Overview

Last week we introduced vectors in Euclidean space and the operations of vector addition, scalar multiplication, dot product, and (for $\mathbb{R}^3$) cross product.

Question

How can we use vectors to describe lines and planes in $\mathbb{R}^3$?

(From Stewart §10.5)

Warm-up

Question

Describe all the vectors in $\mathbb{R}^3$ which are orthogonal to the 0 vector. Can you rephrase your answer as a statement about solutions to some linear equation?

Remember that the statement "$\mathbf{v}$ is orthogonal to $\mathbf{u}$" is equivalent to "$\mathbf{v} \cdot \mathbf{u} = 0$".

This question asks for all the vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$

Using the definition of the dot product, this translates to asking what $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfy the equation $0x + 0y + 0z = 0$...

...the answer is that all vectors in $\mathbb{R}^3$ are orthogonal to the 0 vector. Equivalently, every triple (x, y, z) is a solution to the linear equation $0x + 0y + 0z = 0$.

Lines in $\mathbb{R}^2$

In the xy-plane the general form of the equation of a line is

$$ax + by = c,$$

where a and b are not both zero. If $b \neq 0$ then this equation can be rewritten as

$$y = -(a/b)x + c/b,$$

which has the form $y = mx + k$. (Here m is the slope of the line and the point $(0, k)$ is its y-intercept.)

Example 1

Let L be the line $2x + y = 3$. The line has slope $m = -2$ and the y-intercept is $(0, 3)$.

Alternatively, we could think about this line ($y = -2x + 3$) as the path traced out by a moving particle.

Suppose that the particle is initially at the point $(0, 3)$ at time $t = 0$. Suppose, too, that its x-coordinate changes at a constant rate of 1 unit per second and its y-coordinate changes as a constant rate of -2 units per second.

At $t = 1$ the particle is at $(1, 1)$. If we assume it's always been moving this way, then we also know that at $t = -2$ it was at $(-2, 7)$. In general, we can display the relationship in vector form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t + 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

What is the significance of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$?

In this expression, $\mathbf{v}$ is a vector parallel to the line L, and is called a *direction vector* for L. The previous example shows that we can express L in terms of a direction vector and a vector to specific point on L:

Definition

The equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

is the *vector equation* of the line L. The variable t is called a *parameter*.

Here, $\mathbf{r}_0$ is the vector to a specific point on L; any vector $\mathbf{r}$ which satisfies this equation is a vector to some point on L.

Example 2

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{1}$$

is the *vector equation* of the line L.

If we express the vectors in a vector equation for L in components, we get a collection of equations relating scalars.

Definition

For $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the *parametric equations* of the line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ are

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb. \end{aligned}$$

Lines in $\mathbb{R}^3$

The definitions of the vector and parametric forms of a line carry over perfectly to $\mathbb{R}^3$.

Definition

The *vector form of the equation of the line L* in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ is a specific point on L and $\mathbf{v} \neq \mathbf{0}$ is a direction vector for L. The equations corresponding to the components of the vector form of the equation are called *parametric equations* of L.

Example 3

Let $\mathbf{r}_0 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Then the vector equation of the line L is

$$\mathbf{r} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The line L contains the point $(1, 4, -2)$ and has direction parallel to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. By taking different values of t we can find different points on the line.

Question

For a given line, is the vector equation for the line unique?

No, any vector parallel to the direction vector is another direction vector, and each choice of a point on L will give a different $\mathbf{r}_0$.

Example 4

The line with parametric equations

$$x = 1 + 2t \quad y = -4t \quad z = -3 + 5t.$$

can also be expressed as

$$x = 3 + 2t \quad y = -4 - 4t \quad z = 2 + 5t.$$

or as

$$x = 1 - 4t \quad y = 8t \quad z = -3 - 10t.$$

Note that a fixed value of t corresponds to three different points on L when plugged into the three different systems.

Symmetric equations of a line

Another way of describing a line L is to eliminate the parameter t from the parametric equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

If $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can solve each of the scalar equations for t and obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

These equations are called the *symmetric equations* of the line L through (x_0, y_0, z_0) parallel to $\mathbf{v}$. The numbers a, b and c are called the *direction numbers* of L.

If, for example $a = 0$, the equation becomes

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 5

Find parametric and symmetric equations for the line through $(1, 2, 3)$ and parallel to $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

The line has the vector parametric form

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}),$$

or scalar parametric equations

$$\begin{cases} x = 1 + 2t \\ y = 2 + 3t \\ z = 3 - 4t \end{cases} \qquad (-\infty < t < \infty).$$

Its symmetric equations are

$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{-4}.$$

Example 6

Determine whether the two lines given by the parametric equations below intersect

$$L_1 : x = 1 + 2t, y = 3t, z = 2 - t$$

$$L_2 : x = -1 + s, y = 4 + s, z = 1 + 3s$$

If L_1 and L_2 intersect, there will be values of s and t satisfying

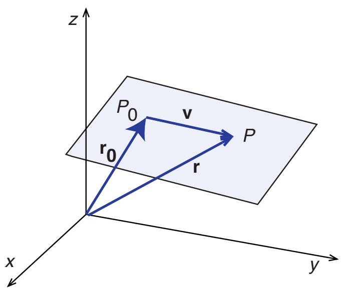
$$\begin{aligned} 1 + 2t &= -1 + s \\ 3t &= 4 + s \\ 2 - t &= 1 + 3s \end{aligned}$$

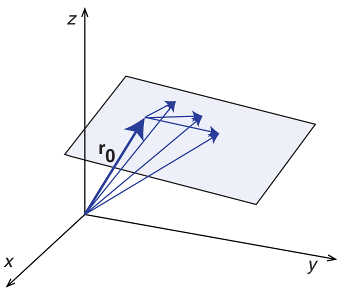
Solving the first two equations gives $s = 14, t = 6$, but these values don't satisfy the third equation. We conclude that the lines L_1 and L_2 don't intersect.
In fact, their direction vectors are not proportional, so the lines aren't parallel, either. They are *skew* lines.

Planes in $\mathbb{R}^3$

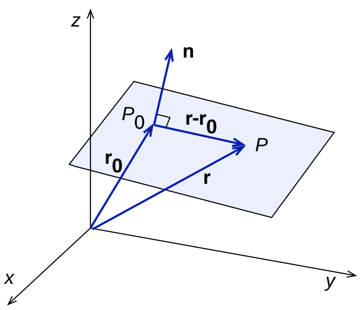
We described a line as the set of position vectors expressible as $\mathbf{r}_0 + \mathbf{v}$, where $\mathbf{r}_0$ was a position vector of a point in L and $\mathbf{v}$ was any vector parallel to L.

We can describe a plane the same way: the set of position vectors expressible as the sum of a position vector to a point in P and an arbitrary vector parallel to P.





Choose a vector $\mathbf{n}$ which is orthogonal to the plane and choose an arbitrary point P_0 in the plane.



How can we use this data to describe all the other points P which lie in the plane?
Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P respectively.
The normal vector $\mathbf{n}$ is orthogonal to every vector in the plane. In particular $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

This equation

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0. \tag{2}$$

can be rewritten as

$$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r}_0. \tag{3}$$

Either of the equations (2) or (3) is called a *vector equation of the plane*.

Example 7

Find a vector equation for the plane passing through $P_0 = (0, -2, 3)$ and normal to the vector $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

We have $\mathbf{r}_0 = \langle 0, -2, 3\rangle$ and $\mathbf{n} = \langle 4, 2, -3\rangle$. Thus the vector form is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

or

$$(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})\cdot [(x - 0)\mathbf{i} + (y + 2)\mathbf{j} + (z - 3)\mathbf{k}] = 0.$$

Expanding this gives us a *scalar equation* for the plane...

Given $\mathbf{n} = \langle A, B, C\rangle, \mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, the vector equation $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0$ becomes

$$\langle A, B, C\rangle\cdot\langle x - x_0, y - y_0, z - z_0\rangle = 0,$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{4}$$

Equation (4) is the *scalar equation of the plane* through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle A, B, C\rangle$.

The equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

can be written more simply in **standard form**

$$Ax + By + Cz + D = 0,$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

If $D = 0$, the plane passes through the origin.

Example 8

Find a scalar equation for the plane passing through $P_0 = (0, -2, 3)$ and normal to the vector $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

The vector form is

$$(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot [(x - 0)\mathbf{i} + (y + 2)\mathbf{j} + (z - 3)\mathbf{k}] = 0,$$

which in scalar form becomes

$$4(x - 0) + 2(y + 2) - 3(z - 3) = 0$$

and this is equivalent to

$$4x + 2y - 3z = -13.$$

Example 9

Find a scalar equation of the plane containing the points

$$P = (1, 1, 2), \quad Q = (0, 2, 3), \quad R = (-1, -1, -4).$$

First, we should find a normal vector $\mathbf{n}$ to the plane, and there are several ways to do this.
The vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ will be perpendicular to $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{PR} = -2\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$. Therefore, we can solve a system of linear equations:

$$0 = \mathbf{n} \cdot (-\mathbf{i} + \mathbf{j} + \mathbf{k}) = -n_1 + n_2 + n_3$$

$$0 = \mathbf{n} \cdot (-2\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}) = -2n_1 - 2n_2 - 6n_3.$$

One solution to this system is $\mathbf{n} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, so this is an example of a normal vector to the plane containing the 3 given points.

We can use this normal vector $\mathbf{n} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, together with any one of the given points to write the equation of the plane. Using $Q = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, the equation is

$$-(x - 0) - 2(y - 2) + 1(z - 3) = 0,$$

which simplifies to

$$x + 2y - z = 1.$$

The first step in this example was finding the normal vector $\mathbf{n}$, but in fact, there's another way to do this.

Recall that *in* $\mathbb{R}^3$ *only*, there is a product of two vectors called a *cross product*. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector denoted $\mathbf{a}\times\mathbf{b}$ which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. If we have two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ parallel to our plane, then $\mathbf{n} = \mathbf{a}\times\mathbf{b}$ is a normal vector.

Example 10

Consider the two planes

$$x - y + z = -1 \quad \text{and} \quad 2x + y + 3z = 4.$$

Explain why the planes above are not parallel and find a direction vector for the line of intersection.

Two planes are parallel if and only if their normal vectors are parallel. Normal vectors for the two planes above are for example

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

respectively. These vectors are not parallel, so the planes can't be parallel and must intersect. A vector $\mathbf{v}$ parallel to the line of intersection is a vector which is orthogonal to both the normal vectors above. We can find such a vector by calculating the cross product of the normal vectors:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -4\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

Example 11

Find the line through the origin and parallel to the line of intersection of the two planes

$$x + 2y - z = 2 \quad \text{and} \quad 2x - y + 4z = 5.$$

The planes have respective normals

$$\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}.$$

A direction vector for their line of intersection is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

A vector parametric equation of the line is

$$\mathbf{r} = t(7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}),$$

since the line passes through the origin.

Parametric equations for this line are, for example,

$$\begin{aligned} x &= 7t \\ y &= -6t \\ z &= -5t \end{aligned}$$

and the corresponding symmetric equations are

$$\frac{x}{7} = \frac{y}{-6} = \frac{z}{-5}.$$

Recommended exercises for review

Stewart §10.5: 1, 3, 15, 19, 25, 29, 35