/2} e^{-|x-y|^2/4t}$ is called the 'heat kernel' on $\mathbb{R}^n$. It may be regarded as the solution to the heat equation with initial condition $f = \delta_y(x)$.