# Warm-up

## Question

*Do you understand the following sentence?*

*The set of $2 \times 2$ symmetric matrices is a subspace of the vector space of $2 \times 2$ matrices.*

## Overview

Last time we defined an abstract vector space as a set of objects that satisfy 10 axioms. We saw that although $\mathbb{R}^n$ is a vector space, so is *the set of polynomials of a bounded degree* and *the set of all $n \times n$ matrices*. We also defined a *subspace* to be a subset of a vector space which is a vector space in its own right.

> *To check if a subset of a vector space is a subspace, you need to check that it contains the zero vector and is closed under addition and scalar multiplication.*

Recall from 1013 that a matrix has two special subspaces associated to it: the *null space* and the *column space*.

### Question

*How do the null space and column space generalise to abstract vector spaces?*

(Lay, §4.2)

## Matrices and systems of equations

Recall the relationship between a matrix and a system of linear equations:

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

The equation $A\mathbf{x} = \mathbf{b}$ corresponds to the system of equations

$$a_1 x + a_2 y + a_3 z = b_1$$
$$a_4 x + a_5 y + a_6 z = b_2.$$

We can find the solutions by row-reducing the augmented matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \end{bmatrix}$$

to reduced echelon form.

# The null space of a matrix

Let $A$ be an $m \times n$ matrix.

## Definition

The **null space** of $A$ is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$:

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

## Example 1

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$.

Then the null space of $A$ is the set of all scalar multiples of $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

We can check easily that $A\mathbf{v} = \mathbf{0}$.

Furthermore, $A(t\mathbf{v}) = tA\mathbf{v} = t\mathbf{0} = \mathbf{0}$, so $t\mathbf{v} \in \text{Nul}A$.

To see that these are the *only* vectors in Nul A, solve the associated homogeneous system of equations.

# The null space theorem

Theorem (Null Space is a Subspace)

*The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$.*

This implies that the set of all solutions to a system of $m$ homogeneous linear equations in $n$ unknowns is a subspace of $\mathbb{R}^n$.

## The null space theorem

**Proof** Since $A$ has $n$ columns, Nul $A$ is a subset of $\mathbb{R}^n$. To show a subset is a subspace, recall that we must verify 3 axioms:

## The null space theorem

**Proof** Since $A$ has $n$ columns, Nul $A$ is a subset of $\mathbb{R}^n$. To show a subset is a subspace, recall that we must verify 3 axioms:

- $\mathbf{0} \in$ Nul $A$ because $A\mathbf{0} = \mathbf{0}$.
- Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in Nul $A$. Then

$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}.$$

Therefore

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

This shows that $\mathbf{u} + \mathbf{v} \in$ Nul $A$.

- If $c$ is any scalar, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

This shows that $c\mathbf{u} \in$ Nul $A$.

This proves that Nul $A$ is a subspace of $\mathbb{R}^n$.

Example 2

Let $W = \left\{ \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} : \begin{array}{l} 3s - 4u = 5r + t \\ 3r + 2s - 5t = 4u \end{array} \right\}$ Show that $W$ is a subspace.

### Example 2

Let $W = \left\{ \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} : \begin{array}{l} 3s - 4u = 5r + t \\ 3r + 2s - 5t = 4u \end{array} \right\}$ Show that $W$ is a subspace.

Hint: Find a matrix $A$ such that Nul $A = W$.

> ### Example 2
>
> Let $W = \left\{ \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} : \begin{array}{l} 3s - 4u = 5r + t \\ 3r + 2s - 5t = 4u \end{array} \right\}$ Show that $W$ is a subspace.
>
> Hint: Find a matrix $A$ such that Nul $A = W$.

If we rearrange the equations given in the description of $W$ we get

$$\begin{array}{rcl} -5r + 3s - t - 4u & = & 0 \\ 3r + 2s - 5t - 4u & = & 0. \end{array}$$

So if $A$ is the matrix $A = \begin{bmatrix} -5 & 3 & -1 & -4 \\ 3 & 2 & -5 & -4 \end{bmatrix}$, then $W$ is the null space of

$A$, and by the Null Space is a Subspace Theorem, $W$ is a subspace of $\mathbb{R}^4$.

# An explicit description of Nul $A$

The span of any set of vectors is a subspace. We can always find a spanning set for Nul A by solving the associated system of equations. (See Lay §1.5).

# The column space of a matrix

Let $A$ be an $m \times n$ matrix.

### Definition

The **column space** of $A$ is the set of all linear combinations of the columns of $A$.
If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

# The column space of a matrix

Let $A$ be an $m \times n$ matrix.

### Definition

The **column space** of $A$ is the set of all linear combinations of the columns of $A$.
If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

### Theorem

*The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.*

Why?

### Example 3

Suppose

$$W = \left\{ \begin{bmatrix} 3a + 2b \\ 7a - 6b \\ -8b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Find a matrix $A$ such that $W = \text{Col } A$.

### Example 3

Suppose

$$W = \left\{ \begin{bmatrix} 3a + 2b \\ 7a - 6b \\ -8b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Find a matrix $A$ such that $W = \text{Col } A$.

$$W = \left\{ a \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{Span } \left\{ \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix} \right\}$$

Put $A = \begin{bmatrix} 3 & 2 \\ 7 & -6 \\ 0 & -8 \end{bmatrix}$. Then $W = \text{Col } A$.

Another equivalent way to describe the column space is

$$\text{Col } A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}.$$

### Example 4

Let

$$\mathbf{u} = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -4 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & -5 & -9 \\ 8 & 8 & -6 \\ -5 & -9 & 3 \\ 3 & -2 & -7 \end{bmatrix}$$

Does $\mathbf{u}$ lie in the column space of $A$?

We just need to answer: *does $A\mathbf{x} = \mathbf{u}$ have a solution?*

Consider the following row reduction:

$$\begin{bmatrix} 5 & -5 & -9 & 6 \\ 8 & 8 & -6 & 7 \\ -5 & -9 & 3 & 1 \\ 3 & -2 & -7 & -4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 11/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the system $A\mathbf{x} = \mathbf{u}$ is consistent.

This means that the vector $\mathbf{u}$ can be written as a linear combination of the columns of $A$.

Consider the following row reduction:

$$\begin{bmatrix} 5 & -5 & -9 & 6 \\ 8 & 8 & -6 & 7 \\ -5 & -9 & 3 & 1 \\ 3 & -2 & -7 & -4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 11/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the system $A\mathbf{x} = \mathbf{u}$ is consistent.

This means that the vector $\mathbf{u}$ can be written as a linear combination of the columns of $A$.

Thus $\mathbf{u}$ is contained in the Span of the columns of $A$, which is the column space of $A$. So the answer is YES!

# Comparing Nul $A$ and Col $A$

### Example 5

Let $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

- The column space of $A$ is a subspace of $\mathbb{R}^k$ where $k =$ _____.
- The null space of $A$ is a subspace of $\mathbb{R}^k$ where $k =$ _____.
- Find a nonzero vector in Col $A$. (There are infinitely many.)
- Find a nonzero vector in Nul $A$.

# Comparing Nul $A$ and Col $A$

### Example 5

Let $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

- The column space of $A$ is a subspace of $\mathbb{R}^k$ where $k = \underline{\quad}$.
- The null space of $A$ is a subspace of $\mathbb{R}^k$ where $k = \underline{\quad}$.
- Find a nonzero vector in Col $A$. (There are infinitely many.)
- Find a nonzero vector in Nul $A$.

For the final point, you may use the following row reduction:

$$\begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{bmatrix}$$

Table: For any $m \times n$ matrix $A$

| Nul $A$ | Col $A$ |
|---|---|
| 1. Nul $A$ is a subspace of $\mathbb{R}^n$. | 1. Col $A$ is a subspace of $\mathbb{R}^m$. |
| 2. Any $\mathbf{v}$ in Nul $A$ has the property that $A\mathbf{v} = \mathbf{0}$. | 2. Any $\mathbf{v}$ in Col $A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent. |
| 3. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. | 3. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$. |

### Question

*How does all this generalise to an abstract vector space?*

An $m \times n$ matrix defines a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and the null space and column space are subspaces of the domain and range, respectively.
We'd like to define the analogous notions for functions between arbitrary vector spaces.

# Linear transformations

### Definition

A *linear transformation* from a vector space $V$ to a vector space $W$ is a function $T : V \to W$ such that

L1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in V$;

L2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for $\mathbf{u} \in V, c \in \mathbb{R}$.

Matrix multiplication always defines a linear transfomation.

## Example 6

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix}$. Then the mapping defined by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.
For example

$$T_A \left( \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

Let $T : \mathbb{P}_2 \to \mathbb{P}_0$ be the map defined by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Then $T$ is a linear transformation.

$$
\begin{aligned}
T((a_0 + a_1 t + a_2 t^2) &+ (b_0 + b_1 t + b_2 t^2)) \\
&= T((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2) \\
&= 2(a_0 + b_0) \\
&= 2a_0 + 2b_0 \\
&= T(a_0 + a_1 t + a_2 t^2) + T(b_0 + b_1 t + b_2 t^2).
\end{aligned}
$$

$$
\begin{aligned}
T(c(a_0 + a_1 t + a_2 t^2)) &= T(ca_0 + ca_1 t + ca_2 t^2) \\
&= 2ca_0 \\
&= cT(a_0 + a_1 t + a_2 t^2)
\end{aligned}
$$

# Kernel of a linear transformation

### Definition

The *kernel* of a linear transformation $T : V \to W$ is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$.
We write

$$\ker T = \{\mathbf{u} \in V : T(\mathbf{u}) = \mathbf{0}\}.$$

The kernel of a linear transformation $T$ is analogous to the null space of a matrix, and ker $T$ is a subspace of $V$.

If ker $T = \{\mathbf{0}\}$, then $T$ is *one to one*.

# The range of a linear transformation

### Definition

The *range* of a linear transformation $T : V \to W$ is the set of all vectors in $W$ of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in $V$.
We write

$$\text{Range } T = \{\mathbf{w} : \mathbf{w} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in V\}.$$

The range of a linear transformation is analogous to the columns space of a matrix, and Range $T$ is a subspace of $W$.

The linear transformation $T$ is *onto* if its range is all of $W$.

### Example 8

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_0$ by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Find the kernel and range of $T$.

### Example 8

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_0$ by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Find the kernel and range of $T$.

The kernel consists of all the polynomials in $\mathbb{P}_2$ satisfying $2a_0 = 0$. This is the set

$$\{a_1 t + a_2 t^2\}.$$

The range of $T$ is $\mathbb{P}_0$.

Example 9

The differential operator $D : \mathbb{P}_2 \to \mathbb{P}_1$ defined by $D(\mathbf{p}(x)) = \mathbf{p}'(x)$ is a linear transformation. Find its kernel and range.

## Example 9

The differential operator $D : \mathbb{P}_2 \to \mathbb{P}_1$ defined by $D(\mathbf{p}(x)) = \mathbf{p}'(x)$ is a linear transformation. Find its kernel and range.

First we see that

$$D(a + bx + cx^2) = b + 2cx.$$

So

$$
\begin{aligned}
\ker D &= \{a + bx + cx^2 : D(a + bx + cx^2) = 0\} \\
&= \{a + bx + cx^2 : b + 2cx = 0\}
\end{aligned}
$$

But $b + 2cx = 0$ if and only if

### Example 9

The differential operator $D : \mathbb{P}_2 \to \mathbb{P}_1$ defined by $D(\mathbf{p}(x)) = \mathbf{p}'(x)$ is a linear transformation. Find its kernel and range.

First we see that

$$D(a + bx + cx^2) = b + 2cx.$$

So

$$
\begin{aligned}
\ker D &= \{a + bx + cx^2 : D(a + bx + cx^2) = 0\} \\
&= \{a + bx + cx^2 : b + 2cx = 0\}
\end{aligned}
$$

But $b + 2cx = 0$ if and only if $b = 2c = 0$, which implies $b = c = 0$. Therefore

$$
\begin{aligned}
\ker D &= \{a + bx + cx^2 : b = c = 0\} \\
&= \{a : a \in \mathbb{R}\}
\end{aligned}
$$

The range of $D$ is all of $\mathbb{P}_1$ since every polynomial in $\mathbb{P}_1$ is the image under $D$ (i.e the derivative) of *some* polynomial in $\mathbb{P}_2$.

To be more specific, if $a + bx$ is in $\mathbb{P}_1$, then

$$a + bx = D\left(ax + \frac{b}{2}x^2\right)$$

### Example 10

Define $S : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$S(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}.$$

That is, if $\mathbf{p}(x) = a + bx + cx^2$, we have

$$S(\mathbf{p}) = \begin{bmatrix} a \\ a + b + c \end{bmatrix}.$$

Show that $S$ is a linear transformation and find its kernel and range.

Leaving the first part as an exercise to try on your own, we'll find the kernel and range of $S$.

- From what we have above, $\mathbf{p}$ is in the kernel of $S$ if and only if

$$S(\mathbf{p}) = \begin{bmatrix} a \\ a + b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For this to occur we must have $a = 0$ and $c = -b$.
So $\mathbf{p}$ is in the kernel of $S$ if

$$\mathbf{p}(x) = bx - bx^2 = b(x - x^2).$$

This gives $\ker S = \text{Span } \{x - x^2\}$.

- The range of $S$.

  Since $S(\mathbf{p}) = \begin{bmatrix} a \\ a + b + c \end{bmatrix}$ and $a, b$ and $c$ are any real numbers, the range of $S$ is all of $\mathbb{R}^2$.

### Example 11

let $F : M_{2 \times 2} \to M_{2 \times 2}$ be the linear transformation defined by taking the transpose of the matrix:

$$F(A) = A^T.$$

We find the kernel and range of $F$.

### Example 11

let $F : M_{2 \times 2} \to M_{2 \times 2}$ be the linear transformation defined by taking the transpose of the matrix:

$$F(A) = A^T.$$

We find the kernel and range of $F$.

We see that

$$
\begin{aligned}
\ker F &= \{A \text{ in } M_{2 \times 2} : F(A) = 0\} \\
&= \{A \text{ in } M_{2 \times 2} : A^T = 0\}
\end{aligned}
$$

But if $A^T = 0$, then $A = (A^T)^T = 0^T = 0$. It follows that $\ker F = 0$.
For any matrix $A$ in $M_{2 \times 2}$, we have $A = (A^T)^T = F(A^T)$. Since $A^T$ is in $M_{2 \times 2}$ we deduce that Range $F = M_{2 \times 2}$.

### Example 12

Let $S : \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$S(\mathbf{p}(x)) = \int_0^1 \mathbf{p}(x)dx.$$

We find the kernel and range of $S$.

In detail, we have

$$
\begin{aligned}
S(a + bx) &= \int_0^1 (a + bx)dx \\
&= \left[ ax + \frac{b}{2}x^2 \right]_0^1 \\
&= a + \frac{b}{2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\ker S &= \{a + bx : S(a + bx) = 0\} \\
&= \left\{a + bx : a + \frac{b}{2} = 0\right\} \\
&= \left\{a + bx : a = -\frac{b}{2}\right\} \\
&= \left\{-\frac{b}{2} + bx\right\}
\end{aligned}
$$

Geometrically, $\ker S$ consists of all those linear polynomials whose graphs have the property that the area between the line and the $x$-axis is equally distributed above and below the axis on the interval $[0, 1]$.

The range of $S$ is $\mathbb{R}$, since every number can be obtained as the image under $S$ of some polynomial in $\mathbb{P}_1$.

For example, if $a$ is an arbitrary real number, then

$$\int_0^1 a \, dx = [ax]_0^1 = a - 0 = a.$$