# Eigenvectors and eigenvalues

From Lay, §5.1

# Overview

Most of the material we've discussed so far falls loosely under two headings:

- geometry of $\mathbb{R}^n$, and
- generalisation of 1013 material to abstract vector spaces.

Today we'll begin our study of eigenvectors and eigenvalues. This is fundamentally different from material you've seen before, but we'll draw on the earlier material to help us understand this central concept in linear algebra. This is also one of the topics that you're most likely to see applied in other contexts.

**Question**

*If you want to understand a linear transformation, what's the smallest amount of information that tells you something meaningful?*

This is a very vague question, but studying eigenvalues and eigenvectors gives us one way to answer it.

From Lay, §5.1

**Definition**

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An *eigenvalue* of an $n \times n$ matrix $A$ is a scalar $\lambda$ such that $A\mathbf{x} = \lambda\mathbf{x}$ has a non-zero solution; such a vector $\mathbf{x}$ is called an *eigenvector corresponding to* $\lambda$.

## Example 1

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Then any nonzero vector $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue 3:

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 3x \\ 0 \end{bmatrix}.$$

Similarly, any nonzero vector $\begin{bmatrix} 0 \\ y \end{bmatrix}$ is an eigenvector for the eigenvalue 2.

Sometimes it's not as obvious what the eigenvectors are.

## Example 2

Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Then any nonzero vector $\begin{bmatrix} x \\ x \end{bmatrix}$ is an eigenvector for the eigenvalue 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 2x \\ 2x \end{bmatrix}.$$

Also, any nonzero vector $\begin{bmatrix} x \\ -x \end{bmatrix}$ is an eigenvector for the eigenvalue 0:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that an eigenvalue can be 0, but an eigenvector must be nonzero.

# Eigenspaces

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Let $A$ be an $n \times n$ matrix, and let $\lambda$ be an eigenvalue of $A$. The collection of all eigenvectors corresponding to $\lambda$, together with the *zero* vector, is called the eigenspace of $\lambda$ and is denoted by $E_\lambda$.

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$$E_\lambda = \mathrm{Nul}\ (A - \lambda I)$$

## Example 3

As before, let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. In the previous example, we verified that the given vectors were eigenvectors for the eigenvalues 2 and 0.
To find the eigenvectors for 2, solve for the null space of $B - 2I$:

$$\mathrm{Nul}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} x \\ x \end{bmatrix}.$$

To find the eigenvectors for the eigenvalue 0, solve for the null space of $B - 0I = B$.

You can always check if you've correctly identified an eigenvector: simply multiply it by the matrix and make sure you get back a scalar multiple.

# Eigenvalues of triangular matrix

**Theorem**

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Then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{33} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

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$(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \text{or} \quad \lambda = a_{33}$$

An $n \times n$ matrix $A$ has eigenvalue $\lambda$ if and only if the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

has a nontrivial solution.

Equivalently, $\lambda$ is an eigenvalue if $A - \lambda I$ is not invertible.

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- The scalar 0 is an eigenvalue of $A$ if and only if $A$ is *not invertible*.

## Theorem

*Let $A$ be an $n \times n$ matrix. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.*

The proof of this theorem is in Lay: Theorem 2, Section 5.1.

## Example 4

Consider the matrix

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}.$$

We are given that $A$ has an eigenvalue $\lambda = 3$ and we want to find a basis for the eigenspace $E_3$.

*Solution* We find the null space of $A - 3I$:

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We are given that $A$ has an eigenvalue $\lambda = 3$ and we want to find a basis for the eigenspace $E_3$.

*Solution* We find the null space of $A - 3I$:

$$A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A - 3I \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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So we get a single equation

$$x + 2y + 3z = 0 \quad \text{or} \quad x = -2y - 3z$$

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$$\mathbf{x} = \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

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Hence $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_3$.