Let $F : M_{2\times 2} \to \mathbb{P}_2$ be a linear transformation given by

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + b + c + (a-b)x + (d-c)x^2$$

Note that the **kernel** of this transformation will be a 2×2 matrix. It is the set of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy

$$\begin{aligned} a + b + c &= 0 \\ a - b &= 0 \\ d - c &= 0 \end{aligned}$$

To find these matrices we solve the system of homogeneous equation given by

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & -1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 0\\ 0 & -2 & -1 & 0\\ 0 & 0 & 1 & -1\end{bmatrix} \to$$

$$\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 1 & 1/2 & 0\\ 0 & 0 & 1 & -1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1/2 & 0\\ 0 & 1 & 1/2 & 0\\ 0 & 0 & 1 & -1\end{bmatrix} \to$$

$$\begin{bmatrix}1 & 0 & 0 & 1/2\\ 0 & 1 & 0 & 1/2\\ 0 & 0 & 1 & -1\end{bmatrix}$$

This gives $a = \frac{-1}{2}d, b = \frac{-1}{2}d$ and $c = d$ so that the matrices we are looking for are of the form

$$\begin{bmatrix}\frac{-1}{2}d & \frac{-1}{2}d\\ d & d\end{bmatrix} = d\begin{bmatrix}-1/2 & -1/2\\ 1 & 1\end{bmatrix}$$

So any matrix that is a scalar multiple of $\begin{bmatrix}-1 & -1\\ 2 & 2\end{bmatrix}$ is in the kernel of F.

The **range** of the transformation is possibly harder to find, though not impossible. The range is a subset of the polynomials of degree at most 2, $\mathbb{P}_2$. What we want to know is whether it is all of $\mathbb{P}_2$ or only part of it.
Essentially we want to know, if we are given a polynomial $a_0 + a_1 x + a_2 x^2$ can we always find a, b, c, d such that

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a_0 + a_1 x + a_2 x^2.$$

This means we need to be able to solve the equations:

$$\begin{aligned} a + b + c &= a_0 \\ a - b &= a_1 \\ d - c &= a_2 \end{aligned}$$

and that means row reducing the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & a_0 \\ 1 & -1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 1 & a_2 \end{bmatrix}$$

We don't need to do as many steps as we did before to show that this will always have a solution. It is sufficient just to make the first step

$$\begin{bmatrix} 1 & 1 & 1 & 0 & a_0 \\ 1 & -1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 1 & a_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & a_0 \\ 0 & -2 & -1 & 0 & a_1 - a_0 \\ 0 & 0 & -1 & 1 & a_2 \end{bmatrix}$$

The matrix is now in echelon form and we know it will always have a solution. This means that the range of F is all of $\mathbb{P}_2$.

We consider the linear transformation $H : \mathbb{P}_2 \to M_{2\times 2}$ given by

$$H(a + bx + cx^2) = \begin{bmatrix} a+b & a-b \\ c & c-a \end{bmatrix}$$

The **kernel** of H is a subset of $\mathbb{P}_2$, the polynomials of degree at most 2, and is the set of polynomials $a + bx + cx^2$ with

$$\begin{aligned} a+b &= 0 \\ a-b &= 0 \\ c &= 0 \\ -a+c &= 0 \end{aligned}$$

It is easy to see that the only solution to this set of equations is $a = b = c = 0$, so the kernel of H is just the zero polynomial.

The **range** of H is a subset of the 2×2 matrices, and again we want to know if it is all of $M_{2\times 2}$ or only part of it. So we want to know if we are given any matrix $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ can we always find a, b and c such that

$$H(a + bx + cx^2) = \begin{bmatrix} x & y \\ z & t \end{bmatrix}.$$

To find out we need to solve the equations

$$\begin{aligned} a + b &= x \\ a - b &= y \\ c &= z \\ -a + c &= t \end{aligned}$$

This gives an augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & x \\ 1 & -1 & 0 & y \\ 0 & 0 & 1 & z \\ -1 & 0 & 1 & t \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & -2 & 0 & y-x \\ 0 & 0 & 1 & z \\ 0 & 1 & 1 & t+x \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 1 & 1 & t+x \\ 0 & -2 & 0 & y-x \\ 0 & 0 & 1 & z \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 1 & 1 & t+x \\ 0 & 0 & 2 & y+x+2t \\ 0 & 0 & 1 & z \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 1 & 1 & t+x \\ 0 & 0 & 0 & y+x+2t-2z \\ 0 & 0 & 1 & z \end{bmatrix}$$

The third row shows that we only have a solution to this set of equations when $y + x + 2t - 2z = 0$. This means that the range of H is the set of all matrices $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ where x, y, z, t satisfy $y + x + 2t - 2z = 0$.

Consider T the linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ given by

$$T(A) = A + A^T$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
More explicitly

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

The **kernel** of T is the set of all matrices for which

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For this to happen we require that $a = d = 0, c = -b$, so that the kernel of T is the set of matrices of the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Finding the **range** of T is not so difficult in this case. We can see immediately from

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

that the effect of T on any matrix is to produce a symmetric matrix (a matrix where $A = A^T$). Furthermore any symmetric matrix can be made by the appropriate choice of a, b, c, d. Thus the range of T is the set of symmetric 2×2 matrices.