# MATH1014

Semester 1

Administrative Overview

Lecturers:

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## Course content

Texts: Stewart *Essential Calculus*, §6 - §12
Lay *Linear Algebra and its Applications*, §4-§6

- integration
- sequences and series
- functions of several variables
- geometry and algebra of vectors
- vector spaces
- eigenvalues and eigenvectors

This course is a continuation of MATH1013, which is a prerequisite.

## Wattle

The Wattle site has important information about the course, including

- lecture notes
- lecture recordings
- tutorial worksheets
- past exams
- discussion board
- contact information for lecturers & course reps
- tutorial registration

Please check this site regularly for updates!

## Assessment

- Midsemester exam (date TBA) (25%)
- Final exam (50%)
- Web Assign quizzes (10%)
- Tutorial quizzes (10%)
- Tutorial participation (5%)

Tips for success:

- Ask questions!
- Make use of the available resources!
- Don't fall behind!

## WebAssign and Quizzes

Assessable quizzes for MATH1014 will be done through the WebAssign interface.

The WebAssign login is
`https://www.webassign.net/login.html`
Hopefully you received information about logging in to the site via email!

- There will an assessable on-line quiz before each tutorial and a number of practice quizzes.
- Your mark for each on-line quiz is contingent on keeping a workbook containing handwritten solutions. Your tutor may ask to look over your workbook to verify that it is your own work.
- These quizzes contribute 10% to your overall grade.
- A further 10% is based on in-tutorial quizzes related to the WebAssign ones. (There will be 10 of these quizzes, and your best 8 will counted.)
- And another 5% is based on your work in your tutorial.

## Other Resources

- The Library!
- The Internet
  - The Math Forum@Drexel
    `http://mathforum.org/library/topics/linear/`
  - Just the Maths
    `http://www.mis.coventry.ac.uk/jtm/contents.htm`
  - Lay student resources for Linear Algebra
    `http://wps.aw.com/aw_lay_linearalg_updated_3/0,10902,2414937-,00.html`

# Feedback

The following lists some the resources available for you to get some feedback.

- Laboratory Sessions.
  - 1.5 hour tutorial laboratory sessions each week.
  - Tutors are generally not available outside set lab times.
- Scheduled office hours (see Wattle for details).
- Discussion board available on Wattle.
- Organisation of self-help tutorial sessions by groups of students is also an excellent idea.
- ANU Counselling. `http://www.anu.edu.au/counsel/`
- Academic Skills and Learning Centre. `http://www.anu.edu.au/academicskills/`

# Calculus

- Numerical integration (§6.5).
- Improper integrals of the first and second kind, comparison test for improper integrals, p-integrals (§6.6).
- Application of integration: Areas, volumes, solids of revolution, volume by slicing, arc length (§7.1, 7.2, 7.4).
- Differential equations: Solution of separable equations, initial value problems, direction fields (§7.6).
- Sequences and series: Limits of sequences, convergence of series, geometric series, telescoping series, p-series, convergence tests, alternating series and other non-positive series, power series, Taylor Series (§8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7).
- Parametric curves, polar coordinates, Area and length in polar coordinates (§9.1, 9.2, 9.3, 9.4)
- Functions of several variables: Contour plots, Partial derivatives, the Chain rule, Directional derivatives, the gradient vector (§11.1, 11.3, 11.5, 11.6).
- Extreme values: Critical points, second derivatives test

# Linear Algebra

- We will be covering most of the material in Stewart, Sections 10.1, 10.2, 10.3 and 10.4, and Lay Chapters 4 and 5, and Chapter 6, Sections 1 - 6.
- Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$, dot products, cross products in $\mathbf{R}^3$, planes and lines in $\mathbf{R}^3$ (Stewart).
- Properties of Vector Spaces and Subspaces.
- Linear Independence, bases and dimension, change of basis.
- Applications to difference equations, Markov chains.
- Eigenvalues and eigenvectors.
- Orthogonality, Gram-Schmidt process. Least squares problem.

## Exams

Please also note:

- While material from MATH1013 will not be directly tested, you will need to know how to apply those techniques.
- Material examinable at the midsemester exam is examinable at the final, though it will not predominate.
- You can use the previous years' exams for revision: note that the syllabus changed slightly in Semester 2 2009. Also look at exercises from Stewart or Lay and the practice quizzes online; the weekly quizzes; and the worksheet questions.

## Coordinates, Vectors and Geometry in $\mathbb{R}^3$

From Stewart, §10.1, §10.2

Question: How do we describe 3-dimensional space?

1. Coordinates
2. Lines, planes, and spheres in $\mathbb{R}^3$
3. Vectors

## Euclidean Space and Coordinate Systems

We identify points in the plane ($\mathbb{R}^2$) and in three-dimensional space ($\mathbb{R}^3$) using coordinates.

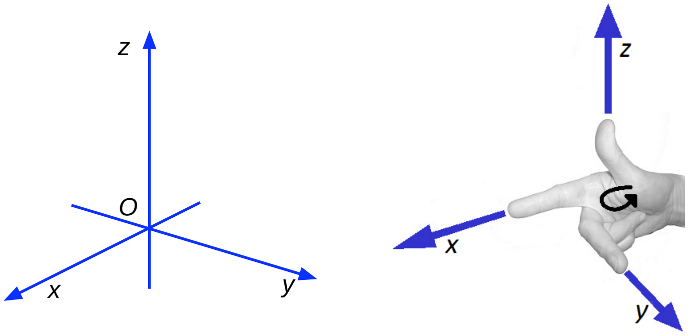
$$\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$$

reads as "$\mathbb{R}^3$ is the set of ordered triples of real numbers".

We first choose a fixed point $\mathbf{O} = (0, 0, 0)$, called the *origin*, and three directed lines through $O$ that are perpendicular to each other. We call these the *coordinate axes* and label them the $x$-axis, the $y$-axis and the $z$-axis.

Usually we think of the $x$- and $y$-axes as being horizontal and the $z$-axis as being vertical.

Together, $\{x, y, z\}$ form a *right-handed coordinate system*.



Compare this to the axes we use to describe $\mathbb{R}^2$, where the $x$-axis is horizontal and the $y$-axis is vertical.

## The Distance Formula

**Definition**

The *distance* $\mid P_1P_2 \mid$ between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$\mid P_1P_2 \mid = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Definition**

The *distance* $\mid P_1P_2 \mid$ between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is

$$\mid P_1P_2 \mid = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

## 1.1 Surfaces in $\mathbb{R}^3$

Lines, planes, and spheres are special sets of points in $\mathbb{R}^3$ which can be described using coordinates.

**Example 1**

The sphere of radius $r$ with centre $C = (c_1, c_2, c_3)$ is the set of all points in $\mathbb{R}^3$ with distance $r$ from $C$:

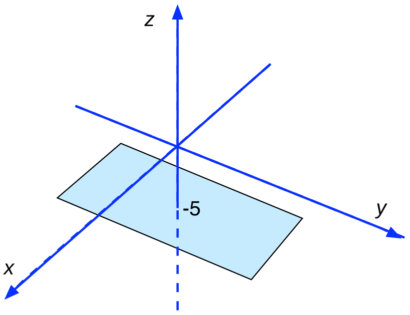
$$S = \{P \ : \ |PC| = r\}.$$

Equivalently, the sphere consists of all the solutions to this equation:

$$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = r^2.$$

**Example 2**

The equation $z = -5$ in $\mathbb{R}^3$ represents the set $\{(x, y, z) \mid z = -5\}$, which is the set of all points whose $z$-coordinate is $-5$. This is a horizontal plane that is parallel to the $xy$-plane and five units below it.



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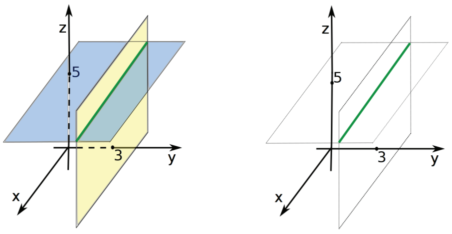
**Example 3**

What does the pair of equations $y = 3, z = 5$ represent? In other words, describe the set of points

$$\{(x, y, z) \ : y = 3 \text{ and } z = 5\} = \{(x, 3, 5)\}.$$



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## Connections with linear equations

Recall from 1013 that a system of linear equations defines a *solution set*. When we think about the unknowns as coordinate variables, we can ask what the solution set looks like.

- A single linear equation with 3 unknowns will **usually** have a solution set that's a plane. (e.g., Example 2 or $3x + 2y - 5z = 1$)
- Two linear equations with 3 unknowns will **usually** have a solution set that's a line. (e.g., Example 3 or $3x + 2y - 5z = 1$ and $x + z = 2$)
- Three linear equations with 3 unknowns will **usually** have a solution set that's a point (i.e., a unique solution).

**Question**

*When do these heuristic guidelines fail?*

## Vectors

We'll study vectors both as formal mathematical objects and as tools for modelling the physical world.

**Definition**

A *vector* is an object that has both magnitude and direction.

Physical quantities such as velocity, force, momentum, torque, electromagnetic field strength are all "vector quantities" in that to specify them requires both a magnitude and a direction.

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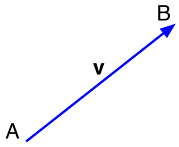
## Vectors

**Definition**

A *vector* is an object that has both magnitude and direction.



We represent vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by arrows. For example, the vector $\mathbf{v}$ has initial point $A$ and terminal point $B$ and we write $\mathbf{v} = \vec{AB}$.
The *zero vector* $\mathbf{0}$ has length zero (and no direction).

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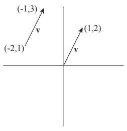
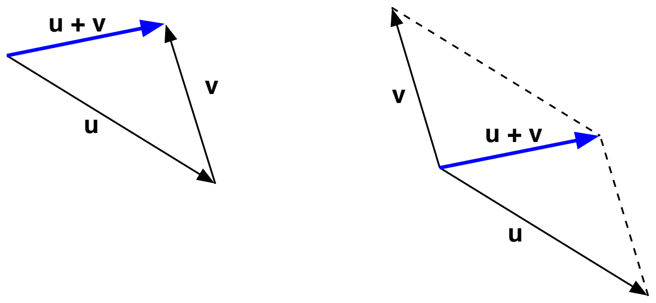
Since a vector doesn't have "location" as one of its properties, we can slide the arrow around as long as we don't rotate or stretch it.



We can describe a vector using the coordinates of its head when its tail is at the origin, and we call these the *components* of the vector. Thus in this

## Vector Addition

If an arrow representing **v** is placed with its tail at the head of an arrow representing **u**, then an arrow from the tail of **u** to the head of **v** represents the sum $\mathbf{u} + \mathbf{v}$.



Suppose that **u** has components $a$ and $b$ and that **v** has components $x$ and $y$. Then $\mathbf{u} + \mathbf{v}$ has components $a + x$ and $b + y$:

$$\mathbf{u} + \mathbf{v} = \langle a, b\rangle + \langle x, y\rangle = \langle a + x, b + y\rangle$$

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## Scalar Multiplication

If **v** is a vector, and $t$ is a real number (*scalar*), then the *scalar multiple* of **v** is a vector with magnitude $|t|$ times that of **v**, and direction the same as **v** if $t > 0$, or opposite to that of **v** if $t < 0$.

If $t = 0$, then $t\mathbf{v}$ is the zero vector $\mathbf{0}$.
If **u** has components $a$ and $b$, then $t\mathbf{v}$ has components $tx$ and $ty$:

$$t\mathbf{v} = t\langle x, y\rangle = \langle tx, ty\rangle.$$

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## Example

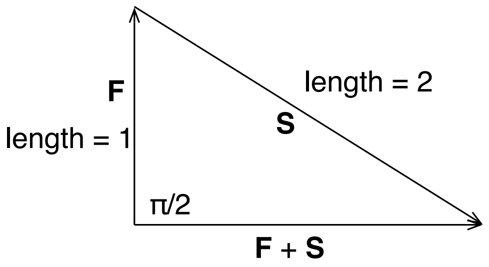
### Example 4

A river flows north at 1km/hr, and a swimmer moves at 2km/hr relative to the water.

- At what angle to the bank must the swimmer move to swim east across the river?
- What is the speed of the swimmer relative to the land?

There are several velocities to be considered:
The velocity of the river, **F**, with $\|\mathbf{F}\| = \mathbf{1}$;
The velocity of the swimmer relative to the water, **S**, so that $\|\mathbf{S}\| = \mathbf{2}$;
The resultant velocity of the swimmer, $\mathbf{F} + \mathbf{S}$, which is to be perpendicular to **F**.

The problem is to determine the *direction* of $\mathbf{S}$ and the *magnitude* of $\mathbf{F} + \mathbf{S}$.



From the figure it follows that the angle between $\mathbf{S}$ and $\mathbf{F}$ must be $2\pi/3$ and the resulting speed will be $\sqrt{3}$ km/hour. □

## Standard basis vectors in $\mathbb{R}^2$

The vector $\mathbf{i}$ has components 1 and 0, and the vector $\mathbf{j}$ has components 0 and 1.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The vector $\mathbf{r}$ from the origin to the point $(x, y)$ has components $x$ and $y$ and can be expressed in the form

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{i} + y\mathbf{j}.$$

The length of of a vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is given by

$$||\mathbf{v}|| = \sqrt{x^2 + y^2}$$

## Standard basis vectors in $\mathbb{R}^3$

In the Cartesian coordinate system in 3-space we define three **standard basis vectors** $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ represented by arrows from the origin to the points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Any vector can be written as a sum of scalar multiples of the standard basis vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}.$$

If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the *length* of $\mathbf{v}$ is defined as

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2} \, .$$

This is just the distance from the origin (with coordinates $0, 0, 0$) of the point with coordinates $a, b, c$.

A vector with length 1 is called a *unit vector*.

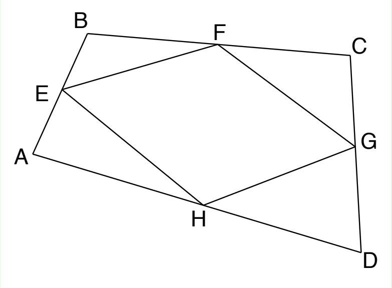
If $\mathbf{v}$ is not zero, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in the same direction as $\mathbf{v}$.

The zero vector is not given a direction.

## Vectors and Shapes

### Example 5

The midpoints of the four sides of any quadrilateral are the vertices of a parallelogram.

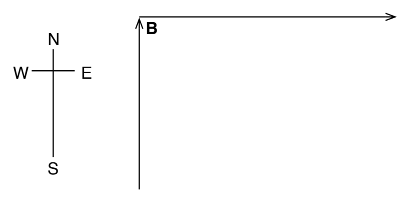


Can you prove this using vectors?

Hint: how can you tell if two vectors are parallel? How can you tell if they have the same length?

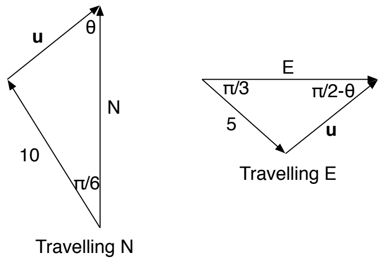
### Example 6

A boat travels due north to a marker, then due east, as shown:



Travelling at a speed of 10 knots with respect to the water, the boat must head 30° west of north on the first leg because of the water current. After rounding the marker and reducing speed to 5 knots with respect to the water, the boat must be steered 60° south of east to allow for the current. Determine the velocity $\mathbf{u}$ of the water current (assumed constant).

A diagram is helpful. The vector **u** represents the velocity of the river current, and has the same magnitude and direction in both diagrams.



Applying the sine rule, we have

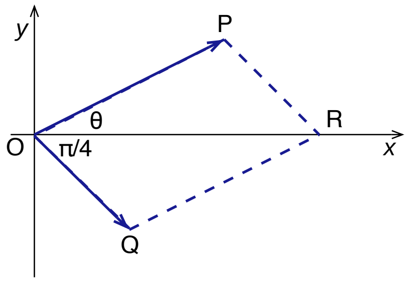
$$\frac{\sin\theta}{10} = \frac{\sin\frac{\pi}{6}}{\|\mathbf{u}\|} \qquad \frac{\cos\theta}{5} = \frac{\sin\frac{\pi}{3}}{\|\mathbf{u}\|}.$$

which are easily solvable for $\|\mathbf{u}\|$ and $\theta$, and hence give **u**. ☐

## Example 7

An aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northwest at 100 km/h?

*Solution* The problem is 2-dimensional, so we can use plane vectors. Choose a coordinate system so that the $x$- and $y$-axes point east and north respectively.



$$\begin{aligned}
\overrightarrow{OQ} &= \mathbf{v}_{air\ \ rel\ \ ground}\\
&= 100\cos(-\pi/4)\mathbf{i} + 100\sin(-\pi/4)\mathbf{j}\\
&= 50\sqrt{2}\mathbf{i} - 50\sqrt{2}\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{OP} &= \mathbf{v}_{aircraft\ \ rel\ \ air}\\
&= 750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{OR} &= \mathbf{v}_{aircraft\ \ rel\ \ ground}\\
&= \overrightarrow{OP} + \overrightarrow{OQ}\\
&= (750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}) + (50\sqrt{2}\mathbf{i} - 50\sqrt{2}\mathbf{j})\\
&= (750\cos\theta + 50\sqrt{2})\mathbf{i} + (750\sin\theta - 50\sqrt{2})\mathbf{j}
\end{aligned}$$

We want $\mathbf{v}_{aircraft\ rel\ ground}$ to be in an easterly direction, that is, in the positive direction of the $x$-axis. So for ground speed of the aircraft $v$, we have

$$\overrightarrow{OR} = v\mathbf{i}.$$

Comparing the two expressions for $\overrightarrow{OR}$ we get

$$v\mathbf{i} = (750\cos\theta + 50\sqrt{2})\mathbf{i} + (750\sin\theta - 50\sqrt{2})\mathbf{j}.$$

This implies that

$$750\sin\theta - 50\sqrt{2} = 0 \quad \leftrightarrow \quad \sin\theta = \frac{\sqrt{2}}{15}.$$

This gives $\theta \approx 0.1$ radians $\approx 5.4^\circ$.
Using this information $v$ can be calculated, as well as the time to travel a given distance.