Warm-up

Question

Do you understand the following sentence?

> *The set of 2×2 symmetric matrices is a subspace of the vector space of 2×2 matrices.*

Overview

Last time we defined an abstract vector space as a set of objects that satisfy 10 axioms. We saw that although $\mathbb{R}^n$ is a vector space, so is *the set of polynomials of a bounded degree* and *the set of all $n \times n$ matrices*. We also defined a *subspace* to be a subset of a vector space which is a vector space in its own right.

> *To check if a subset of a vector space is a subspace, you need to check that it contains the zero vector and is closed under addition and scalar multiplication.*

Recall from 1013 that a matrix has two special subspaces associated to it: the *null space* and the *column space*.

Question

How do the null space and column space generalise to abstract vector spaces?

(Lay, §4.2)

Matrices and systems of equations

Recall the relationship between a matrix and a system of linear equations:

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

The equation $A\mathbf{x} = \mathbf{b}$ corresponds to the system of equations

$$\begin{aligned} a_1x + a_2y + a_3z &= b_1 \\ a_4x + a_5y + a_6z &= b_2. \end{aligned}$$

We can find the solutions by row-reducing the augmented matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \end{bmatrix}$$

to reduced echelon form.

The null space of a matrix

Let A be an $m \times n$ matrix.

Definition

The **null space** of A is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$:

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Example 1

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$.

Then the null space of A is the set of all scalar multiples of $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

We can check easily that $A\mathbf{v} = \mathbf{0}$.
Furthermore, $A(t\mathbf{v}) = tA\mathbf{v} = t\mathbf{0} = \mathbf{0}$, so $t\mathbf{v} \in \text{Nul} A$.
To see that these are the *only* vectors in Nul A, solve the associated homogeneous system of equations.

The null space theorem

Theorem (Null Space is a Subspace)

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

This implies that the set of all solutions to a system of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

The null space theorem

Proof Since A has n columns, Nul A is a subset of $\mathbb{R}^n$. To show a subset is a subspace, recall that we must verify 3 axioms:

- $\mathbf{0} \in \text{Nul } A$ because $A\mathbf{0} = \mathbf{0}$.
- Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in Nul A. Then

  $$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}.$$

  Therefore
  $$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

  This shows that $\mathbf{u}+\mathbf{v} \in \text{Nul } A$.
- If c is any scalar, then

  $$A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

  This shows that $c\mathbf{u} \in \text{Nul } A$.

This proves that Nul A is a subspace of $\mathbb{R}^n$.

Example 2

Let $W = \left\{ \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} : \begin{array}{c} 3s - 4u = 5r + t \\ 3r + 2s - 5t = 4u \end{array} \right\}$ Show that W is a subspace.

Hint: Find a matrix A such that Nul A=W.

If we rearrange the equations given in the description of W we get

$$\begin{aligned} -5r + 3s - t - 4u &= 0 \\ 3r + 2s - 5t - 4u &= 0. \end{aligned}$$

So if A is the matrix $A = \begin{bmatrix} -5 & 3 & -1 & -4 \\ 3 & 2 & -5 & -4 \end{bmatrix}$, then W is the null space of A, and by the Null Space is a Subspace Theorem, W is a subspace of $\mathbb{R}^4$.

An explicit description of Nul A

The span of any set of vectors is a subspace. We can always find a spanning set for Nul A by solving the associated system of equations. (See Lay §1.5).

The column space of a matrix

Let A be an $m \times n$ matrix.

Definition

The **column space** of A is the set of all linear combinations of the columns of A.
If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Why?

Example 3

Suppose

$$W = \left\{ \begin{bmatrix} 3a + 2b \\ 7a - 6b \\ -8b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ a \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{Span } \left\{ \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix} \right\}$$

Put $A = \begin{bmatrix} 3 & 2 \\ 7 & -6 \\ 0 & -8 \end{bmatrix}$. Then $W = \text{Col } A$.

Another equivalent way to describe the column space is

$$\text{Col } A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}.$$

Example 4

Let

$$\mathbf{u} = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -4 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & -5 & -9 \\ 8 & 8 & -6 \\ -5 & -9 & 3 \\ 3 & -2 & -7 \end{bmatrix}$$

Does $\mathbf{u}$ lie in the column space of A?

We just need to answer: *does $A\mathbf{x} = \mathbf{u}$ have a solution?*

Consider the following row reduction:

$$\left[\begin{array}{cccc|c} 5 & -5 & -9 & 6 \\ 8 & 8 & -6 & 7 \\ -5 & -9 & 3 & 1 \\ 3 & -2 & -7 & -4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7/2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the system $A\mathbf{x} = \mathbf{u}$ is consistent.

This means that the vector $\mathbf{u}$ can be written as a linear combination of the columns of A.

Thus $\mathbf{u}$ is contained in the Span of the columns of A, which is the column space of A. So the answer is YES!

Comparing Nul A and Col A

Example 5

Let $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

- The column space of A is a subspace of $\mathbb{R}^k$ where $k =$ ____.
- The null space of A is a subspace of $\mathbb{R}^k$ where $k =$ ____.
- Find a nonzero vector in Col A. (There are infinitely many.)
- Find a nonzero vector in Nul A.

For the final point, you may use the following row reduction:

$$\begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{bmatrix}$$

Table: For any $m \times n$ matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1.Col A is a subspace of $\mathbb{R}^m$.
2. Any $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	2. Any $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
3. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	3. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

Question

How does all this generalise to an abstract vector space?

An $m \times n$ matrix defines a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and the null space and column space are subspaces of the domain and range, respectively.
We'd like to define the analogous notions for functions between arbitrary vector spaces.

Linear transformations

Definition

A *linear transformation* from a vector space V to a vector space W is a function $T : V \to W$ such that

L1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in V$;

L2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for $\mathbf{u} \in V, c \in \mathbb{R}$.

Matrix multiplication always defines a linear transfomation.

Example 6

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix}$. Then the mapping defined by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.
For example

$$T_A\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

Example 7

Let $T : \mathbb{P}_2 \to \mathbb{P}_0$ be the map defined by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Then T is a linear transformation.

$$\begin{aligned} T((a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t + b_2 t^2)) \\ &= T((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2) \\ &= 2(a_0 + b_0) \\ &= 2a_0 + 2b_0 \\ &= T(a_0 + a_1 t + a_2 t^2) + T(b_0 + b_1 t + b_2 t^2). \end{aligned}$$

$$\begin{aligned} T(c(a_0 + a_1 t + a_2 t^2)) &= T(ca_0 + ca_1 t + ca_2 t^2) \\ &= 2ca_0 \\ &= cT(a_0 + a_1 t + a_2 t^2) \end{aligned}$$

Kernel of a linear transformation

Definition

The *kernel* of a linear transformation $T : V \to W$ is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$.
We write

$$\ker T = \{\mathbf{u} \in V : T(\mathbf{u}) = \mathbf{0}\}.$$

The kernel of a linear transformation T is analogous to the null space of a matrix, and $\ker T$ is a subspace of V.

If $\ker T = \{\mathbf{0}\}$, then T is *one to one*.

The range of a linear transformation

Definition

The *range* of a linear transformation $T : V \to W$ is the set of all vectors in W of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in V.
We write

$$\text{Range } T = \{\mathbf{w} : \mathbf{w} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in V\}.$$

The range of a linear transformation is analogous to the columns space of a matrix, and Range T is a subspace of W.

The linear transformation T is *onto* if its range is all of W.

Example 8

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_0$ by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Find the kernel and range of T.

The kernel consists of all the polynomials in $\mathbb{P}_2$ satisfying $2a_0 = 0$. This is the set

$$\{a_1 t + a_2 t^2\}.$$

The range of T is $\mathbb{P}_0$.

Example 9

The differential operator $D : \mathbb{P}_2 \to \mathbb{P}_1$ defined by $D(\mathbf{p}(x)) = \mathbf{p}'(x)$ is a linear transformation. Find its kernel and range.

First we see that

$$D(a + bx + cx^2) = b + 2cx.$$

So

$$\begin{aligned} \ker D &= \{a + bx + cx^2 : D(a + bx + cx^2) = 0\} \\ &= \{a + bx + cx^2 : b + 2cx = 0\} \end{aligned}$$

But $b + 2cx = 0$ if and only if $b = 2c = 0$, which implies $b = c = 0$. Therefore

$$\begin{aligned} \ker D &= \{a + bx + cx^2 : b = c = 0\} \\ &= \{a : a \in \mathbb{R}\} \end{aligned}$$

The range of D is all of $\mathbb{P}_1$ since every polynomial in $\mathbb{P}_1$ is the image under D (i.e the derivative) of *some* polynomial in $\mathbb{P}_2$.
To be more specific, if $a + bx$ is in $\mathbb{P}_1$, then

$$a + bx = D\left(ax + \frac{b}{2}x^2\right)$$

Example 10

Define $S : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$S(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}.$$

That is, if $\mathbf{p}(x) = a + bx + cx^2$, we have

$$S(\mathbf{p}) = \begin{bmatrix} a \\ a + b + c \end{bmatrix}.$$

Show that S is a linear transformation and find its kernel and range.

Leaving the first part as an exercise to try on your own, we'll find the kernel and range of S.

- From what we have above, $\mathbf{p}$ is in the kernel of S if and only if

  $$S(\mathbf{p}) = \begin{bmatrix} a \\ a + b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  For this to occur we must have $a = 0$ and $c = -b$.
  So $\mathbf{p}$ is in the kernel of S if

  $$\mathbf{p}(x) = bx - bx^2 = b(x - x^2).$$

  This gives $\ker S = \text{Span } \{x - x^2\}$.

- The range of S.
  Since $S(\mathbf{p}) = \begin{bmatrix} a \\ a + b + c \end{bmatrix}$ and a, b and c are any real numbers, the range of S is all of $\mathbb{R}^2$.

Example 11

let $F : M_{2\times 2} \to M_{2\times 2}$ be the linear transformation defined by taking the transpose of the matrix:

$$F(A) = A^T.$$

We find the kernel and range of F.

We see that

$$\begin{aligned} \ker F &= \{A \text{ in } M_{2\times 2} : F(A) = 0\} \\ &= \{A \text{ in } M_{2\times 2} : A^T = 0\} \end{aligned}$$

But if $A^T = 0$, then $A = (A^T)^T = 0^T = 0$. It follows that $\ker F = 0$.
For any matrix A in $M_{2\times 2}$, we have $A = (A^T)^T = F(A^T)$. Since A^T is in $M_{2\times 2}$ we deduce that Range $F = M_{2\times 2}$.

Example 12

Let $S : \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$S(\mathbf{p}(x)) = \int_0^1 \mathbf{p}(x)dx.$$

We find the kernel and range of S.

In detail, we have

$$\begin{aligned} S(a + bx) &= \int_0^1 (a + bx)dx \\ &= \left[ax + \frac{b}{2}x^2\right]_0^1 \\ &= a + \frac{b}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \ker S &= \{a + bx : S(a + bx) = 0\} \\ &= \left\{a + bx : a + \frac{b}{2} = 0\right\} \\ &= \left\{a + bx : a = -\frac{b}{2}\right\} \\ &= \left\{-\frac{b}{2} + bx\right\} \end{aligned}$$

Geometrically, $\ker S$ consists of all those linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis on the interval $[0, 1]$.

The range of S is $\mathbb{R}$, since every number can be obtained as the image under S of some polynomial in $\mathbb{P}_1$.
For example, if a is an arbitrary real number, then

$$\int_0^1 a\ dx = [ax]_0^1 = a - 0 = a.$$