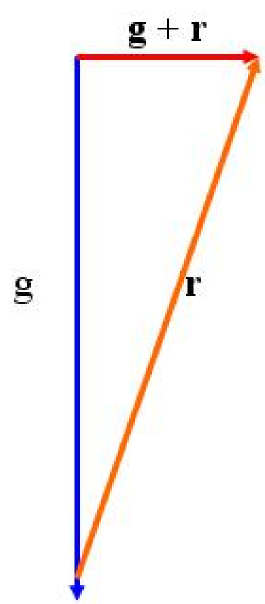
4 Planes and lines – Examples and Exercises

4.1 Examples

1. **Finding the Net Force Acting on a Skydiver**

   At a certain point during a jump, there are two principal forces acting on a skydiver: gravity exerting a force of 180 pounds straight down, and air resistance exerting a force of 180 pounds up and 30 pounds to the right. What is the net force acting on the skydiver?

   Solution

   Forces on a Skydiver

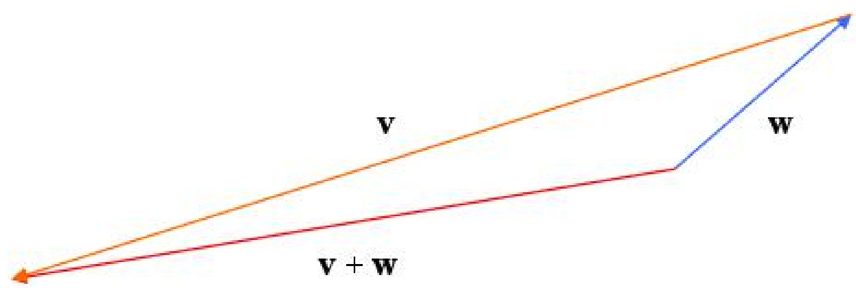
   We write the gravity force as $\mathbf{g} = (0, -180)$ and the air resistance force vector as $\mathbf{r} = (30, 180)$. The net force on the skydiver is the sum of the two forces, $\mathbf{g}+\mathbf{r} = (30, 0)$.

   Notice that at this point, the vertical forces are balanced, producing a "free-fall" vertically, so that the skydiver is neither accelerating nor decelerating vertically. The net force is purely horizontal, combating the horizontal motion of the skydiver after jumping from the plane.

2. **Steering an Aircraft in a Headwind and a Crosswind**

An airplane has an airspeed of 400 mph. Suppose that the wind velocity is given by the vector $\mathbf{w} = (20, 30)$. In what direction should the airplane head in order to fly due west (that is, in the direction of the unit vector $-\mathbf{i} = (-1, 0)$)?

Solution



Forces on an airplane

The velocity vectors for the airplane and the wind are illustrated in the diagram above. Let the airplane's velocity vector be $\mathbf{v} = (x, y)$. The effective velocity of the plane is then $\mathbf{v} + \mathbf{w}$, which we set equal to $(c, 0)$ for some negative constant c. Since

$$\mathbf{v} + \mathbf{w} = (x + 20, y + 30) = (c, 0),$$

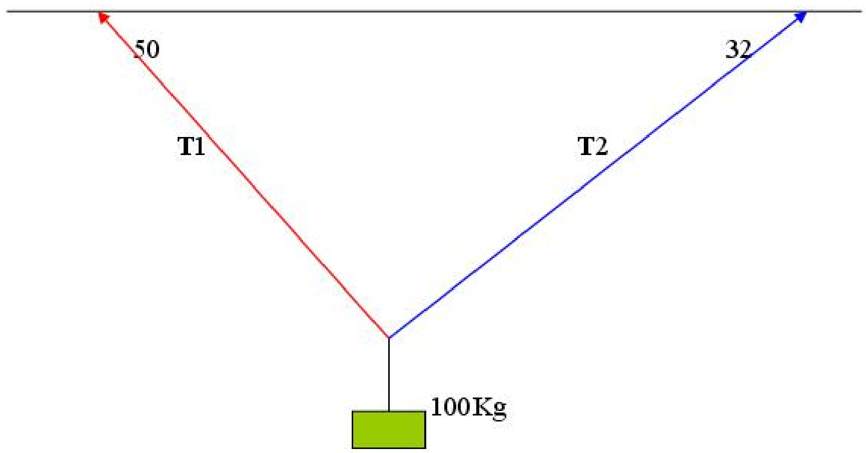
we must have $x + 20 = c$ and $y + 30 = 0$, so that $y = -30$.

Further since the plane's airspeed is 400 mph, we must have

$$|\mathbf{v}| = \sqrt{x^2 + y^2} = \sqrt{x^2 + 900} = 400.$$

This gives $x^2 + 900 = 160,000$ so that $x = -\sqrt{159,100}$. (We take the negative square root so that the plane flies westward). Consequently, the plane should head in the direction of $\mathbf{v} = (-\sqrt{159.100}, -30)$, which points left and down, or southwest, at an angle of $\tan^{-}1(30/\sqrt{159,100}) \approx 4°$ below due west.

3. A load with mass of 100 kg hangs from two wires as shown in the figure below



Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires, and their magnitudes.

Solution We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components.

$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos 50°\mathbf{i} + |\mathbf{T}_1|\sin 50°\mathbf{j} \tag{1}$$
$$\mathbf{T}_2 = |\mathbf{T}_2|\cos 32°\mathbf{i} + |\mathbf{T}_2|\sin 32°\mathbf{j} \tag{2}$$

The force of gravity acting on the load is

$$\mathbf{F} = -100(9.8)\mathbf{j} = -980\mathbf{j}.$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{F} = 980\mathbf{j}.$$

Thus

$$\begin{aligned}&(-|\mathbf{T}_1|\cos 50°\mathbf{i} + |\mathbf{T}_1|\sin 50°\mathbf{j}) + \\ &\quad (|\mathbf{T}_2|\cos 32°\mathbf{i} + |\mathbf{T}_2|\sin 32°\mathbf{j}) = 980\mathbf{j}.\end{aligned}$$

Equating components:

$$\begin{aligned}-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32° &= 0\\ |\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32° &= 980\end{aligned}$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50° + \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°}\sin 32° = 980.$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{980}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 839N,$$

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \approx 636N.$$

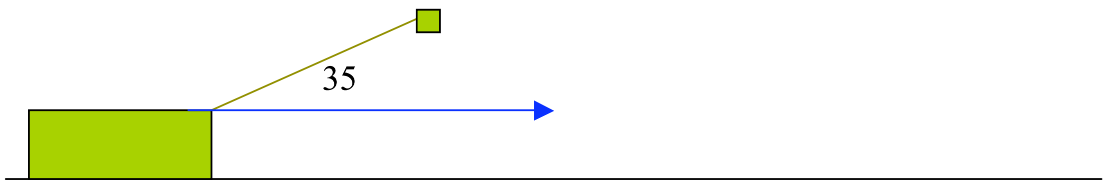
Substituting in (1) and (2) we obtain the tension vectors

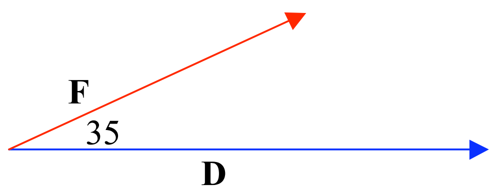
$$\mathbf{T}_1 \approx -539\mathbf{i} + 643\mathbf{j} \quad \mathbf{T}_2 \approx 539\mathbf{i} + 337\mathbf{j}.$$

4. **Pulling a Wagon**

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution





If **F** and **D** are the force and displacement vectors, as shown above, then the work done is

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 35^\circ \\ &= (70)(100)\cos 35^\circ \approx 5734 \text{ N}\cdot\text{m} = 5734 \text{ J} \end{aligned}$$

5. Suppose two airplanes fly paths described by the parametric equations:

$$P_1 : \begin{cases} x &= 3 \\ y &= 6 - 2t \\ z &= 3t + 1 \end{cases} \quad \text{and } P_2 : \begin{cases} x &= 1 + 2s \\ y &= 3 + s \\ z &= 2 + 2s \end{cases}$$

Describe the shape of the flight paths.

Determine whether the paths intersect.

Determine if the planes collide.

6. Explore the geometric object determined by the parametric equations

$$\begin{cases} x &= 2s+3t \\ y &= 3s+2t \\ z &= s+t \end{cases}$$

Given that there are two parameters, what dimension do you expect the object to have?

Given that the individual parametric equations are linear, what do you expect the object to be?

Show that the points $(0,0,0), (2,3,1)$ and $(3,2,1)$ are on the object.

Find the equation of the plane containing these three points.

Substitute in the equations for x, y and z and show that the object lies in the plane.

Argue that the object is, in fact, the entire plane.

7. Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$.

   Solution

   The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is thereofre perpendicular to the plane through P, Q and R. We have

$$\begin{aligned}
\overrightarrow{PQ} &= (-2-1)\mathbf{i} + (5-4)\mathbf{j} + -(1-6)\mathbf{k} \\
&= -3\mathbf{i} + \mathbf{j} - 7\mathbf{k} \\
\overrightarrow{PR} &= (1-1)\mathbf{i} + (-1-4)\mathbf{j} + (1-6)\mathbf{k} \\
&= -5\mathbf{j} - 5\mathbf{k}.
\end{aligned}$$

   We compute the cross product of these vectors:

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\
&= (-5-35)\mathbf{i} - (15-0)\mathbf{j} + (15-0)\mathbf{k} \\
&= -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}.
\end{aligned}$$

   So the vector (-40, -15, 15) is perpendicular to the given plane. Any non zero scalar multiple of this vector, such as (-8, -3, 3), is also perpendicular to the plane.

8. **Torque**

The idea of a cross product often occurs in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance , if we tighten a bolt by applying a force to a wrench we produce a turning effect.) The torque τ (relative to the orogon) is defined to be the cross product of the position and force vectors

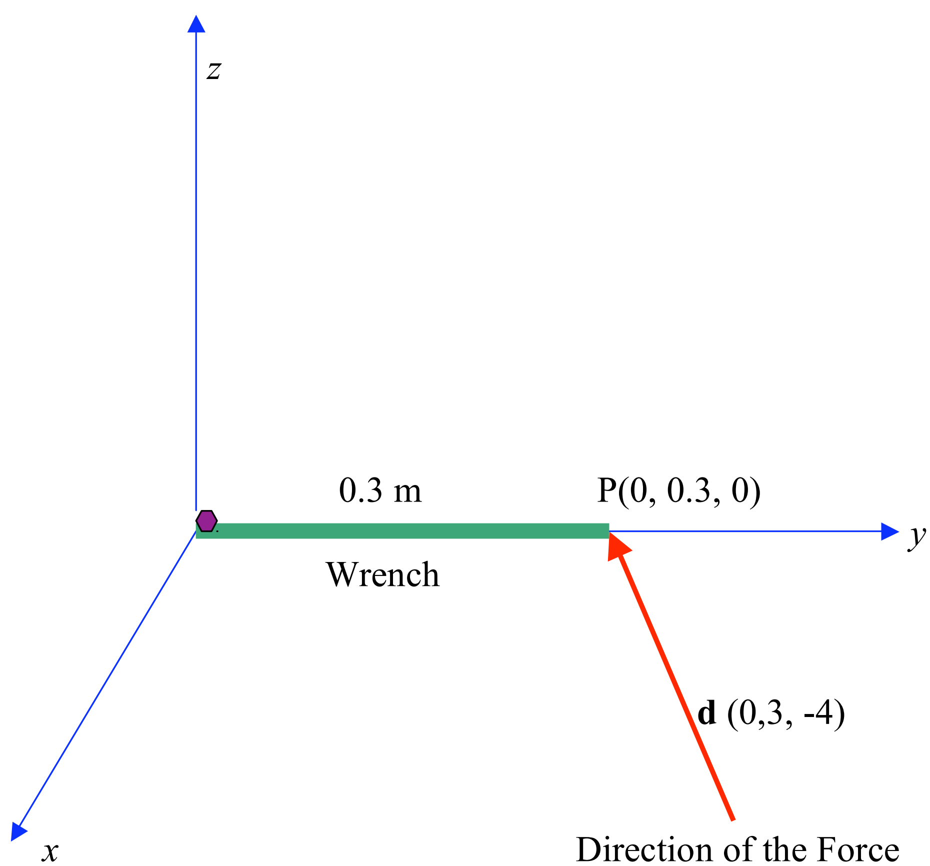
$$\tau = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque indicates the axis of rotation. the magnitude of the torque is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

where θ is the angle between the position and force vectors. Notice that the only component of **F** that can cause a rotation is the component perpendicular to **r**, that is $|\mathbf{F}| \sin\theta$. The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

Example A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $(0, 3, -4)$ at the end of the wrench. Find the magnitude of the force needed to supply 100 N·m of torque to the bolt.



Solution The first task is to find the sin of the angle between the direction of the force and the y-axis. The force is applied to the wrench at position P so the displacement vector is

$$\mathbf{r} = (0, 0.3, 0).$$

We use the formula

$$|\mathbf{d} \times \mathbf{r}| = |\mathbf{d}||\mathbf{r}| \sin\theta$$

Now

$$|\mathbf{d} \times \mathbf{r}| = |\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -4 \\ 0 & .3 & 0 \end{vmatrix}| = |1.2\mathbf{i}| = 1.2$$

and

$$|\mathbf{d}||\mathbf{r}| = |5||0.3| = 1.5$$

so

$$\sin\theta = \frac{|\mathbf{d} \times \mathbf{r}|}{|\mathbf{d}||\mathbf{r}|} = \frac{1.2}{1.5} = 0.8$$

We now use the formula

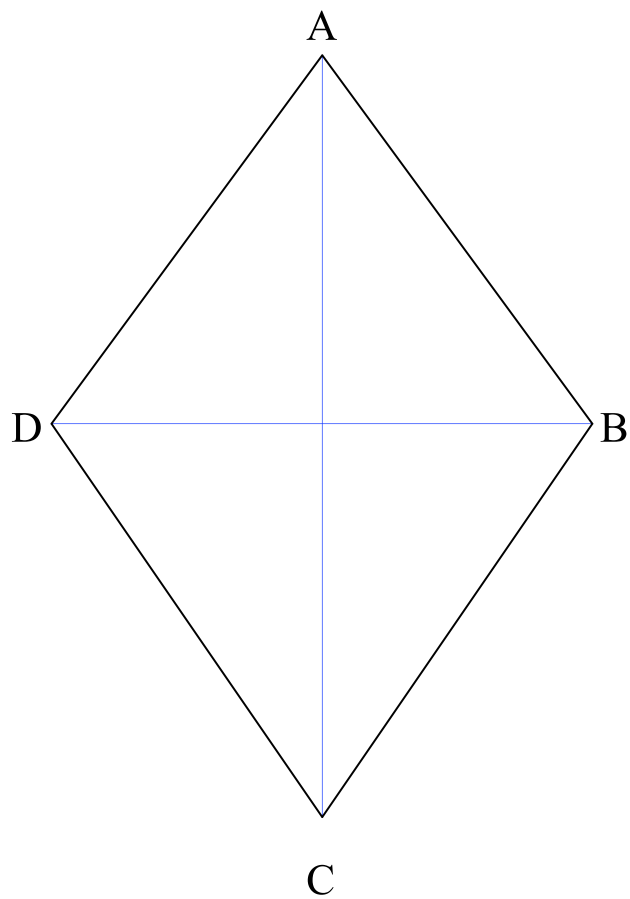
$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

so that

$$|\mathbf{F}| = \frac{|\tau|}{|\mathbf{r}| \sin\theta} = \frac{100}{0.3 \times 0.8} = 416.66 \approx 417\text{N}$$

9. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

   Solution Consider the quadrilateral $ABCD$:



Note that $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CD}| = |\overrightarrow{DA}|$ and

$$\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} \quad \text{and} \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

To show that $\overrightarrow{DB}$ and $\overrightarrow{AC}$ are perpendicular we aim to show that $\overrightarrow{DB}\cdot\overrightarrow{AC} = 0$. Now

$$\begin{aligned}
\overrightarrow{DB}\cdot\overrightarrow{AC} &= (\overrightarrow{DA} + \overrightarrow{AB})\cdot(\overrightarrow{AB} + \overrightarrow{BC}) \\
\\
&= \overrightarrow{DA}\cdot\overrightarrow{AB} + \overrightarrow{DA}\cdot\overrightarrow{BC} \\
&\qquad + \overrightarrow{AB}\cdot\overrightarrow{AB} + \overrightarrow{AB}\cdot\overrightarrow{BC}
\end{aligned}$$

We now use that $\overrightarrow{BC} = -\overrightarrow{DA}$ to get

$$\begin{aligned}
\overrightarrow{DB}\cdot\overrightarrow{AC} &= \overrightarrow{DA}\cdot\overrightarrow{AB} - \overrightarrow{DA}\cdot\overrightarrow{DA} \\
&\qquad + \overrightarrow{AB}\cdot\overrightarrow{AB} - \overrightarrow{AB}\cdot\overrightarrow{DA} \\
\\
&= \overrightarrow{AB}\cdot\overrightarrow{AB} - \overrightarrow{DA}\cdot\overrightarrow{DA} \\
&= |\overrightarrow{AB}|^2 - |\overrightarrow{DA}|^2 \\
&= 0.
\end{aligned}$$

This shows that the diagonals of $ABCD$ are perpendicular.

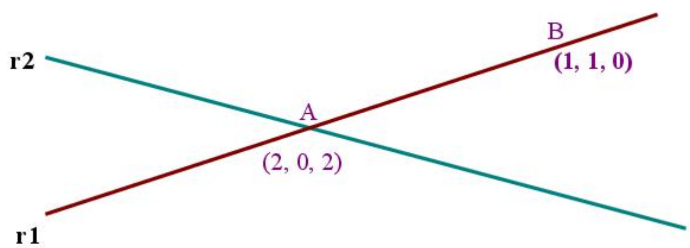
10. (a) Find the point at which the given lines intersect

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution We need to find s and t so that we get the same point on both lines. So we solve

$$\begin{aligned} 1+t &= 2-s \\ 1-t &= s \\ 2t &= 2 \end{aligned}$$

It is clear this time (though not always so immediately) that $t = 1$ and $s = 0$. This gives the point of intersection as $(2, 0, 2)$.



If we found that such equations were inconsistent, then it would indicate that the lines did not intersect.

(b) We now find an equation of the plane containing the lines $\mathbf{r}_1$ and $\mathbf{r}_2$.

Solution

We first find a normal to the plane by finding a vector perpendicular to the direction vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Note

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

The normal to the plane is given by

$$\begin{aligned} \mathbf{n} &= \mathbf{u}_1 \times \mathbf{u}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ -1 & 1 & 0 \end{vmatrix} \\ &= -2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \end{aligned}$$

The equation of the plane is therefore given by

$$-2x - 2y - z = D$$

Where $D = -2x_0 - 2y_0 - z_0$ for some point (x_0, y_0, z_0) on the plane. We can use either the point $(1, 1, 0)$ or the point $(2, 0, 2)$ to get $D = -4$. So the equation of the plane is

$$-2x - 2y - z = -4 \quad \text{or} \quad 2x + 2y + z = 4.$$

11. We find the equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $\mathbf{r} = \begin{bmatrix} 1+t \\ 1-t \\ 2t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Solution

The vector normal to the plane is given by $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the direction of the line $\mathbf{r}$ is given by $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. We need to find the direction vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the new line $\mathbf{r}_1$ satisfying the conditions in the question.

Now $\mathbf{r}_1$ must be parallel to the plane, which means it must be perpendicular to $\mathbf{n}$: that is $\mathbf{v}\cdot\mathbf{n} = 0$. Secondly it must be perpendicular to $\mathbf{r}$ that is $\mathbf{v}\cdot\mathbf{u} = 0$.

$$\begin{aligned} \mathbf{v}\cdot\mathbf{n} = 0 &\implies x + y + z = 0 \\ \mathbf{v}\cdot\mathbf{n} = 0 &\implies x - y + 2z = 0 \end{aligned}$$

So

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{bmatrix}$$

This gives $\mathbf{v} = z \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix}$, so let us take $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$. Then

$$\mathbf{r}_1 = P_0 + t\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}.$$

Alternatively $\mathbf{r}_1$ is given by

$$\frac{x}{-3} = y - 1 = \frac{z-2}{2}.$$

12. **Find equations for the line through the point $(0,1,2)$ that is perpendicular to the line $x = 1+t, y = 1-t, z = 2t$ and intersects the line.

Solution

The line $x = 1+t, y = 1-t, z = 2t$ is the same as the line $\mathbf{r} = \begin{bmatrix} 1+t \\ 1-t \\ 2t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ as in the previous question, with direction vector $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Again, let us call the new line $\mathbf{r}_1$ with direction vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, so that $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Again we require $\mathbf{v} \cdot \mathbf{u} = 0$, which gives

$$x - y + 2z = 0 \quad \text{or} \quad x = y - 2z.$$

To satisfy the requirement that the two lines intersect we need to be able to solve (for s and t)

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} y-2z \\ y \\ z \end{bmatrix}$$

This is equivalent to

$$t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - s \begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

This gives a system of equations:

$$\begin{aligned} t + (2z-y)s &= -1 \\ t + ys &= 0 \\ 2t - zs &= 2 \end{aligned}$$

with augmented matrix

$$\begin{bmatrix} 1 & 2z-y & -1 \\ 1 & y & 0 \\ 2 & -z & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2z-y & -1 \\ 0 & 2y-2z & 1 \\ 0 & 2y-5z & 4 \end{bmatrix}$$

The last two rows of this matrix mean

$$\begin{aligned} (2y-2z)s &= 1 \\ (2y-5z)s &= 4 \end{aligned}$$

If we multiply the first equation by 4, for the equations to be consistent we require that

$$4(2y-2z) = 2y-5z.$$

This implies that $z = 2y$. So we can write

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y \\ y \\ 2y \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

So the equation of the line is

$$\mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

or

$$\frac{x}{-3} = y - 1 = \frac{z - 1}{2}$$

13. Find the distance between the parallel planes

$$2x - 3y + z = 4 \quad \text{and} \quad 4x - 6y + 2z = 3.$$

Solution

Let us rewrite the equations:

$$2x - 3y + z = 4 \quad \text{and} \quad 2x - 3y + z = 3/2$$

so that is is clear that they are parallel and that they each have normal vector $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Now the formula for the distance from a point $P(x_0, y_0, z_0)$ to the plane $Ax+By+Cz = D$ is given by

$$s = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Let us take $2x-3y+z = 4$ as the plane $Ax+By+Cz = 4$, so that $A = 2, B = -3, C = 1$ and $D = 4$.

Also let us take (x_0, y_0, z_0) as a point on the plane $2x - 3y + z = 3/2$.

Then $Ax_0 + By_0 + Cz_0 = 3/2$ precisely because (x_0, y_0, z_0) is a point on the plane $2x - 3y + z = 3/2$.

With this information

$$s = \frac{|3/2 - 4|}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{1}{2\sqrt{14}}.$$

4.2 Exercises

1. A clothesline is tied between two poles 8 m apart. The line is quite taught and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

2. A woman is walking due west on the deck of a ship at 5 Km/h. The ship is moving north at a speed of 35 km/h. Find the speed and direction of the woman relative to the surface of the water.

3. Find unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point (2, 4).

4. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

5. Find the angle between a diagonal of a cube and one of its edges.

6. Find two unit vectors orthogonal to both $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$.

7. Find a non-zero vector $\mathbf{n}$ orthogonal to the plane through the points P, Q and R, and find the area of the triangle PQR

   (a) $P(1, 0, 0), Q(0, 2, 0), R(0, 0, 3)$

   (b) $P(0, -2, 0), Q(4, 1, -2), R(5, 3, 1)$

8. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are all non zero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

9. Find parametric and standard equations for the following lines:

   (a) The line through the points $(1, 3, 2)$ and $(-4, 3, 0)$.

   (b) The line through the points $(0, 1/2, 1)$ and $(2, 1, -3)$.

   (c) The line through $(1, -1, 1)$ and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$.

   (d) The line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.

10. Find an equation of the plane:

    (a) The plane through the point $(6, 3, 2)$ and perpendicular to the vector $-2\mathbf{i}+\mathbf{j}+5\mathbf{k}$.

    (b) The plane through the point $(1, -1, 1)$ and with normal vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

    (c) The plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$.

    (d) The plane through the points $(3, -1, 2)$,
    $(8, 2, 4)$ and $(-1, -2, -3)$.

(e) The plane that passes through the point $(6, 0, -2)$ and contains the line: $x = 4 - 2t, y = 3 + 5t, z = 7 + 4t$.

(f) The plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

11. Find the point at which the line intersect the given plane:

(a) The line $x = 1 + t, y = 2t, z = 3t$ and the plane $x + y + z = 1$.

(b) The line $x = y - 1 = 2z$ and the plane $4x - y + 3z = 8$.

12. Find the parametric equations for the line of intersection of the planes and find the angle between the planes:

$$x + y + z = 1 \quad \text{and} \quad x + 2y + 2z = 1.$$

13. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, 02)$ and $(3, 4, 0)$.

14. Find the distance from the point $(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$.

15. Find the distance between the parallel planes

$$6z = 4y - 2x \quad \text{and} \quad 9z = 1 - 3x + 6y.$$

16. Show that the lines with standard equations

$$x = y = z \quad \text{and} \quad x + 1 = \frac{y}{2} = \frac{z}{3}$$

are skew, and find the distance between these lines.

4.3 Answers to Exercises

1. $\mathbf{T}_1 \approx -196\mathbf{i} + 3.92\mathbf{j}, \mathbf{T}_2 \approx 196\mathbf{i} + 3.92\mathbf{j}$.

2. The speed is $\sqrt{1250} \approx 35.4$ km/h, and the direction is N8°W.

3. $\pm(\mathbf{i} + 4\mathbf{j})/\sqrt{17}$.

4.

5. $\cos^{-1}(1/\sqrt{3}) \approx 55°$.

6. $\pm\left(\frac{-2\mathbf{i}}{\sqrt{6}} + \frac{-\mathbf{j}}{\sqrt{6}} + \frac{\mathbf{k}}{\sqrt{6}}\right)$

7. (a) $\mathbf{n} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, and the area of PQR is $7/2$.

(b) $\mathbf{n} = 13\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}$ and the area of PQR is $\frac{1}{2}\sqrt{390}$

8. Proof.

9. (a) Parametric form: $\begin{cases} x = 1 - 5t \\ y = 3 \\ z = 2 - 2t \end{cases}$

   Standard form: $\dfrac{x-1}{-5} = \dfrac{z-2}{-2}, y = 3.$

   (b) Parametric form: $\begin{cases} x = 2 + 2t \\ y = 1 + \frac{1}{2}t \\ z = -3 - 4t \end{cases}$

   Standard form: $\dfrac{x-2}{2} = 2y - 2 = \dfrac{z+3}{-4}.$

   (c) Parametric form: $\begin{cases} x = 1 + t \\ y = -1 + 2t \\ z = 1 + t \end{cases}$

   Standard form: $x - 1 = \dfrac{y+1}{2} = z - 1.$

   (d) Parametric form: $\begin{cases} x = 1 + t \\ y = 1 \\ z = -1 - t \end{cases}$

   Standard form: $x - 1 = \dfrac{z+1}{-1}, y = 1.$

10. (a) $-2x + y + 5z = 1.$

    (b) $x + y - z = -1$

    (c) $3x - 7z = -9.$

    (d) $-13x + 17y + 7z = -42.$

    (e) $33x + 10y + 4z = 190.$

    (f) $x - 2y + 4z = -1.$

11. (a) $(1, 0, 0).$

    (b) $(2, 3, 1).$

12. The line of intersection is given by

$$\begin{cases} x = 1 \\ y = -t \\ z = t \end{cases}$$

Note that the direction vector is $-\mathbf{j}+\mathbf{k}$ and a point on the line is $(1,0,0)$. If you choose a different point on the line of intersection e.g $(1,1,-1)$ then the line of intersection could be expressed

$$\begin{cases} x = 1 \\ y = 1 - t \\ z = -1 + t \end{cases}$$

The angle between the two lines is given by $\cos^{-1} \dfrac{5}{3\sqrt{3}} \approx 15.8°$. The angle can also be given by $\sin^{-1} \dfrac{\sqrt{2}}{3\sqrt{3}}$, but this does not indicate whether the angle is between 0 and $\pi/2$ or between $\pi/2$ and π.

13. $x + 2y + z = 5$.

14. $\sqrt{\dfrac{61}{14}}$.

15. $\dfrac{1}{3\sqrt{14}}$.

16. $\dfrac{5}{2\sqrt{14}}$.