Math 3325, 2014

Problem set 1 — discuss in Tutorial on July 28.

1. (a) Show that the set
$$\{f \in L^2([0,4]) \mid \int_0^4 f(x)\,dx = 0\}$$
is a closed subspace of $L^2([0,4])$.

   (b) Find the closest point in this subspace to the characteristic function of the interval $[0,1]$, in the L^2 metric.
2. Show that the set
$$\{f \in L^2(\mathbb{R}) \mid \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 0\}$$
is not a closed subspace of $L^2(\mathbb{R})$.
3. Find the projection of a function f in $L^2([0,1])$ onto the subspace of continuous functions that are linear on $[0,1/2]$ and $[1/2,1]$.
4. Consider the function $f = \chi_{[0,1]}$ as an element of $L^1([0,4])$ (note here we look at the L^1 norm, not the L^2 norm). Show that there are lots of functions g that are "halfway between 0 and f", in the sense that $\|g\|_1 = \|f-g\|_1 = 1/2$. On the other hand, show that in a Hilbert space, given $f \neq 0$, there is only one element g satisfying $\|g\| = \|f-g\| = \|f\|/2$, namely $f/2$.
5. Show that the examples of BLTs in lecture 2 are bounded, namely the maps
$$f(x) \mapsto \int_{-\infty}^{\infty} k(x-y)f(y)\,dy : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$
for $k \in L^1(\mathbb{R})$, and integration on $L^2([0,1])$:
$$f(x) \mapsto \int_0^x f(s)\,ds.$$
6. Let H be a Hilbert space. Suppose that a sequence P_j of orthogonal projections converges to the BLT T in operator norm, i.e. $\|T - P_j\| \to 0$ as $j \to \infty$. Show that T is an orthogonal projection.
7. Suppose that F is a finite rank operator on H. (This means, by definition, that the range of F is finite dimensional.) Show that there exist $x_1, \dots, x_N, y_1, \dots, y_N \in H$, where N is the dimension of the range of F, such that
$$Fx = \sum_{j=1}^{N} (x_i, x)y_i.$$