Overview

We've studied the geometric and algebraic behaviour of vectors in Euclidean space. This week we turn to an abstract model that has many of the same algebraic properties.

The importance of this is two-fold:

- Many models of physical processes do not sit in $\mathbb{R}^3$, or indeed in $\mathbb{R}^n$ for any n.
- Apparently different situations often turn out to be "essentially" the same; studying the abstract case solves many problems at once.

(Lay, §4.1)

Let's review vector operations in language that will help set up our generalisation:

- Vectors are objects which can be added together or multiplied by scalars; both operations give back a vector.
- Vector addition is commutative and associative; scalar multiplication and vector addition are distributive.
- Adding the zero vector to $\mathbf{v}$ doesn't change $\mathbf{v}$.
- Multiplying a vector $\mathbf{v}$ by the scalar 1 doesn't change $\mathbf{v}$.
- Adding $\mathbf{v}$ to $(-1)\mathbf{v}$ gives the zero vector.

(Notice that we haven't included the dot product. This does have a role to play in our abstract setting, but we'll come to it later in the term.)

Definition

A *vector space* is a non-empty set V of objects called *vectors* on which are defined operations of *addition* and *multiplication by scalars*. These objects and operations must satisfy the following ten axioms for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

For now, we'll take the set of scalars to be the real numbers. In a few weeks, we'll consider vector spaces where the scalars are complex numbers instead.

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The axioms for a vector space

1. $\mathbf{u}+\mathbf{v}$ is in V;
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$; (commutativity)
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$; (associativity)
4. there is an element $\mathbf{0}$ in V, $\mathbf{0}+\mathbf{u}=\mathbf{u}$;
5. there is $-\mathbf{u}\in V$ with $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$;
6. $c\mathbf{u}$ is in V;
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$;
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$;
9. $c(d\mathbf{u})=(cd)\mathbf{u}$;
10. $1\mathbf{u}=\mathbf{u}$.

Example 1

Let $M_{2\times 2}=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} : a,b,c,d\in\mathbb{R}\right\}$, with the usual operations of addition of matrices and multiplication by a scalar.

In this context the the zero vector $\mathbf{0}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$.

The negative of the vector $\mathbf{v}=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ is $-\mathbf{v}=\begin{bmatrix} -a & -b \\ -c & -d\end{bmatrix}$.

For the same vector $\mathbf{v}$ and $t\in\mathbb{R}$ we have $t\mathbf{v}=\begin{bmatrix} ta & tb \\ tc & td\end{bmatrix}$.

If $\mathbf{v}=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix} e & f \\ g & h\end{bmatrix}$ then $\mathbf{u}+\mathbf{w}=\begin{bmatrix} a+e & b+f \\ c+g & d+h\end{bmatrix}$.

Example 2

Let $\mathbb{P}_2$ be the set of all polynomials of degree at most 2 with coefficients in $\mathbb{R}$. Elements of $\mathbb{P}_2$ have the form

$$\mathbf{p}(t)=a_0+a_1t+a_2t^2$$

where a_0, a_1 and a_2 are real numbers and t is a real variable. You are already familiar with adding two polynomials or multiplying a polynomial by a scalar.

The set $\mathbb{P}_2$ is a vector space.

We will just verify 3 out of the 10 axioms here.

Let $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ and $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2$, and let c be a scalar.

Axiom 1: $\mathbf{v} + \mathbf{u}$ is in V

The polynomial $\mathbf{p} + \mathbf{q}$ is defined in the usual way: $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$$

which is also a polynomial of degree at most 2. So $\mathbf{p} + \mathbf{q}$ is in $\mathbb{P}_2$.

Axiom 4: $\mathbf{v} + \mathbf{0} = \mathbf{v}$

The zero vector $\mathbf{0}$ is the zero polynomial $\mathbf{0} = 0 + 0t + 0t^2$.

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0}(t) = (a_0 + 0) + (a_1 + 0)t + (a_2 + 0)t^2 = \mathbf{p}(t).$$

So $\mathbf{p} + \mathbf{0} = \mathbf{p}$.

Axiom 6: $c\mathbf{u}$ is in V

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (ca_0) + (ca_1)t + (ca_2)t^2.$$

This is again a polynomial in $\mathbb{P}_2$.

The remaining 7 axioms also hold, so $\mathbb{P}_2$ is a vector space.

In fact, the previous example generalises:

Example 3

Let $\mathbb{P}_n$ be the set of polynomials of degree at most n with coefficients in $\mathbb{R}$. Elements of $\mathbb{P}_n$ are polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + \ldots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable.

As in the example above, the usual operations of addition of polynomials and multiplication of a polynomial by a real number make $\mathbb{P}_n$ a vector space.

Example 4

The set $\mathbb{Z}$ of integers with the usual operations *is not* a vector space. To demonstrate this it is enough to to find that *one* of the ten axioms fails and to give a specific instance in which it fails (i.e., a *counterexample*).

In this case we find that we do not have closure under scalar multiplication (Axiom 6). For example, the multiple of the integer 3 by the scalar $\frac{1}{4}$ is

$$\left(\frac{1}{4}\right)(3) = \frac{3}{4}$$

which is not an integer. Thus it is not true that cx is in $\mathbb{Z}$ for *every* x in $\mathbb{Z}$ and *every* scalar c.

Example 5

Let $\mathcal{F}$ denote the set of real valued functions defined on the real line. If f and g are two such functions and c is a scalar, then $f+g$ and cf are defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x).$$

This means that the *value* of $f+g$ at x is obtained by adding together the values of f and g at x. So if f is the function $f(x) = \cos x$ and g is $g(x) = e^x$ then

$$(f+g)(0) = f(0) + g(0) = \cos 0 + e^0 = 1 + 1 = 2.$$

We find cf in a similar way. This means axioms 1 and 6 are true. The other axioms need to be verified, and with that verification $\mathcal{F}$ is a vector space.

Sometimes we have vector spaces with *unintuitive* operations for addition and scalar multiplication.

Example 6

Consider $\mathbb{R}_{>0}$, the positive real numbers, under the following operations:

- $\mathbf{v} \oplus \mathbf{w} = \mathbf{vw}$
- $c \otimes \mathbf{v} = \mathbf{v}^c$.

Counterintuitively, this is a vector space! For example, we can check Axiom 7:

$$c \otimes (\mathbf{u} \oplus \mathbf{v}) = (\mathbf{uv})^c$$

while

$$(c \otimes \mathbf{u}) \oplus (c \otimes \mathbf{v}) = \mathbf{u}^c \mathbf{v}^c.$$

To make things work out, we find $\mathbf{0} = \mathbf{1}$, and $-\mathbf{u} = \mathbf{u}^{-1}$

What's going on here?

The following theorem is a direct consequence of the axioms.

Theorem

Let V be a vector space, $\mathbf{u}$ a vector in V and c a scalar.

1. $\mathbf{0}$ *is unique;*
2. $-\mathbf{u}$ *is the unique vector that satisfies* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
3. $0\mathbf{u} = \mathbf{0}$; *(note difference between* 0 *and* $\mathbf{0}$*)*
4. $c\mathbf{0} = \mathbf{0}$;
5. $(-1)\mathbf{u} = -\mathbf{u}$.

Exercises 4.1.25 - 29 of Lay outline the proofs of these results.

Subspaces

Some of the vector space examples we've seen "sit inside" others.
For example, we sketched the proof that $\mathbb{P}_2$ and $\mathbb{P}_4$ are both vector spaces. Any polynomial of degree at most two can also be viewed as a polynomial of degree at most 4:

$$a_0 + a_1 t + a_2 t^2 = a_0 + a_1 t + a_2 t^2 + 0t^3 + 0t^4.$$

If you have a subset H of a vector space V, some of the axioms are satisfied for free. For example, you don't need to check that scalar multiplication in H distributes through vector addition: you already know this is true in H because it's true in V.

Subspaces

This idea is formalised in the notion of a *subspace*.

Definition

A *subspace* of a vector space V is a subset H of V such that

1. The zero vector is in H: $\mathbf{0} \in H$;
2. whenever $\mathbf{u}, \mathbf{v}$ are in H, $\mathbf{u} + \mathbf{v}$ is in H.
   " H is closed under vector addition."
3. $c\mathbf{u}$ is in H whenever $\mathbf{u}$ is in H and c is in $\mathbb{R}$.
   "H is closed under scalar multiplication."

This is not a new idea: in MATH1013 the same definition is given for subspaces of $\mathbb{R}^n$.

Examples

Example 7

If V is any vector space, the subset $\{\mathbf{0}\}$ of V containing only the zero vector $\mathbf{0}$ is a subspace of V.

This is called the *zero subspace* or the *trivial subspace*.

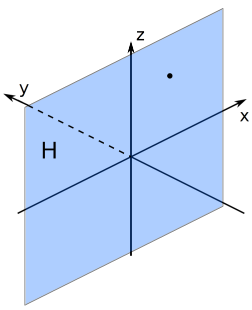
Example 8

Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show that H is a subspace of $\mathbb{R}^3$.

- The zero vector of $\mathbb{R}^3$ is in H: set $a = 0$ and $b = 0$.
- H is closed under addition: adding two vectors in H always produces another vector whose second entry is 0 and therefore in H.
- H is closed under scalar multiplication: multiplying a vector in H by a scalar produces another vector in H.

Since all three properties hold, H is a subspace of $\mathbb{R}^3$.

If we identify vectors in $\mathbb{R}^3$ with points in 3D space as usual, then H is the plane through the origin given by the *homogeneous* equation $y = 0$.

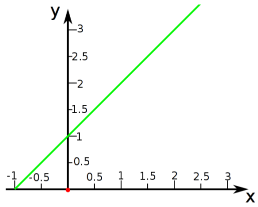


H is a plane, but H is NOT EQUAL to $\mathbb{R}^2$!
(The set $\mathbb{R}^2$ is not contained in $\mathbb{R}^3$.)

Example 9

Is $H = \left\{ \begin{bmatrix} s \\ s+1 \end{bmatrix} : s \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

We can identify H with the line whose equation is $y = x + 1$.



Clearly, the zero vector is not in H, so H is not a subspace of $\mathbb{R}^2$.
(Observe that the equation $y = x + 1$ is *not* homogeneous).
As you saw in MATH1013, lines and planes through the origin are subspaces of $\mathbb{R}^n$ while lines and planes that do not pass through the origin are not subspaces.

Example 10

Let W be the set of symmetric 2×2 matrices:

$$W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \right\} = \left\{ A \mid A^T = A \right\}.$$

Then W is a subspace of $M_{2\times 2}$.

- The zero matrix satisfies the condition: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- Let A and B be in W. Then $A^T = A$ and $B^T = B$, from which it follows that
$$(A + B)^T = A^T + B^T = A + B.$$
Therefore $A + B$ is symmetric and is in W.
- Similarly, $(cA)^T = cA^T = cA$, so cA is symmetric and is in W.

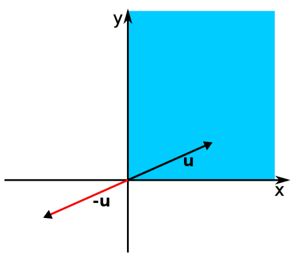
Example 11

Let V be the first quadrant in the xy-plane:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}.$$

Is V a subspace of $\mathbb{R}^2$?

The answer is NO. Look at the picture below for example



Example 12

Let H be the set of all polynomials (with coefficients in $\mathbb{R}$) of degree at most two that have value 0 at $t = 1$

$$H = \{\mathbf{p} \in \mathbb{P}_2 \ : \ \mathbf{p}(1) = 0\}.$$

Is H a subspace of $\mathbb{P}_2$?

- The zero polynomial satisfies $\mathbf{0}(t) = 0$ for every t, so in particular $\mathbf{0}(1) = 0$.
- Let $\mathbf{p}$ and $\mathbf{q}$ be in H. Then $\mathbf{p}(1) = 0$ and $\mathbf{q}(1) = 0$
  Thus
  $$(\mathbf{p} + \mathbf{q})(1) = \mathbf{p}(1) + \mathbf{q}(1) = 0 + 0 = 0.$$
- If c is in $\mathbb{R}$ and $\mathbf{p}$ is in H we have
  $$(c\mathbf{p})(1) = c(\mathbf{p}(1)) = c0 = 0.$$

Yes, H is a subspace of $\mathbb{P}_2$!

Example 13

Let U be the set of all polynomials (with coefficients in $\mathbb{R}$) of degree at most two that have value 2 at $t = 1$

$$U = \{\mathbf{p} \in \mathbb{P}_2 \ : \ \mathbf{p}(1) = 2\}.$$

Is U a subspace of $\mathbb{P}_2$?

NO! In fact, the subset U doesn't satisfy any of the three subspace axioms.

Span: a recipe for building a subspace

Definition

Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in V, then the set of all vectors that can be written as linear combinations of the vectors is S is called Span(S):

$$\text{Span}(S) = \{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \ : \ c_1, \dots, c_p \text{ are real numbers}\}$$

Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be a set of vectors in a vector space V. Then Span(S) is a subspace of V.

The subspace Span(S) is the "smallest" subspace of V that contains S, in the sense that if H is a subspace of V that contains all the vectors in S then Span$(S) \subset H$.

Example 14

Let $V = \{\langle a+3b, 3a-2b\rangle : a, b \in \mathbb{R}\}$. Is V a subspace of $\mathbb{R}^2$?

Write the vectors in V in column form:

$$\begin{aligned} \begin{bmatrix} a+3b \\ 3a-2b \end{bmatrix} &= \begin{bmatrix} a \\ 3a \end{bmatrix} + \begin{bmatrix} 3b \\ -2b \end{bmatrix} \\ &= a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ -2 \end{bmatrix} \end{aligned}$$

So $V = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and it is therefore a subspace of $\mathbb{R}^2$.
(In fact, it's all of $\mathbb{R}^2$, but that still counts as a subspace!)

Example 15

Let W be the set of all vectors in $\mathbb{R}^4$ of the form

$$\begin{bmatrix} 4a-2b \\ a+b+c \\ 0 \\ -2c-6a \end{bmatrix} \qquad (a, b, c \in \mathbb{R}) \qquad (W)$$

Show that W is a subspace of $\mathbb{R}^4$.

Since

$$\begin{bmatrix} 4a-2b \\ a+b+c \\ 0 \\ -2c-6a \end{bmatrix} = a\begin{bmatrix} 4 \\ 1 \\ 0 \\ -6 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix},$$

it follows that W is the subspace of $\mathbb{R}^4$ spanned by the three vectors

$$\begin{bmatrix} 4 \\ 1 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}.$$

Suggested exercises for review

Lay §4.1: 3, 9, 13, 33