Overview

Yesterday we introduced equations to describe lines and planes in $\mathbb{R}^3$:

- $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$
  The vector equation for a line describes arbitrary points $\mathbf{r}$ in terms of a specific point $\mathbf{r_0}$ and the direction vector $\mathbf{v}$.
- $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$
  The vector equation for a plane describes arbitrary points $\mathbf{r}$ in terms of a specific point $\mathbf{r_0}$ and the normal vector $\mathbf{n}$.

Question

How can we find the distance between a point and a plane in $\mathbb{R}^3$? Between two lines in $\mathbb{R}^3$? Between two planes? Between a plane and a line?

(From Stewart §10.5)

Distances in $\mathbb{R}^3$

The distance between two points is the length of the line segment connecting them. However, there's more than one line segment from a point P to a line L, so what do we mean by the *distance* between them?

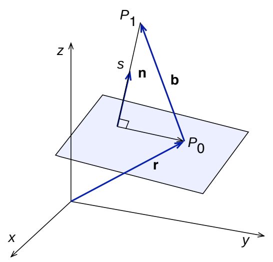
The distance between any two subsets A, B of $\mathbb{R}^3$ is the smallest distance between points a and b, where a is in A and b is in B.

- To determine the distance between a point P and a line L, we need to find the point Q on L which is closest to P, and then measure the length of the line segment PQ.
  This line segment is *orthogonal* to L.
- To determine the distance between a point P and a plane S, we need to find the point Q on S which is closest to P, and then measture the length of the line segment PQ.
  Again, this line segment is *orthogonal* to S.

In both cases, the key to computing these distances is drawing a picture and using one of the vector product identitites.

Distance from a point to a plane

We find a formula for the distance s from a point $P_1 = (x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$.



Let $P_0 = (x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$

The distance s from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle A, B, C \rangle$.

$$
\begin{aligned}
s &= |\operatorname{comp}_{\mathbf{n}}\mathbf{b}| \\
&= \frac{|\mathbf{n}\cdot\mathbf{b}|}{||\mathbf{n}||} \\
&= \frac{|A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|Ax_1+By_1+Cz_1-(Ax_0+By_0+Cz_0)|}{\sqrt{A^2+B^2+C^2}}
\end{aligned}
$$

Since P_0 is on the plane, its coordinates satisfy the equation of the plane and so we have $Ax_0+By_0+Cz_0+D=0$. Thus the formula for s can be written

$$s = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

Example 1

We find the distance from the point $(1, 2, 0)$ to the plane $3x-4y-5z-2=0$.

From the result above, the distance s is given by

$$s = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

where $(x_0, y_0, z_0) = (1, 2, 0)$,

$$A = 3, B = -4, C = -5 \text{ and } D = -2.$$

This gives

$$
\begin{aligned}
s &= \frac{|3\cdot 1+(-4)\cdot 2+(-5)\cdot 0-2|}{\sqrt{3^2+(-4)^2+(-5)^2}} \\
&= \frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}.
\end{aligned}
$$

Distance from a point to a line

Question

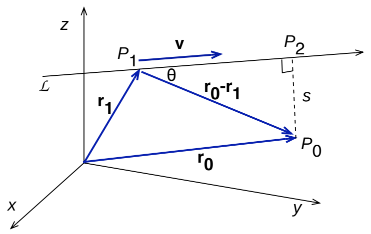
Given a point $P_0 = (x_0, y_0, z_0)$ and a line L in $\mathbb{R}^3$, what is the distance from P_0 to L?

Tools:

- describe L using vectors
- $||\mathbf{u}\times\mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}||\sin\theta$

Distance from a point to a line

Let $P_0 = (x_0, y_0, z_0)$ and let L be the line through P_1 and parallel to the nonzero vector $\mathbf{v}$. Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be the position vectors of P_0 and P_1 respectively. P_2 on L is the point closest to P_0 if and only if the vector $\overrightarrow{P_2P_0}$ is perpendicular to L.



The distance from P_0 to L is given by

$$s = ||\overrightarrow{P_2P_0}|| = ||\overrightarrow{P_1P_0}|| \sin\theta = ||\mathbf{r}_0 - \mathbf{r}_1|| \sin\theta$$

where θ is the angle between $\mathbf{r}_0 - \mathbf{r}_1$ and $\mathbf{v}$

Since

$$||(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}|| = ||\mathbf{r}_0 - \mathbf{r}_1||\,||\mathbf{v}|| \sin\theta$$

we get the formula

$$\begin{aligned} s &= ||\mathbf{r}_0 - \mathbf{r}_1|| \sin\theta \\ &= \frac{||(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}||}{||\mathbf{v}||} \end{aligned}$$

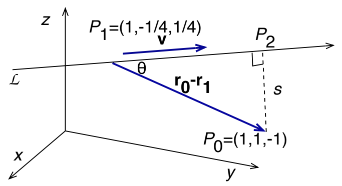
Example 2

Find the distance from the point $(1, 1, -1)$ to the line of intersection of the planes

$$x + y + z = 1, \quad 2x - y - 5z = 1.$$

The direction of the line is given by $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ where $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$.

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = -4\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}.$$



In the diagram, P_1 is an arbitrary point on the line. To find such a point, put $x = 1$ in the first equation. This gives $y = -z$ which can be used in the second equation to find $z = 1/4$, and hence $y = -1/4$.

Here $\overrightarrow{P_1P_0} = \mathbf{r}_0 - \mathbf{r}_1 = \frac{5}{4}\mathbf{j} - \frac{5}{4}\mathbf{k}$. So

$$\begin{aligned} s &= \frac{||(\mathbf{r}_0 - \mathbf{r}_1)\times\mathbf{v}||}{||\mathbf{v}||} \\ &= \frac{||(\frac{5}{4}\mathbf{j} - \frac{5}{4}\mathbf{k})\times(-4\mathbf{i} + 7\mathbf{j} - 3\mathbf{k})||}{\sqrt{(-4)^2 + 7^2 + (-3)^2}} \\ &= \frac{||5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}||}{\sqrt{74}} \\ &= \sqrt{\frac{75}{74}}. \end{aligned}$$

Distance between two lines

Let L_1 and L_2 be two lines in $\mathbb{R}^3$ such that

- L_1 passes through the point P_1 and is parallel to the vector $\mathbf{v}_1$
- L_2 passes through the point P_2 and is parallel to the vector $\mathbf{v}_2$.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of P_1 and P_2 respectively.
Then parametric equation for these lines are

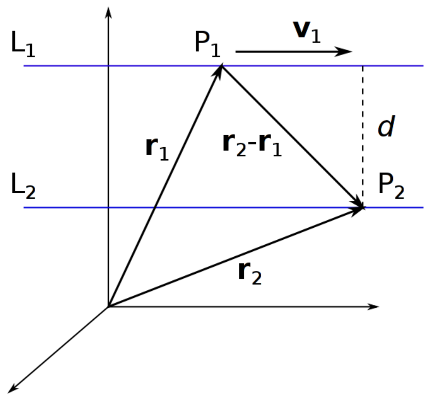
$$\begin{array}{ll} L_1 & \mathbf{r} = \mathbf{r}_1 + t\mathbf{v}_1 \\ L_2 & \tilde{\mathbf{r}} = \mathbf{r}_2 + s\mathbf{v}_2 \end{array}$$

Note that $\mathbf{r}_2 - \mathbf{r}_1 = \overrightarrow{P_1P_2}$.
We want to compute the smallest distance d (simply called the distance) between the two lines.
If the two lines intersect, then $d = 0$. If the two lines do not intersect we can distinguish two cases.

Case 1: L_1 and L_2 are parallel and do not intersect.

In this case the distance d is simply the distance from the point P_2 to the line L_1 and is given by

$$d = \frac{||\overrightarrow{P_1P_2} \times \mathbf{v}_1||}{||\mathbf{v}_1||} = \frac{||(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{v}_1||}{||\mathbf{v}_1||}$$

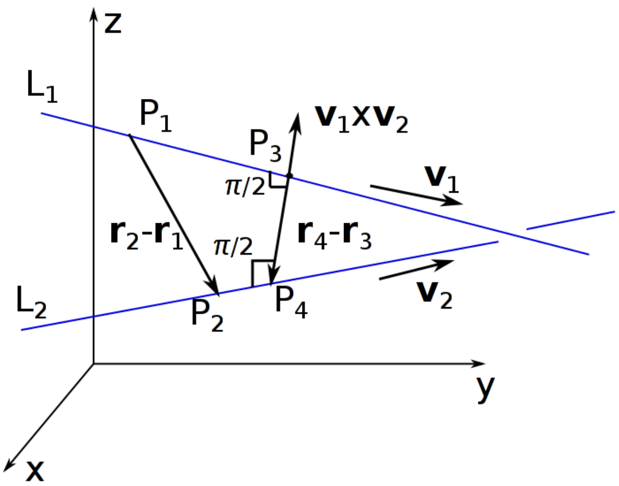


Case 2: L_1 and L_2 are skew lines.

If P_3 and P_4 (with position vectors $\mathbf{r}_3$ and $\mathbf{r}_4$ respectively) are the points on L_1 and L_2 that are closest to one another, then the vector $\overrightarrow{P_3P_4}$ is perpendicular to both lines (i.e. to both $\mathbf{v}_1$ and $\mathbf{v}_2$) and therefore parallel to $\mathbf{v}_1 \times \mathbf{v}_2$. The distance d is the length of $\overrightarrow{P_3P_4}$.
Now $\overrightarrow{P_3P_4} = \mathbf{r}_4 - \mathbf{r}_3$ is the vector projection of $\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1$ along $\mathbf{v}_1 \times \mathbf{v}_2$.
Thus the distance d is the absolute value of the scalar projection of $\mathbf{r}_2 - \mathbf{r}_1$ along $\mathbf{v}_1 \times \mathbf{v}_2$

$$d = ||\mathbf{r}_4 - \mathbf{r}_3|| = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{||\mathbf{v}_1 \times \mathbf{v}_2||}$$

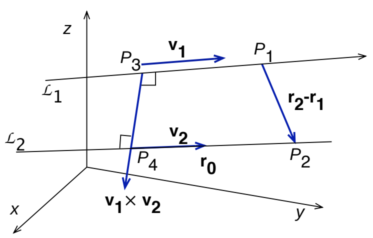
Observe that if the two lines are parallel then $\mathbf{v}_1$ and $\mathbf{v}_2$ are proportional and thus $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ (the zero vector) and the above formula does not make sense.



Example 3

Find the distance between the skew lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$



We can take $P_1 = (1, 1, 1)$, a point on the first line, and $P_2 = (1, 2, 3)$ a point on the second line. This gives $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{j} + 2\mathbf{k}$.

Now we need to find $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{v}_1 = (\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$

and

$$\mathbf{v}_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

This gives

$$\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}.$$

The required distance d is the length of the projection of $\mathbf{r}_2 - \mathbf{r}_1$ in the direction of $\mathbf{v}_1 \times \mathbf{v}_2$, and is given by

$$\begin{aligned} d &= \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{||\mathbf{v}_1 \times \mathbf{v}_2||} \\ &= \frac{|(\mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 8\mathbf{k})|}{\sqrt{(-1)^2 + 2^2 + 8^2}} \\ &= \frac{18}{\sqrt{69}}. \end{aligned}$$