Overview

Last time, we used coordinate axes to describe points in space and we introduced vectors. We saw that vectors can be added to each other or multiplied by scalars.

Question: Can two vectors be multiplied?

- dot product
- cross product

(From Stewart, §10.3, §10.4)

The dot product

The *dot* or *scalar product* of two vectors is a scalar:

Definition

Given $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} = \mathbf{a}^T\mathbf{b} &= \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= a_1b_1 + a_2b_2 + \cdots + a_nb_n \end{aligned}$$

Example 1

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$, then

$$\mathbf{u}\cdot\mathbf{v} = (1)(-4) + (4)(5) + (-2)(-1) = 18.$$

The following properties come directly from the definition:

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
3. $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$, $k \in \mathbb{R}$

Magnitude and the dot product

Recall that if $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the *length* (or *magnitude*) of $\mathbf{v}$ is defined as

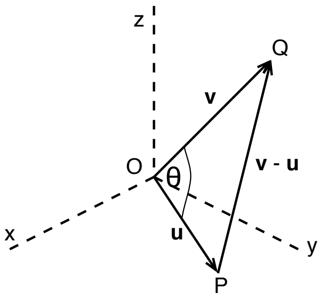
$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2} \, .$$

The dot product is a convenient way to compute length:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Direction and the dot product

The dot product $\mathbf{u} \cdot \mathbf{v}$ is useful for determining the relative directions of $\mathbf{u}$ and $\mathbf{v}$.
Suppose $\mathbf{u} = \overrightarrow{OP}, \mathbf{v} = \overrightarrow{OQ}$. The *angle* θ between $\mathbf{u}$ and $\mathbf{v}$ is the angle at O in the triangle POQ.



Necessarily $\theta \in [0, \pi]$.

Calculating:

$$\begin{aligned} \|\overrightarrow{PQ}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\ &= \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} \, . \end{aligned}$$

But the cosine rule, applied to triangle POQ, gives

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

whence

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta \tag{1}$$

If either $\mathbf{u}$ or $\mathbf{v}$ are zero then the angle betwen them is not defined. In this case, however, (1) still holds in the sense that both sides are zero.

Theorem

If θ is the angle between the directions of $\mathbf{u}$ *and* $\mathbf{v}$ $(0 \leq \theta \leq \pi)$*, then*

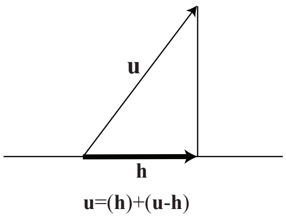
$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

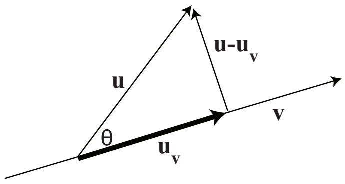
Definition

Two vectors are called *orthogonal* or *perpendicular* or *normal* if $\mathbf{u}\cdot\mathbf{v} = 0$, that is, $\theta = \pi/2$.

Scalar and vector projections

Just as we can write a vector in $\mathbb{R}^2$ as a sum of its horizontal and vertical components, we can write any vector as a sum of piece parallel to and perpendicular to a fixed vector.





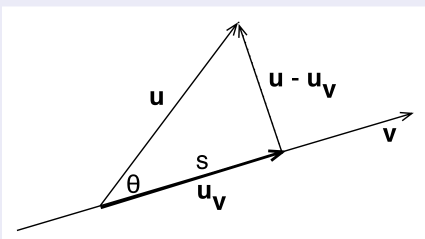
Scalar and vector projections

Definition

The *scalar projection* $s = \mathrm{comp}_{\mathbf{v}}\mathbf{u}$ of any vector $\mathbf{u}$ in the direction of the nonzero vector $\mathbf{v}$ is the scalar product of $\mathbf{u}$ with a unit vector in the direction of $\mathbf{v}$.

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \mathbf{u}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\|\cos\theta$$

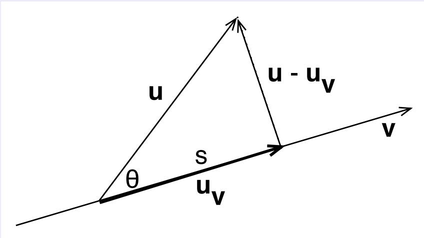
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.



Definition

The *vector projection* $\mathbf{u_v} = \mathrm{proj}_\mathbf{v}\mathbf{u}$ of $\mathbf{u}$ in the direction of the nonzero vector $\mathbf{v}$ is the scalar multiple of a unit vector $\hat{\mathbf{v}}$ in the direction of $\mathbf{v}$, by the scalar projection of $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\mathrm{proj}_\mathbf{v}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\hat{\mathbf{v}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$



In words:

- The scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ is...
- The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is...

Remember that we can write $\mathbf{u}$ as a sum of a vector parallel to $\mathbf{v}$ and a vector perpendicular to $\mathbf{v}$. We call the summand parallel to $\mathbf{v}$ the *component* in the $\mathbf{v}$ direction.

- The scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ is the length of the component of $\mathbf{u}$ in the $\mathbf{v}$ direction.
- The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is the component of $\mathbf{u}$ in the $\mathbf{v}$ direction.

Definition of the cross product

In $\mathbb{R}^3$ *only*, there is a product of two vectors called a *cross product* or *vector product*. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector denoted $\mathbf{a}\times\mathbf{b}$.

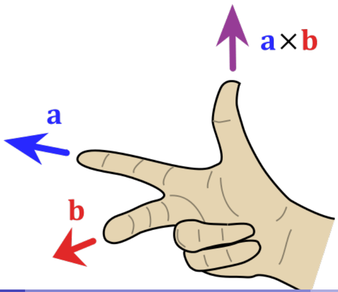
To specify a vector in $\mathbb{R}^3$, we need to give its magnitude and direction.

Definition of the cross product

Definition

Given **a** and **b** in $\mathbb{R}^3$ with $\theta \in [0, \pi]$ the angle between them, the cross product $\mathbf{a} \times \mathbf{b}$ is the vector defined by the following properties:

- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
- $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**
- $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form a right-handed coordinate system



Computing cross products

Given $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, how can we find the coordinates of $\mathbf{a} \times \mathbf{b}$?

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a}\times\mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

You should check that this formula gives a vector satisfying the definition on the previous slide! Alternatively, we could give this formula as the definition and then prove those properties as a theorem.

In order to make the definition easier to remember we use the notation of determinants. Recall that a **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Further a **determinant of order 3** can be defined in terms of second order determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

We now rewrite the cross product using determinants of order 3 and the standard basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

In view of the similarity of the last two equations we often write

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{2}$$

Although the first row of the symbolic determinant in Equation 2 consists of vectors, it can be expanded as if it were an ordinary determinant.

Example 2

Find a vector with positive $\mathbf{k}$ component which is perpendicular to both $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Solution The vector $\mathbf{a}\times\mathbf{b}$ will be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} \\ &= -7\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

Now we require a vector with a positive $\mathbf{k}$. It is given by $\langle 7, 6, 4\rangle$.

Properties of the cross product

Lemma

Two non zero vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel (or antiparallel) if and only if*

$$\mathbf{a}\times\mathbf{b} = \mathbf{0}.$$

Properties of the cross product

If **u** **v** and **w** are any vectors in $\mathbb{R}^3$, and t is a real number, then

1. $\mathbf{u}\times\mathbf{v} = -\ldots.$
2. $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \ldots.$
3. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \ldots.$
4. $(t\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(t\mathbf{v}) = \ldots.$
5. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \ldots.$
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \ldots$

Properties of the cross product

If **u** **v** and **w** are any vectors in $\mathbb{R}^3$, and t is a real number, then...

1. $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}.$
2. $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}.$
3. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}.$
4. $(t\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(t\mathbf{v}) = t(\mathbf{u}\times\mathbf{v}).$
5. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}.$
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

Note the absence of an associative law. The cross product is not associative. In general

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}!$$

Comparing the dot and cross product

- Where is each defined?
- What is the output?
- What's the significance of zero?
- Is it commutative?

Example 3

A triangle ABC has vertices $(2,-1,0)$, $(5,-4,3)$, $(1,-3,2)$. Is it a right triangle?

The sides are $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$, $\overrightarrow{BC} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$.

Since

$$\cos\theta_C = \frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|} = \frac{(-1)(-4)+(-2)(1)+(2)(-1)}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|} = \frac{0}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|} = 0,$$

the sides $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are orthogonal.

Example 4

For what value of k do the four points $A = (1,1,-1)$, $B = (0,3,-2)$, $C = (-2,1,0)$ and $D = (k,0,2)$ all lie in a plane?

Solution The points A, B and C form a triangle and all lie in the plane containing this triangle. We need to find the value of k so that D is in the same plane.
One way of doing this is to find a vector $\mathbf{u}$ perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and then find k so that $\overrightarrow{AD}$ is perpendicular to $\mathbf{u}$.
A suitable vector $\mathbf{u}$ is given by $\overrightarrow{AB}\times\overrightarrow{AC}$. We then require that

$$\mathbf{u}\cdot\overrightarrow{AD} = 0.$$

Putting this together we require that

$$(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD} = \mathbf{0}.$$

Example (continued)

For what value of k do the four points $A = (1,1,-1)$, $B = (0,3,-2)$, $C = (-2,1,0)$ and $D = (k,0,2)$ all lie in a plane?

Now

$$\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \overrightarrow{AC} = -3\mathbf{i} + \mathbf{k}, \quad \overrightarrow{AD} = (k-1)\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

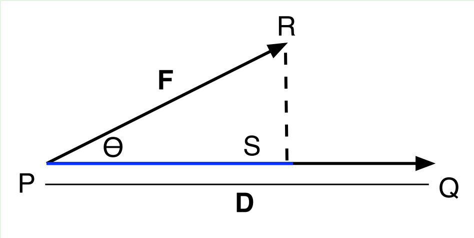
Then

$$\begin{aligned}
(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD} &= \overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) \\
&= \begin{vmatrix} k-1 & -1 & 3 \\ -1 & 2 & -1 \\ -3 & 0 & 1 \end{vmatrix} \\
&= (k-1)2 - (-1)(-4) + 3(6) \\
&= 2k - 2 - 4 + 18 \\
&= 2k + 12
\end{aligned}$$

So $(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD} = 0$ when $k = -6$, and D lies on the required plane

Example 5

One use of projections occurs in physics in calculating work.



Suppose a constant force $\mathbf{F} = \vec{PR}$ moves an object from P to Q. The **displacement vector** is $\mathbf{D} = \vec{PQ}$. The **work** done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:

$$W = (\|\mathbf{F}\| \cos\theta)\,\|\mathbf{D}\| = \mathbf{F}\cdot\mathbf{D}.$$

Example 6

Let $\mathbf{a} = \langle 1, 3, 0\rangle$ and $\mathbf{b} = \langle -2, 0, 6\rangle$, Then

$$\begin{aligned}
\mathrm{comp}_{\mathbf{a}}\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\\
&= \frac{-2+0+0}{\sqrt{1+9+0}} = \frac{-2}{\sqrt{10}}.\\
\mathrm{proj}_{\mathbf{a}}\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\hat{\mathbf{a}}\\
&= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|}\\
&= \frac{-2}{\sqrt{10}}\frac{\langle 1,3,0\rangle}{\sqrt{10}}\\
&= \frac{\langle -2,-6,0\rangle}{10} = \langle -1/5, -3/5, 0\rangle.
\end{aligned}$$