Overview

Last week we introduced the notion of an abstract vector space, and we saw that apparently different sets like polynomials, continuous functions, and symmetric matrices all satisfy the 10 axioms defining a vector space. We also discussed *subspaces*, subsets of a vector space which are vector spaces in their own right. To any **linear transformation** between vector spaces, one can associate two special subspaces:

- the kernel
- the range.

Today we'll talk about linearly independent vectors and bases for abstract vector spaces. The definitions are the same for abstract vector spaces as for Euclidean space, so you may find it helpful to review the material covered in 1013.

(Lay, §4.3, §4.4)

Linear independence

Definition (Linear Independence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be *linearly independent* if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (1)$$

has *only* the trivial solution, $c_1 = c_2 = \cdots = c_p = 0$.

Definition

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be *linearly dependent* if it is not linearly independent, i.e., if there are some weights $c_1, c_2, \ldots, c_p$, **not all zero**, such that (1) holds.

Here's a recipe for proving a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent:

1. Write the equation
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$
2. Manipulate the equation to prove that all the $c_i = 0$. Done!
3. If you find a different solution, then you've instead proven that the set is linearly dependent.

!

If you start by assuming the c_i are all zero, you can't prove anything!

Example 1

Show that the vectors $2x+3$, $4x^2$, and $1+x$ are linearly independent in $\mathbb{P}_2$.

1. Set a linear combination of the given vectors equal to $\mathbf{0}$:

$$a(2x+3)+b(4x^2)+c(1+x)=0.$$

2. Now manipulate the equation to see what coefficients are possible:

$$(3a+c)+(2a+c)x+4bx^2=0.$$

This implies

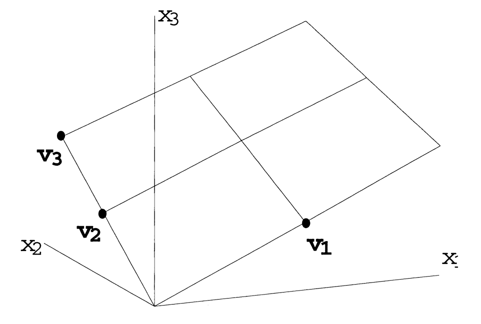
$$\begin{aligned} 3a+c&=0\\ 2a+c&=0\\ 4b&=0 \end{aligned}$$

But the only solution to this system is $a=b=c=0$, so the given vectors are linearly independent.

Span of a set

Example 2

Consider the plane H illustrated below:



Which of the following are valid descriptions of H?

(a) $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_3\}$

(c) $H = \text{Span}\ \{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\, \mathbf{v}_3\}$

The spanning set theorem

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ equals H:

$$H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}.$$

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Theorem (The spanning set theorem)

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \mathit{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

(a) *If the vector $\mathbf{v_k}$ in S is a linear combination of the remaining vectors of S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.*

(b) *If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.*

Example 3

Find a basis for $\mathbb{P}_2$ which is a subset of $S = \{1, x, 1 + x, x + 3, x^2\}$.

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Example 3

Find a basis for $\mathbb{P}_2$ which is a subset of $S = \{1, x, 1+x, x+3, x^2\}$.

First, let's check if we have any hope: does S span $\mathbb{P}_2$?
The spanning set theorem says that if any vector in S is a linear combination of the other vectors in S, we can remove it without changing the span.

$$\text{Span } \{1, x, 1+x, x+3, x^2\} = \text{Span } \{1, x, x^2\}.$$

The set $\{1, x, x^2\}$ spans $\mathbb{P}_2$ and is linearly independent, so it's a basis.

Other correct answers are $\{1, 1+x, x^2\}$, $\{1, x+3, x^2\}$, $\{x+3, 1+x, x^2\}$, $\{x, x+3, x^2\}$, and $\{x, 1+x, x^2\}$.

Bases for Nul A and Col A

Given any subspace V, it's natural to ask for a basis of V.

When a subspace is defined as the null space or column space of a matrix, there is an algorithm for finding a basis.

Recall the following example from the last lecture:

Example 4

Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row reducing the matrix gives

$$\begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r1 \to r1 - 5r2} \begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is equivalent to the system of equations

$$\begin{array}{ccccccccccc} x_1 & & & + & 6x_3 & - & 8x_4 & + & x_5 & = & 0 \\ & & x_2 & - & 2x_3 & + & x_4 & & & = & 0 \end{array}$$

The general solutions is $x_1 = -6x_3 + 8x_4 - x_5$, $x_2 = 2x_3 - x_4$. The free variables are x_3, x_4 and x_5.

We express the general solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\uparrow \, \mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\uparrow \, \mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\uparrow \, \mathbf{w}}$$

We get a vector for each free variable, and these form a spanning set for Nul A. In fact, this spanning set is linearly independent, so it's a basis.

A basis for Col A

Theorem

The pivot columns of a matrix A form a basis for Col A.

Although we won't prove this is true, we'll see why it should be plausible using this example.

Example 5

We find a basis for Col A, where

$$\begin{aligned} A &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 4 & 3 & 33 & -6 & 8 \\ 2 & -1 & 9 & -8 & -4 \\ -2 & 2 & -6 & 10 & 2 \end{bmatrix} \end{aligned}$$

We row reduce A to get

$$A = \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 4 & 3 & 33 & -6 & 8 \\ 2 & -1 & 9 & -8 & -4 \\ -2 & 2 & -6 & 10 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

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$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} \to \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix}$$

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Note that

$$\mathbf{b}_3 = 6\mathbf{b}_1 + 3\mathbf{b}_2 \text{ and } \mathbf{b}_4 = -3\mathbf{b}_1 + 2\mathbf{b}_2$$

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We can check that

$$\mathbf{a}_3 = 6\mathbf{a}_1 + 3\mathbf{a}_2 \text{ and } \mathbf{a}_4 = -3\mathbf{a}_1 + 2\mathbf{a}_2$$

Elementary row operations do not affect the linear dependence relationships among the columns of the matrix.

$$B = \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Looking at the columns of B, we can guess that $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_5}$ form a basis for Col B.

We check

1. $\mathbf{b}_2$ is not a multiple of $\mathbf{b}_1$.
2. $\mathbf{b}_5$ is not a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

Elementary row operations do not affect the linear dependence relationships among the columns of the matrix.

Since $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_5}\}$ is a basis for Col B,

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} \text{ is a basis for Col } A.$$

Review

1. To find a basis for Nul A, use elementary row operations to transform $[A\ \mathbf{0}]$ to an equivalent reduced row echelon form $[B\ \mathbf{0}]$. Use the row reduced echelon form to find a parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. If Nul $A \neq \{\mathbf{0}\}$, the vectors found in this parametric form of the general solution are automatically linearly independent and form a basis for Nul A.

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1. To find a basis for Nul A, use elementary row operations to transform $[A\ \mathbf{0}]$ to an equivalent reduced row echelon form $[B\ \mathbf{0}]$. Use the row reduced echelon form to find a parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. If Nul $A \neq \{\mathbf{0}\}$, the vectors found in this parametric form of the general solution are automatically linearly independent and form a basis for Nul A.
2. A basis for Col A is is formed from the pivot columns of A. The matrix B determines the pivot columns, but it is important to return to the matrix A.

The unique representation theorem

Theorem (The Unique Representation Theorem)

Suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V. Then each $\mathbf{x} \in V$ has a unique expansion

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n \tag{2}$$

where $c_1, \ldots, c_n$ are in $\mathbb{R}^n$.

We say that the c_i are the *coordinates* of $\mathbf{x}$ relative to the basis $\mathcal{B}$, and we write $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Example 6

We found several bases for $\mathbb{P}_2$, including

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \{1, x+3, x^2\}.$$

Find the coordinates for $5 + 2x + 3x^2$ with respect to $\mathcal{B}$ and $\mathcal{C}$.

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We have

$$5 + 2x + 3x^2 = 5(1) + 2(x) + 3(x^2),$$

so $[5 + 2x + 3x^2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$.

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so $[5 + 2x + 3x^2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$.

Similarly,

$$5 + 2x + 3x^2 = -1(1) + 2(x+3) + 3(x^2)$$

so $[5 + 2x + 3x^2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

Why is the Unique Representation Theorem true?

Suppose that $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a basis for V, and that we can write

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$$
$$\mathbf{x} = d_1\mathbf{b_1} + \cdots + d_n\mathbf{b_n}.$$

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$$\mathbf{x} = d_1\mathbf{b_1} + \cdots + d_n\mathbf{b_n}.$$

We'd like to show that this implies $c_i = d_i$ for all i. Subtract the second line from the first to get

$$\mathbf{0} = (c_1 - d_1)\mathbf{b_1} + \cdots + (c_n - d_n)\mathbf{b_n}.$$

Since $\mathcal{B}$ is a basis, the $\mathbf{b_i}$ are linearly independent. This implies all the coefficients $c_i - d_i$ are equal to 0.
Thus, $c_i = d_i$ for all i.

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However, we can express a vector in $\mathbb{R}^n$ in terms of any basis.

Example 7

Suppose $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{E}}\}$. Then $\mathbf{i} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{E}}$, so $\mathbf{i} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}_{\mathcal{B}}$.